Subgroups of Finite Wreath Product Groups for \( p=3 \)

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SUBGROUPS OF FINITE WREATH PRODUCT GROUPS FOR $p = 3$

A Thesis

Presented to

The Graduate Faculty of The University of Akron

In Partial Fulfillment

of the Requirements for the Degree

Master of Science

Jessica Lynn Gonda

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ABSTRACT

Let $M$ be the additive abelian group of 3-by-3 matrices whose entries are from the ring of integers modulo 9. The problem of determining all the normal subgroups of the regular wreath product group $P = \mathbb{Z}_9 \wr (\mathbb{Z}_3 \times \mathbb{Z}_3)$ that are contained in its base subgroup is equivalent to the problem of determining the subgroups of $M$ that are invariant under two particular endomorphisms of $M$. In this thesis we give a partial solution to the latter problem by implementing a systematic approach using concepts from group theory and linear algebra.
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CHAPTER I
INTRODUCTION

The work of this thesis is prompted by Dr. Riedl’s research [1] on the regular wreath product group \( P = \mathbb{Z}_9 \wr (\mathbb{Z}_3 \times \mathbb{Z}_3) \) and in particular by his efforts to identify all the normal subgroups of \( P \) that are contained in the base subgroup of \( P \). Determining all the normal subgroups is equivalent to determining all the subgroups of a particular matrix group that are invariant under two specific endomorphisms. The latter will be the focus of this thesis.

Dr. Riedl has develop a systematic method that splits this extensive computational problem into 98 more manageable and non-overlapping subproblems. The goal in this paper is to solve one of these subproblems. Several definitions and some background information are necessary to state the problem explicitly.

In this chapter along with Chapter 2, definitions are taken from Dr. Riedl’s notes [1] and Stacie Wyles’s Thesis [2].

**Definition 1.0.1.** Let \( M \) to be the additive abelian group consisting of all 3-by-3 matrices with entries taken from the ring of integers modulo 9, namely \( \mathbb{Z}_9 = \{0, 1, ..., 8\} \).

We refer to \( M \) as the matrix group.
Definition 1.0.2. Now we will define two endomorphisms $\partial_i : M \to M$ for $i \in \{1, 2\}$.

Let $x \in M$ be arbitrary and write

$$x = \begin{bmatrix} x_{0,0} & x_{0,1} & x_{0,2} \\ x_{1,0} & x_{1,1} & x_{1,2} \\ x_{2,0} & x_{2,1} & x_{2,2} \end{bmatrix},$$

which we call the **vertical derivative** of the matrix $x$. We also define the matrix

$$\partial_1 x = \begin{bmatrix} x_{1,0} & x_{1,1} & x_{1,2} \\ x_{2,0} & x_{2,1} & x_{2,2} \\ -3(x_{1,0} + x_{2,0}) & -3(x_{1,1} + x_{2,1}) & -3(x_{1,2} + x_{2,2}) \end{bmatrix},$$

We define the matrix

$$\partial_2 x = \begin{bmatrix} x_{0,1} & x_{0,2} & -3(x_{0,1} + x_{0,2}) \\ x_{1,1} & x_{1,2} & -3(x_{1,1} + x_{1,2}) \\ x_{2,1} & x_{2,2} & -3(x_{2,1} + x_{2,2}) \end{bmatrix},$$

which we call the **horizontal derivative** of the matrix $x$.

Definition 1.0.3. For $i \in \{1, 2\}$, a subgroup $H$ of $M$ is said to be $\partial_i$-invariant in case $\partial_i H \subseteq H$, which is to say that $\partial_i x \in H$ for every $x \in H$. A subgroup $H$ of $M$ is said to be **doubly-invariant** provided that $H$ is both $\partial_1$-invariant and $\partial_2$-invariant.

Let $\mathcal{V}$ denote the set consisting of all the double-invariant subgroups of $M$.

Dr. Riedl’s research goal is to determine all the subgroups of $M$ that are members of the set $\mathcal{V}$. As stated earlier, Dr. Riedl has developed a systematic approach which breaks this enormous computational problem into smaller non-overlapping sub-
problems that are more manageable. In this thesis we obtain a complete solution to one of these subproblems.

In order to describe what we have done in this thesis, we now give an overview of Dr. Riedl’s approach. (For more details, see Chapters 1 and 2 of Stacie’s Wyles’s thesis [2]). First, we need a definition.

**Definition 1.0.4.** Let \( \theta \) and \( \varphi \) be endomorphisms of the group \( M \). The composition of \( \theta \) and \( \varphi \) is an endomorphism of \( M \), which we denote by \( \theta \varphi \). For any nonnegative integer \( k \), let \( \theta^k \) denote the \( k \)-fold composition of \( \theta \) with itself. Finally, let \( \theta(M) \) denote the image of \( M \) under \( \theta \), which is a subgroup of \( M \).

We mention that \( \partial_1 \partial_2 = \partial_2 \partial_1 \), which is to say that the endomorphisms \( \partial_1 \) and \( \partial_2 \) commute.

Dr. Riedl’s approach begins by partitioning the set \( V \) into a collection of nonempty subsets. Hence the problem of determining the members of \( V \) is broken into the smaller subproblems of determining the members of each of these nonempty subsets. We now describe the manner in which the set \( V \) is partitioned, but omitting many of the details.

A subgroup \( H \) of \( M \) is said to be an **Engel subgroup** if \( H = \partial_1^{a_1} \partial_2^{a_2}(M) \) for some pair of nonnegative integers \( a_1 \) and \( a_2 \). Let \( \mathcal{E} \) denote the set consisting of all the Engel subgroups of \( M \). It can be shown that every Engel subgroup is doubly-invariant, which is to say that \( \mathcal{E} \subseteq V \). Let \( \mathcal{B} \) denote the set consisting of all subgroups \( B \) of \( M \) such that \( B \) is equal to the product of some collection of Engel subgroups. It
can be shown that every subgroup belonging to $\mathcal{B}$ is doubly-invariant, which is to say that $\mathcal{B} \subseteq \mathcal{V}$. For each doubly-invariant subgroup $H \in \mathcal{V}$ we define the infimum of $H$, denoted $\inf(H)$, to be the product of all those subgroups of $H$ that are members of $\mathcal{B}$. It is not difficult to show that $\inf(H) \in \mathcal{B}$ for every $H \in \mathcal{V}$, and that $\inf(B) = B$ for every $B \in \mathcal{B}$. This gives us a surjective map $\mathcal{V} \to \mathcal{B}$ defined by the rule $H \mapsto \inf(H)$. For each subgroup $B \in \mathcal{B}$, we define the set $\mathcal{V}(B) = \{H \in \mathcal{V} \mid \inf(H) = B\}$, which is the pre-image of $B$ under the aforementioned map. It is now clear that

$$\mathcal{V} = \bigcup_{B \in \mathcal{B}} \mathcal{V}(B)$$

is a disjoint union.

In Chapter 2 of [2] it is shown that $|\mathcal{B}| = 98$. Hence the problem of determining the members of $\mathcal{V}$ is broken into the 98 smaller non-overlapping subproblems of determining the members of $\mathcal{V}(B)$ for the various subgroups of $B \in \mathcal{B}$.

In this thesis we determine the members of $\mathcal{V}(B)$ for one particular subgroup $B$ belonging to $\mathcal{B}$, namely the subgroup $B$ consisting of all matrices

$$\begin{bmatrix}
a & b & 0 \\
c & 0 & 0 \\
0 & 0 & 0
\end{bmatrix}$$

for which each of the three entries $a, b, c$ is a multiple of 3. We will find $|\mathcal{V}(B)| = 2,889.$
CHAPTER II
PRELIMINARIES

In this chapter we give an overview of the process of determining all the subgroups belonging to the set $\mathcal{V}(B)$ for an arbitrary $B \in \mathcal{B}$. We also introduce some notation and terminology that will be used in this thesis.

We want to determine for $B$ the members of the set $\mathcal{V}(B) = \{ V \in \mathcal{V} \mid \inf(V) = B \}$. We will now describe a systematic method that produces a tree structure under inclusion to find all the subgroups of $\mathcal{V}$. The subgroup $B$ is the root of the tree, which is referred to as Level 0. Level 1 consists of the subgroups or branches that are directly above $B$. These subgroups contain $B$ as a subgroup and do not contain any other subgroups of $\mathcal{V}(B)$. If a subgroup has a branch extending above it we refer to it as a nonterminal subgroup. If a subgroup has no branches extending above it, then it is referred to as terminal. If all subgroups at a level are terminal then we have found all the members of $\mathcal{V}(B)$.

\textbf{Definition 2.0.5.} Let $B, B'$ be members of $\mathcal{B}$. We say that $B'$ is an immediate successor of $B$ in case $B < B'$ and there does not exist any $B'' \in \mathcal{B}$ such that $B < B'' < B'$.

Since $\partial_1^0 \partial_2^0 : M \to M$ is the identity map, we have $M = \partial_1^0 \partial_2^0(M)$. Thus $M$ itself is an Engel subgroup, and in particular $M \in \mathcal{B}$. Thus for each $B \in \mathcal{B}$ satisfying
$B < M$ it is clear from the fact that $B$ is a finite set that there exists at least one immediate successor of $B$.

**Definition 2.0.6.** Let $H$ be a subgroup of $M$. Since $\partial_1$ and $\partial_2$ are endomorphisms of $M$, the preimage of $H$ under $\partial_1$ and $\partial_2$, namely $\partial_1^{-1}H$ and $\partial_2^{-1}H$, are both subgroups of $M$. The **pullback** of $H$, denoted as $\partial^{-1}H$, is defined by $\partial^{-1}H = \partial_1^{-1}H \cap \partial_2^{-1}H$.

For arbitrary $B \in \mathcal{B}$, we now briefly describe an algorithm to determine the members of $\mathcal{V}(B)$ and to describe its tree structure. (For more details, see Chapter 2 of [2]).

For each nonnegative integer $k$, we shall let $\mathcal{L}_k$ denote the set consisting of all those subgroups $H \in \mathcal{V}(B)$ that are located at level $k$ in the tree diagram. The root of the tree is the subgroup $B$ itself, which we denote as $W_0$. This is the only subgroup at Level 0. Thus we have $\mathcal{L}_0 = \{W_0\}$. The subgroup $W_0$ is always nonterminal.

For each positive integer $k$ we shall iteratively define $\mathcal{L}_k$ to be the union of the sets $\mathcal{L}_k(W_{k-1})$ where $W_{k-1}$ runs over all the nonterminal subgroups belonging to the set $\mathcal{L}_{k-1}$. This union will always be a disjoint union. In the tree diagram, we always draw an edge between $W_{k-1}$ and every subgroup belonging to $\mathcal{L}_k(W_{k-1})$ whenever $W_{k-1}$ is a nonterminal subgroup belonging to $\mathcal{L}_{k-1}$.

Let $\mathcal{S}$ be the set consisting of all the immediate successors of $B \in \mathcal{B}$. We now describe the subgroups at Level 1. Let $\mathcal{L}_1(W_0)$ be the set consisting of all subgroups $W_1$ of $M$ such that $W_0 < W_1 < \partial^{-1}W_0$ and $B_j \not\in W_1$ for all $B_j \in \mathcal{S}$. In order to find all the members of $\mathcal{L}_1(W_0)$ we must first calculate the pullback $\partial^{-1}W_0$. Since $W_0$ is
the only member of \( L_0 \), the preceding paragraph tells us that the set \( L_1 \) is equal to \( L_1(W_0) \).

A subgroup \( W_1 \in L_1 \) is said to be nonterminal if and only if

\[
\text{rank}(\partial^{-1}W_0/W_1) < \text{rank}(\partial^{-1}W_1/W_1).
\]

Let \( \hat{L}_1 \) denote the set of all the nonterminal members of \( L_1 \). We now describe the subgroups at Level 2. Recall that \( L_2 \) will be defined as the union of the sets \( L_2(W_1) \) as \( W_1 \) runs over the members of \( \hat{L}_1 \). Thus it suffices to define the members of \( L_2(W_1) \) where \( W_1 \in \hat{L}_1 \) is arbitrary.

Let \( L_2(W_1) \) be the set of all subgroups \( W_2 \) of \( M \) such that \( W_1 < W_2 < \partial^{-1}W_1 \) and \( \partial^{-1}W_0 \cap W_2 = W_1 \). Notice that these last two conditions are equivalent to \( W_2/W_1 \) being a nontrivial proper subgroup of \( \partial^{-1}W_1/W_1 \) whose intersection with \( \partial^{-1}W_0/W_1 \) is trivial. The existence of at least one such subgroup \( W_2/W_1 \) is guaranteed by the condition \( \text{rank}(\partial^{-1}W_0/W_1) < \text{rank}(\partial^{-1}W_1/W_1) \) which holds since we are assuming \( W_1 \) is nonterminal. Hence \( L_2(W_1) \) is nonempty, which justifies the terminology of saying \( W_1 \) is nonterminal. Thus the set \( L_2 \) is defined.

A subgroup \( W_2 \in L_2 \) is said to be nonterminal if and only if

\[
\text{rank}(\partial^{-1}W_1/W_2) < \text{rank}(\partial^{-1}W_2/W_2).
\]

Let \( \hat{L}_2 \) be the set of all the nonterminal members of \( L_2 \). One then defines, for each \( W_2 \in \hat{L}_2 \), the set \( L_3(W_2) \) in a manner analogous to the definition of \( L_2(W_1) \) for \( W_1 \in \hat{L}_1 \), namely that \( L_3(W_2) \) is the set of all subgroups \( W_3 \) of \( M \) such that \( W_2 < W_3 < \partial^{-1}W_2 \) and \( \partial^{-1}W_1 \cap W_3 = W_2 \). One continues in this manner to determine the members of \( \mathcal{V}(B) \) at each level.

As mentioned earlier, every subgroup contained in \( \mathcal{V}(B) \) appears exactly once in the tree structure just described. The following theorem states this.
Theorem 2.0.1. If $B \in \mathcal{B}$ and $B < M$ then $\mathcal{V}(B) = \bigcup_{k=0}^{\infty} \mathcal{L}_k$ is a disjoint union. In other words $\mathcal{V}(B)$ consists of exactly all the subgroups in the tree.

Whenever we encounter a subgroup $W_i$ for $i \geq 2$ in this process, we need to determine whether $W_i$ is terminal or nonterminal, and this involves the ranks of the abelian 3-groups $\partial^{-1}W_{i-1}/W_i$ and $\partial^{-1}W_i/W_i$. This requires that we calculate the pullback $\partial^{-1}W_i$, which can be a time consuming task. The following result enables us, in certain situations in this thesis for which $i = 3$, to recognize immediately that these two ranks are equal, and therefore that $W_i$ is terminal.

Lemma 2.0.2. Terminal Lemma Let $W_{i-1}$ and $W_i$ be subgroups of $M$ such that $W_{i-1} < W_i$ and $|W_i/W_{i-1}| = 3$ and $W_i = \langle W_{i-1}, m \rangle$ for some element $m \in M$. Suppose that the (2,0)-entry and the (0,2)-entry of every element of $W$ is divisible by 3, but that neither the (2,0)-entry nor the (0,2)-entry of $m$ is divisible by 3. Then $\partial^{-1}W_{i-1} = \partial^{-1}W_i$.

We now introduce convenient notation that will be used throughout the rest of this thesis. For each $r \in \mathbb{Z}_9 = \{0, 1, 2, ..., 8\}$, there exist unique values $s, t \in \{0, 1, 2\}$ such that $r = 3s + t$. Thus, an arbitrary matrix $x \in M$ has the form

$$x = \begin{bmatrix}
3s_{0,0} + t_{0,0} & 3s_{0,1} + t_{0,1} & 3s_{0,2} + t_{0,2} \\
3s_{1,0} + t_{1,0} & 3s_{1,1} + t_{1,1} & 3s_{1,2} + t_{1,2} \\
3s_{2,0} + t_{2,0} & 3s_{2,1} + t_{2,1} & 3s_{2,2} + t_{2,2}
\end{bmatrix},$$

where each value $s_{i,j}$ and $t_{i,j}$ is taken from the set $\{0, 1, 2\}$. Using this notation, we
may express the vertical and horizontal derivatives of $x$ respectively as

$$\partial_1 x = \begin{bmatrix}
3s_{1,0} + t_{1,0} & 3s_{1,1} + t_{1,1} & 3s_{1,2} + t_{1,2} \\
3s_{2,0} + t_{2,0} & 3s_{2,1} + t_{2,1} & 3s_{2,2} + t_{2,2} \\
-3(t_{1,0} + t_{2,0}) & -3(t_{1,1} + t_{2,1}) & -3(t_{1,2} + t_{2,2})
\end{bmatrix},$$

$$\partial_2 x = \begin{bmatrix}
3s_{0,1} + t_{0,1} & 3s_{0,2} + t_{0,2} & -3(t_{0,1} + t_{0,2}) \\
3s_{1,1} + t_{1,1} & 3s_{1,2} + t_{1,2} & -3(t_{1,1} + t_{1,2}) \\
3s_{2,1} + t_{2,1} & 3s_{2,2} + t_{2,2} & -3(t_{2,1} + t_{2,2})
\end{bmatrix}.$$

**Definition 2.0.7.** We define the subgroups $\Omega_0(Z_9) = \{0\}$, $\Omega_1(Z_9) = \{0, 3, 6\}$, and $\Omega_2(Z_9) = \{0, 1, \ldots, 8\}$ of the additive group $Z_9 = \{0, 1, \ldots, 8\}$ of integers modulo 9.

For any given collection of values $\alpha(i,j) \in \{0, 1, 2\}$ for $0 \leq i, j \leq 2$, we define the **pattern subgroup**

$$\begin{bmatrix}
\alpha(0, 0) & \alpha(0, 1) & \alpha(0, 2) \\
\alpha(1, 0) & \alpha(1, 1) & \alpha(1, 2) \\
\alpha(2, 0) & \alpha(2, 1) & \alpha(2, 2)
\end{bmatrix}$$

to be the set of all matrices

$$\begin{bmatrix}
x_{0,0} & x_{0,1} & x_{0,2} \\
x_{1,0} & x_{1,1} & x_{1,2} \\
x_{2,0} & x_{2,1} & x_{2,2}
\end{bmatrix} \in M$$

such that $x_{i,j} \in \Omega_{\alpha(i,j)}(Z_9)$. 
For our problem, we take

\[
B = \begin{bmatrix}
1 & 1 & 0 \\
1 & 0 & 0 \\
0 & 0 & 0
\end{bmatrix}.
\]

The immediate successors of \(B\) are

\[
B_1 = \begin{bmatrix}
1 & 1 & 0 \\
1 & 0 & 0 \\
1 & 0 & 0
\end{bmatrix},
B_2 = \begin{bmatrix}
1 & 1 & 0 \\
1 & 1 & 0 \\
0 & 0 & 0
\end{bmatrix},
B_3 = \begin{bmatrix}
1 & 1 & 1 \\
1 & 0 & 0 \\
0 & 0 & 0
\end{bmatrix},
B_4 = \begin{bmatrix}
2 & 1 & > \\
1 & 0 & 0 \\
\lor & 0 & 0
\end{bmatrix}.
\]

The symbol \(\lor\) in \(B_4\) denotes that the value of the \(x_{2,0}\) entry is determined by the other two entries in its column. In particular, \(x_{0,2} = -3(x_{0,0} + x_{1,0})\). The symbol \(>\) denotes that the value of the \(x_{0,2}\) entry is determined by the other two entries in its row. In particular, \(x_{0,2} = -3(x_{0,0} + x_{0,1})\). \(B_4\) is an example of a **generalized pattern subgroup**. All the subgroups in \(\mathcal{B}\) are generalized pattern subgroups.
CHAPTER III
DOUBLY-INVARIANT SUBGROUPS FOR $B$

Now that we have covered the necessary definitions and background information we can show the doubly-invariant subgroups for our particular $B$. We will see that $|\mathcal{V}(B)| = 2,889.$

Let $W_0 = B$. We determine the pullback of $W_0$ as

$$\partial^{-1}W_0 = \left[\begin{array}{ccc} 2 & 2 & 2 \\ 1 & 1 & 0 \\ 1 & 0 & 0 \end{array}\right] \cap \left[\begin{array}{ccc} 2 & 1 & 1 \\ 2 & 1 & 0 \\ 2 & 0 & 0 \end{array}\right] = \left[\begin{array}{ccc} 2 & 1 & 1 \\ 1 & 1 & 0 \\ 1 & 0 & 0 \end{array}\right].$$

Note that $|W_0| = 3^3$, $|\partial^{-1}W_0| = 3^7$, and $\partial^{-1}W_0/W_0 \cong \mathbb{Z}_3 \times \mathbb{Z}_3 \times \mathbb{Z}_3 \times \mathbb{Z}_3$. A basis for the vector space $\partial^{-1}W_0/W_0$ is $y_1 + W_0$, $y_2 + W_0$, $y_3 + W_0$, $y_4 + W_0$ where

$$y_1 = \left[\begin{array}{ccc} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 3 & 0 & 0 \end{array}\right], \quad y_2 = \left[\begin{array}{ccc} 0 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 0 \end{array}\right], \quad y_3 = \left[\begin{array}{ccc} 0 & 0 & 3 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{array}\right], \quad y_4 = \left[\begin{array}{ccc} 1 & 0 & -3 \\ 0 & 0 & 0 \\ -3 & 0 & 0 \end{array}\right].$$

The 1-dimensional subspaces of $\partial^{-1}W_0/W_0$ generated by the basis vectors $y_1 + W_0$, $y_2 + W_0$, $y_3 + W_0$, $y_4 + W_0$ are $B_1/W_0$, $B_2/W_0$, $B_3/W_0$, $B_4/W_0$ respectively.

The set $\mathcal{L}_1$ consists of all subgroups $W_1$ that satisfy $W_0 < W_1 < \partial^{-1}W_0$ and $B_k \notin W_1$ for $k \in \{1, 2, 3, 4\}$. The subgroups $W_1$ belonging to $\mathcal{L}_1$ correspond to the nontrivial proper subspaces $W_1/W_0$ of $\partial^{-1}W_0/W_0$ that contain none of the
four 1-dimensional subspaces \(B_1/W_0, B_2/W_0, B_3/W_0, B_4/W_0\). Since \(\partial^{-1}W_0/W_0\) has
dimension 4, every such subspace \(W_1/W_0\) has dimension 1, 2, or 3.

To help us define the subgroups \(W_1\) belonging to \(\mathcal{L}_1\), it will be convenient to
identify each element of the vector space \(\partial^{-1}W_0/W_0\) with its coordinate vector with
respect to the ordered basis \(y_1 + W_0, y_2 + W_0, y_3 + W_0, y_4 + W_0\). In this way we identify
\(\partial^{-1}W_0/W_0\) with the vector space \(\mathbb{Z}_3 \times \mathbb{Z}_3 \times \mathbb{Z}_3 \times \mathbb{Z}_3\) consisting of row vectors. Under
this identification, the elements \(y_1 + W_0, y_2 + W_0, y_3 + W_0, y_4 + W_0\) in \(\partial^{-1}W_0/W_0\) are
associated with the standard basis vectors \([1, 0, 0, 0], [0, 1, 0, 0], [0, 0, 1, 0], [0, 0, 0, 1]\)
in \(\mathbb{Z}_3 \times \mathbb{Z}_3 \times \mathbb{Z}_3 \times \mathbb{Z}_3\).

The subgroups \(W_1\) belonging to \(\mathcal{L}_1\) are in one-to-one correspondence with
the nontrivial proper subspaces \(W_1/W_0\) of \(\partial^{-1}W_0/W_0\) that contain none of \(y_1 + W_0, y_2 + W_0, y_3 + W_0, y_4 + W_0\). Under our identification, each such subspace \(W_1/W_0\)
is associated with a subspace \(S\) of \(\mathbb{Z}_3 \times \mathbb{Z}_3 \times \mathbb{Z}_3 \times \mathbb{Z}_3\) that contains none of the
standard basis vectors \([1, 0, 0, 0], [0, 1, 0, 0], [0, 0, 1, 0], [0, 0, 0, 1]\). Let \(m\) denote the
unique matrix in reduced row-echelon form that is row-equivalent to the matrix whose
rows are the members of an arbitrarily-chosen basis of such a subspace \(S\).

If \(W_1/W_0\) is 1-dimensional then there are three possible forms for the matrix
\(m\). The first form is

\[
m = [0, 0, 1, c_1] \quad \text{for } c_1 \in \{1, 2\}
\]

(2 possibilities), which is considered in Case 1. The second form is

\[
m = [0, 1, c_1, c_2] \quad \text{for } c_1, c_2 \in \{0, 1, 2\}, \ (c_1, c_2) \neq (0, 0)
\]
(8 possibilities), which is considered in Case 2. The third form is

\[ m = [1, c_1, c_2, c_3] \quad \text{for} \quad c_1, c_2, c_3 \in \{0, 1, 2\}, \ (c_1, c_2, c_3) \neq (0, 0, 0) \]

(26 possibilities), which is considered in Case 3.

If \( W_1/W_0 \) is 2-dimensional then there are three possible forms for the matrix \( m \). The first form is

\[
m = \begin{bmatrix} 0 & 1 & 0 & c_1 \\ 0 & 0 & 1 & c_2 \end{bmatrix} \quad \text{for} \quad c_1, c_2 \in \{1, 2\}
\]

(4 possibilities), which is considered in Case 4. The second form is

\[
m = \begin{bmatrix} 1 & c_1 & 0 & c_2 \\ 0 & 0 & 1 & c_3 \end{bmatrix} \quad \text{for} \quad c_1, c_2, c_3 \in \{0, 1, 2\}, \ (c_1, c_2) \neq (0, 0), \ c_3 \neq 0
\]

(16 possibilities) which is considered in Case 5. The third form is

\[
m = \begin{bmatrix} 1 & 0 & c_1 & c_2 \\ 0 & 1 & c_3 & c_4 \end{bmatrix} \quad \text{for} \quad c_1, c_2, c_3, c_4 \in \{0, 1, 2\}, \ (c_1, c_2) \neq (0, 0), \ (c_3, c_4) \neq (0, 0)
\]

(64 possibilities) which is considered in Case 6.

If \( W_1/W_0 \) is 3-dimensional then the matrix \( m \) has the form

\[
m = \begin{bmatrix} 1 & 0 & 0 & c_1 \\ 0 & 1 & 0 & c_2 \\ 0 & 0 & 1 & c_3 \end{bmatrix} \quad \text{for} \quad c_1, c_2, c_3 \in \{1, 2\}
\]

(8 possibilities), which is considered in Case 7.

We now summarize our results. In Case 1 we find 2 subgroups \( W_1 \in \mathcal{L}_1 \), both satisfying \(|W_1| = 3^4\). Exactly 1 of these 2 members of \( \mathcal{L}_1 \) is nonterminal. We find
that 1 nonterminal member of $\mathcal{L}_1$ is contained in 27 members of $\mathcal{L}_2$. Thus we find 27 subgroups $W_2 \in \mathcal{L}_2$, each of which satisfies $|W_2| = 3^5$. All 27 members of $\mathcal{L}_2$ are terminal. So in Case 1 we find a total of $2 + 27 = 29$ subgroups.

In Case 2 we find 8 subgroups $W_1 \in \mathcal{L}_1$, all satisfying $|W_1| = 3^4$. Exactly 2 of these 8 members are nonterminal. Each of these 2 nonterminal members of $\mathcal{L}_1$ is contained in 27 members of $\mathcal{L}_2$. Thus we find 54 subgroups $W_2$ of $\mathcal{L}_2$, all satisfying $|W_2| = 3^5$. Every member of $\mathcal{L}_2$ is terminal. So in Case 2 we find a total of $8 + 54 = 62$ subgroups.

In Case 3 we find 26 subgroups $W_1 \in \mathcal{L}_1$, all satisfying $|W_1| = 3^4$. Exactly 1 of these 26 is nonterminal. This 1 nonterminal member of $\mathcal{L}_1$ is contained in 27 members of $\mathcal{L}_2$. Thus we find 27 subgroups $W_2$ of $\mathcal{L}_2$, all satisfying $|W_2| = 3^5$. Every member of $\mathcal{L}_2$ is terminal. So in Case 3 we find a total of $26 + 27 = 53$ subgroups.

In Case 4 we find 4 subgroups $W_1 \in \mathcal{L}_1$, all of which satisfy $|W_1| = 3^5$ and all of which are nonterminal. We find that 2 of these members of $\mathcal{L}_1$ are each contained in 9 members of $\mathcal{L}_2$ and 2 of these members of $\mathcal{L}_1$ are each contained in 117 members of $\mathcal{L}_2$. Thus we find 252 subgroups $W_2 \in \mathcal{L}_2$, 90 of which satisfy $|W_2| = 3^6$ and 162 of which satisfy $|W_2| = 3^7$. Each of the 90 members of $\mathcal{L}_2$ that satisfy $|W_2| = 3^6$ is terminal. Exactly 134 members of $\mathcal{L}_2$ that satisfy $|W_2| = 3^7$ are terminal. Exactly 28 members of $\mathcal{L}_2$ that satisfy $|W_2| = 3^7$ are nonterminal. Each of these 28 nonterminal members of $\mathcal{L}_2$ is contained in 9 members of $\mathcal{L}_3$. Thus we find 252 subgroups $W_3 \in \mathcal{L}_3$, all satisfying $|W_3| = 3^8$. Every member of $\mathcal{L}_3$ is terminal. So in Case 4 we find a total of $4 + 252 + 252 = 508$ subgroups.
In Case 5 we find 16 subgroups $W_1 \in \mathcal{L}_1$, all satisfying $|W_1| = 3^5$. Exactly 11 of these 16 members of $\mathcal{L}_1$ are nonterminal. 10 of these 11 nonterminal members of $\mathcal{L}_1$ are contained in 9 members of $\mathcal{L}_2$. The 11th nonterminal member of $\mathcal{L}_1$ is contained in 36 members of $\mathcal{L}_2$. Thus we find 216 subgroups $W_2 \in \mathcal{L}_2$, 125 of which satisfy $|W_2| = 3^6$ and 81 of which satisfy $|W_2| = 3^7$. Every member of $\mathcal{L}_2$ that satisfies $|W_2| = 3^6$ is terminal. Exactly 5 members of $\mathcal{L}_2$ that satisfies $|W_2| = 3^7$ are nonterminal. Each of these 5 members of $\mathcal{L}_2$ is contained in 9 members of $\mathcal{L}_3$. Thus we find 45 subgroups $W_3 \in \mathcal{L}_3$, all satisfying $|W_3| = 3^8$. Every member of $\mathcal{L}_3$ is terminal. So in Case 5 we find a total of $16 + 216 + 45 = 277$ subgroups.

In Case 6 we find 64 subgroups of $W_1 \in \mathcal{L}_1$, all of which satisfy $|W_1| = 3^5$. Exactly 32 of these 64 members of $\mathcal{L}_1$ are nonterminal. We find that 30 of these 32 nonterminal members of $\mathcal{L}_1$ are each contained in 9 members of $\mathcal{L}_2$ and 2 of the 32 nonterminal members are each contained in 117 members of $\mathcal{L}_2$. Thus we find 504 subgroups $W_2 \in \mathcal{L}_2$, 342 of which satisfy $|W_2| = 3^6$ and 162 of which satisfy $|W_2| = 3^7$. Every member of $\mathcal{L}_2$ is terminal. So in Case 6 we find a total of $64 + 504 = 568$ subgroups.

In Case 7 we find 8 subgroups $W_1 \in \mathcal{L}_1$, all of which satisfy $|W_1| = 3^6$ and all of which are nonterminal. We find that 6 of these nonterminal members of $\mathcal{L}_1$ are each contained in 21 members of $\mathcal{L}_2$ and 2 of these nonterminal members of $\mathcal{L}_2$ are each contained in 183 members of $\mathcal{L}_2$. Thus we find 492 subgroups $W_2$ of $\mathcal{L}_2$, 150 of which satisfy $|W_2| = 3^7$, 288 of which satisfy $|W_2| = 3^8$, and 54 of which satisfy $|W_2| = 3^9$. Each of the 150 members of $\mathcal{L}_2$ that satisfy $|W_2| = 3^7$ is terminal.
Exactly 69 members of $L_2$ that satisfy $|W_2| = 3^8$ are nonterminal. Each of these 69 nonterminal members of $L_2$ is contained in 9 members of $L_3$. Exactly 48 members of $L_2$ that satisfy $|W_2| = 3^9$ are nonterminal and are each contained in 3 members of $L_3$. Exactly 6 members of $L_2$ that satisfy $|W_2| = 3^9$ are nonterminal and are each contained in 21 members of $L_3$. Thus we find 891 subgroups $W_3 \in L_3$, 621 of which satisfy $|W_3| = 3^9$, 216 of which satisfy $|W_3| = 3^{10}$, and 54 of which satisfy $|W_3| = 3^{11}$. Every member of $L_3$ is terminal. So in Case 7 we find a total of $8 + 492 + 891 = 1,391$ subgroups.

In the seven separate cases that we have considered, the number of subgroups that we have found are 29, 62, 53, 508, 277, 568, and 1,391 respectively. Thus remembering to include $B$ itself (the unique member of $L_0$) in our count, we obtain $|\mathcal{V}(B)| = 1 + 29 + 62 + 53 + 508 + 277 + 568 + 1,391 = 2,889$. 
CHAPTER IV

CASE 1

We fix an arbitrary value $c_1 \in \{1, 2\}$. Let $m_1 = y_3 + c_1 y_4$. Thus

$$m_1 = \begin{bmatrix} c_1 & 0 & -3(c_1 - 1) \\ 0 & 0 & 0 \\ -3c_1 & 0 & 0 \end{bmatrix}.$$ 

Let $W_1 = < W_0, m_1 > \in \mathcal{L}_1$. The number of subgroups of this type is 2. Note that $|W_1| = 3^4$ and $\partial^{-1} W_0 / W_1 \cong \mathbb{Z}_3 \times \mathbb{Z}_3 \times \mathbb{Z}_3$ and rank $(\partial^{-1} W_0 / W_1) = 3$. We now calculate the pullback $\partial^{-1} W_1$. The subgroup $W_1$ is contained in the pattern subgroup

$$\begin{bmatrix} 2 & 1 & 1 \\ 1 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix}.$$ 

Thus the pullback $\partial^{-1} W_1$ is contained in the pattern subgroup

$$\begin{bmatrix} 2 & 2 & 2 \\ 2 & 1 & 1 \\ 1 & 0 & 0 \end{bmatrix} \cap \begin{bmatrix} 2 & 2 & 1 \\ 2 & 1 & 0 \\ 2 & 1 & 0 \end{bmatrix} = \begin{bmatrix} 2 & 2 & 1 \\ 2 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}.$$ 

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Let

\[
x = \begin{bmatrix}
3s_{0,0} + t_{0,0} & 3s_{0,1} + t_{0,1} & 3s_{0,2} \\
3s_{1,0} + t_{1,0} & 3s_{1,1} & 0 \\
3s_{2,0} & 0 & 0
\end{bmatrix}
\in \begin{bmatrix}
2 & 2 & 1 \\
2 & 1 & 0 \\
1 & 0 & 0
\end{bmatrix}.
\]

Our strategy for computing $\partial^{-1}W_1$ is to determine conditions on the variables $s_{i,j}$ and $t_{i,j}$ appearing in the above expression for $x$ that are equivalent to $x$ being contained in $\partial^{-1}W_1$. Of course $x \in \partial^{-1}W_1$ if and only if $\partial_1 x, \partial_2 x \in \partial^{-1}W_1$. Note that

\[
\partial_1 x = \begin{bmatrix}
3s_{1,0} + t_{1,0} & 3s_{1,1} & 0 \\
3s_{2,0} & 0 & 0 \\
-3t_{1,0} & 0 & 0
\end{bmatrix}
\quad\text{and}\quad
\partial_2 x = \begin{bmatrix}
3s_{0,1} + t_{0,1} & 3s_{0,2} & -3t_{0,1} \\
3s_{1,1} & 0 & 0 \\
0 & 0 & 0
\end{bmatrix}.
\]

By equating entry by entry each of the matrix forms $\partial_1 x, \partial_2 x$ with a matrix expression representing a general element of $W_1$, we shall obtain a system of congruences modulo 3 whose unknowns are some of the variables $s_{i,j}$ and $t_{i,j}$. Solving this system of congruences will give us a useful description of those matrices $x$ that belong to $\partial^{-1}W_1$.

Recall

\[
\partial^{-1}W_0 = \begin{bmatrix}
2 & 1 & 1 \\
1 & 1 & 0 \\
1 & 0 & 0
\end{bmatrix}.
\]
We want to calculate $\partial^{-1}W_1$. As we mentioned earlier, $\partial^{-1}W_1$ is contained in the pattern subgroup
\[
\begin{bmatrix}
2 & 2 & 1 \\
2 & 1 & 0 \\
1 & 0 & 0
\end{bmatrix},
\]
which we shall call $P$. We know that $W_0 \subseteq W_1$. Therefore $\partial^{-1}W_0 \subseteq \partial^{-1}W_1$. Comparing the entries of $\partial^{-1}W_0$ and $P$ we see that only differ at the $(1, 0)$ and $(0, 1)$ entries. Looking at the $(0, 0)$ entry, since both pullbacks have a 2 as the entry we see that $s_{0,0}$ and $t_{0,0}$ will not appear in the congruences. Since the $(1, 0)$ and $(0, 1)$ entries are the only entries that differ they are the only variables we consider. Looking at the $(0, 1)$ entry, we see $\partial^{-1}W_0$ has a 1 and $\partial^{-1}W_1$ has a 2. Therefore $t_{0,1}$ may appear in the congruences. Likewise, from the $(1, 0)$ entry we will consider the variable $t_{1,0}$.

These are the only variables that are able to appear in the system of congruences. We refer to these as the "variables in play". It is important to identify the variables in play prior to creating the system of congruences because any of them that does not appear in the system will be a free variable and we want to be sure to take note of that. In the current situation, the variables that are in play are those appearing in the matrix
\[
\begin{bmatrix}
0 & t_{0,1} & 0 \\
t_{1,0} & 0 & 0 \\
0 & 0 & 0
\end{bmatrix}.
\]
To make the resulting system of congruences as easy to work with as possible, when equating certain entries it is no loss to work modulo 3 rather than modulo 9. We
define $I$ to be the largest pattern subgroup of $M$ that is contained in $B$. Since $B$ itself is a pattern subgroup, we obtain $I = B$. Hence when equating $(0, 0)$, $(1, 0)$, and $(0, 1)$ entries we only need to work modulo 3 rather than modulo 9.

We want to identify a value $a_1 \in \mathbb{Z}_9$ such that $\partial_1 x \equiv a_1 m_1 (\mod I)$. A formal expression for $a_1 m_1$ is

$$ a_1 \begin{bmatrix} c_1 & 0 & -3(c_1 - 1) \\ 0 & 0 & 0 \\ -3c_1 & 0 & 0 \end{bmatrix} = \begin{bmatrix} a_1 c_1 & 0 & -3a_1(c_1 - 1) \\ 0 & 0 & 0 \\ -3a_1c_1 & 0 & 0 \end{bmatrix}. $$

Comparing $(0, 1)$-entries, we get $3s_{1,1} \equiv 0$ which gives no information.

Comparing $(1, 0)$-entries, we get $3s_{2,0} \equiv 0$ which gives no information.

Comparing $(0, 0)$-entries, we get $t_{1,0} \equiv a_1 c_1$. Since $c_1^2 \equiv 1$, we get $c_1 t_{1,0} \equiv a_1$.

Comparing $(2, 0)$-entries, we get $c_1 t_{1,0} \equiv a_1$.

Comparing $(0, 2)$-entries, we get $0 \equiv a_1(c_1 - 1)$. Since $a_1 \equiv c_1 t_{1,0}$, we get $0 \equiv (c_1 - 1)t_{1,0}$.

We see that $\partial_1 x \in W_1$ if and only if

$$ 0 \equiv (c_1 - 1)t_{1,0} \quad (A1). $$

We want to identify a value $b_1 \in \mathbb{Z}_9$ such that $\partial_2 x \equiv b_1 m_1 (\mod I)$. A formal expression for $b_1 m_1$ is

$$ b_1 \begin{bmatrix} c_1 & 0 & 3(c_1 - 1) \\ 0 & 0 & 0 \\ -3c_1 & 0 & 0 \end{bmatrix} = \begin{bmatrix} b_1 c_1 & 0 & 3b_1(c_1 - 1) \\ 0 & 0 & 0 \\ -3b_1c_1 & 0 & 0 \end{bmatrix}. $$

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Comparing (0, 1)-entries, we get $3s_{0,2} \equiv 0$ which gives no information.

Comparing (1, 0)-entries, we get $3s_{1,1} \equiv 0$ which gives no information.

Comparing (0, 0)-entries, we get $t_{0,1} \equiv b_1 c_1$. Since $c_1^2 \equiv 1$, we get $c_1 t_{0,1} \equiv b_1$.

Comparing (2, 0)-entries, we get $0 \equiv b_1 c_1$.

Comparing (0, 2)-entries, we get $b_1 (c_1 - 1) \equiv t_{0,1}$. Since $b_1 \equiv c_1 t_{0,1}$ we get $t_{1,0} (c_1 - 1) \equiv t_{0,1}$.

We see that $\partial_2 x \in W_1$ if and only if

$$t_{0,1} \equiv 0 \quad \text{(B1)}.$$

Hence $x \in \partial^{-1} W_1$ if and only if

$$(c_1 - 1) t_{1,0} \equiv 0 \quad \text{(A1)}$$

$$t_{0,1} \equiv 0 \quad \text{(B1)}.$$

It is convenient to consider the cases $c_1 = 2$ and $c_1 = 1$ separately.

4.1 Case 1.1

Let us examine the case when $c_1 = 2$. Then $x \in \partial^{-1} W_1$ if and only if

$$t_{1,0} \equiv 0$$

$$t_{0,1} \equiv 0.$$

So $\partial^{-1} W_1 = \partial^{-1} W_0$. Thus $\text{rank}(\partial^{-1} W_1 / W_1) = \text{rank}(\partial^{-1} W_0 / W_1)$ and $W_1$ is terminal and $W_1 \notin \hat{\mathcal{L}}_1$.  

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4.2 Case 1.2

Let us examine the case when \( c_1 = 1 \). Our expression for \( m_1 \) becomes

\[
\begin{bmatrix}
1 & 0 & 0 \\
0 & 0 & 0 \\
-3 & 0 & 0
\end{bmatrix}.
\]

Since \( c_1 \equiv 1 \), the congruence stating that \( (c_1 - 1)t_{1,0} \equiv 0 \) is automatically satisfied regardless of the value of \( t_{1,0} \).

Hence \( x \in \partial^{-1}W_1 \) if and only if

\[ t_{0,1} \equiv 0. \]

We regard \( t_{1,0} \) as the free variable. Taking \( t_{1,0} \equiv 1 \), and the matrix \( x \) becomes

\[
v_1 = \begin{bmatrix}
0 & 0 & 0 \\
1 & 0 & 0 \\
0 & 0 & 0
\end{bmatrix}.
\]

Recall that

\[
\partial^{-1}W_0 = \begin{bmatrix}
2 & 1 & 1 \\
1 & 1 & 0 \\
1 & 0 & 0
\end{bmatrix}, \quad y_1 = \begin{bmatrix}
0 & 0 & 0 \\
0 & 0 & 0 \\
3 & 0 & 0
\end{bmatrix}, \quad y_2 = \begin{bmatrix}
0 & 0 & 0 \\
0 & 0 & 0 \\
0 & 0 & 0
\end{bmatrix}, \quad y_3 = \begin{bmatrix}
0 & 0 & 3 \\
0 & 0 & 0 \\
0 & 0 & 0
\end{bmatrix}.
\]

We see that \( \partial^{-1}W_1 = \langle \partial^{-1}W_0, v_1 \rangle \). Since \( v_1 \notin \partial^{-1}W_0 \) and \( 3v_1 \in \partial^{-1}W_0 \), we know \( |\partial^{-1}W_1/\partial^{-1}W_0| = 3 \). Since \( |\partial^{-1}W_0| = 3^7 \) then \( |\partial^{-1}W_1| = 3^8 \). Because \( |W_1| = 3^4 \) then \( |\partial^{-1}W_1/W_1| = 3^4 \). Also, \( v_1, y_1, y_2, y_3 \in \partial^{-1}W_1 \) but are not contained in \( W_1 \). Each
of \(3v_1, 3y_1, 3y_2, 3y_3\) is contained in \(W_1\). So \(v_1 + W_1, y_1 + W_1, y_2 + W_1, y_3 + W_1\) are elements of order 3 in the group \(\partial^{-1}W_1/W_1\). These four elements form a generating set for the group \(\partial^{-1}W_1/W_1\). Recall that \(|\partial^{-1}W_1/W_1| = 3^4\), we obtain \(\partial^{-1}W_1/W_1 \cong \mathbb{Z}_3 \times \mathbb{Z}_3 \times \mathbb{Z}_3 \times \mathbb{Z}_3\). Since \(\text{rank}(\partial^{-1}W_1/W_1) = 4\) and \(\text{rank}(\partial^{-1}W_0/W_1) = 3\), then \(\text{rank}(\partial^{-1}W_0/W_1) < \text{rank}(\partial^{-1}W_1/W_1)\). Hence \(W_1\) is nonterminal and \(W_1 \in \hat{\mathcal{L}}_1\). A basis for \(\partial^{-1}W_1/W_1\) is \(v_1 + W_1, y_1 + W_1, y_2 + W_1, y_3 + W_1\). A basis for \(\partial^{-1}W_0/W_1\) is \(y_1 + W_1, y_2 + W_1, y_3 + W_1\).

We fix arbitrary values \(c_2, c_3, c_4 \in \{0, 1, 2\}\). There are \(3^3 = 27\) ways to choose these values. Let \(m_2 = c_2y_1 + c_3y_2 + c_4y_3 + v_1\). Thus

\[
m_2 = \begin{bmatrix}
0 & 0 & 3c_4 \\
1 & 3c_3 & 0 \\
3c_2 & 0 & 0
\end{bmatrix}.
\]

Let \(W_2 = < W_1, m_2 >\). There are 27 subgroups of \(W_2\) of this type. Since \(m_2 \notin W_1\) and \(3m_2 \in W_1\) then \(|W_2/W_1| = 3\). Also, since \(|W_1| = 3^4\) then \(|W_2| = 3^5\). Recall \(|\partial^{-1}W_1| = 3^8\), then \(\partial^{-1}W_1/W_2 \cong \mathbb{Z}_3 \times \mathbb{Z}_3 \times \mathbb{Z}_3\) and \(\text{rank}(\partial^{-1}W_1/W_2) = 3\).

The subgroup \(W_2\) is contained in the pattern subgroup

\[
\begin{bmatrix}
2 & 1 & 1 \\
2 & 1 & 0 \\
1 & 0 & 0
\end{bmatrix}.
\]
The pullback $\partial^{-1}W_2$ is contained in the pattern subgroup

$$\left[\begin{array}{ccc} 2 & 2 & 2 \\ 2 & 1 & 1 \\ 2 & 1 & 0 \end{array}\right] \cap \left[\begin{array}{ccc} 2 & 2 & 1 \\ 2 & 2 & 1 \\ 2 & 1 & 0 \end{array}\right] = \left[\begin{array}{ccc} 2 & 2 & 1 \\ 2 & 1 & 1 \\ 2 & 1 & 0 \end{array}\right].$$

Let

$$x = \left[\begin{array}{ccc} 3s_{0,0} + t_{0,0} & 3s_{0,1} + t_{0,1} & 3s_{0,2} \\ 3s_{1,0} + t_{1,0} & 3s_{1,1} & 3s_{1,2} \\ 3s_{2,0} + t_{2,0} & 3s_{2,1} & 0 \end{array}\right] \in \left[\begin{array}{ccc} 2 & 2 & 1 \\ 2 & 1 & 1 \\ 2 & 1 & 0 \end{array}\right].$$

Thus

$$\partial_1 x = \left[\begin{array}{ccc} 3s_{1,0} + t_{1,0} & 3s_{1,1} & 3s_{1,2} \\ 3s_{2,0} + t_{2,0} & 3s_{2,1} & 0 \\ -3(t_{1,0} + t_{2,0}) & 0 & 0 \end{array}\right] \text{ and } \partial_2 x = \left[\begin{array}{ccc} 3s_{0,1} + t_{0,1} & 3s_{0,2} & -3t_{0,1} \\ 3s_{1,1} & 3s_{1,2} & 0 \\ 3s_{2,1} & 0 & 0 \end{array}\right].$$

The variables that are in play are those appearing in the matrix

$$\left[\begin{array}{ccc} 0 & t_{0,1} & 0 \\ t_{1,0} & 0 & 3s_{1,2} \\ t_{2,0} & 3s_{2,1} & 0 \end{array}\right].$$

We want to identify a value $a_1, a_2 \in \mathbb{Z}_9$ such that $\partial_1 x \equiv a_1 m_1 + a_2 m_2 (\text{mod } I)$. A formal expression for $a_1 m_1 + a_2 m_2$ is

$$a_1 \left[\begin{array}{ccc} 1 & 0 & 0 \\ 0 & 0 & 3c_4 \\ -3 & 0 & 0 \end{array}\right] + a_2 \left[\begin{array}{ccc} 0 & 0 & 3c_4 \\ 1 & 3c_3 & 0 \\ 3c_2 & 0 & 0 \end{array}\right] = \left[\begin{array}{ccc} a_1 & 0 & 3a_2 c_4 \\ a_2 & 3a_2 c_3 & 0 \\ -3(a_2 c_2 - a_1) & 0 & 0 \end{array}\right].$$
Comparing $(0, 1)$-entries, we get $3s_{1,1} \equiv 0$ which gives no information.

Comparing $(1, 0)$-entries, we get $a_{2} \equiv t_{2,0}$.

Comparing $(0, 0)$-entries, we get $a_{1} \equiv t_{1,0}$.

Comparing $(0, 2)$-entries, we get $a_{2}c_{4} \equiv s_{1,2}$. Since $a_{2} \equiv t_{2,0}$, we get $c_{4}t_{2,0} \equiv s_{1,2}$.

Comparing $(1, 1)$-entries, we get $a_{2}c_{3} \equiv s_{2,1}$. Since $a_{2} \equiv t_{2,0}$, we get $c_{3}t_{2,0} \equiv s_{2,1}$.

Comparing $(2, 0)$-entries, we get $a_{1} - a_{2}c_{2} \equiv t_{1,0} + t_{2,0}$. Plugging in $a_{1}$ and $a_{2}$, we get $t_{1,0} - c_{2}t_{2,0} \equiv t_{1,0} + t_{2,0}$, which says that $(c_{2} + 1)t_{2,0} \equiv 0$.

We see that $\partial_{1}x \in W_{2}$ if and only if

\[ c_{4}t_{2,0} \equiv s_{1,2} \quad (A1) \]
\[ c_{3}t_{2,0} \equiv s_{2,1} \quad (A2) \]
\[ (c_{2} + 1)t_{2,0} \equiv 0 \quad (A3). \]

We want to identify a value $b_{1}, b_{1} \in \mathbb{Z}_{9}$ such that $\partial_{2}x \equiv b_{1}m_{1} + b_{2}m_{2} (\text{mod } I)$.

A formal expression for $b_{1}m_{1} + b_{2}m_{2}$ is

\[
\begin{bmatrix}
1 & 0 & 0 \\
0 & 0 & 0 \\
-3 & 0 & 0
\end{bmatrix} b_{1} + \begin{bmatrix}
0 & 0 & 3c_{4} \\
1 & 3c_{3} & 0 \\
3c_{2} & 0 & 0
\end{bmatrix} b_{2} = \begin{bmatrix}
b_{1} & 0 & 3b_{2}c_{4} \\
b_{2} & 3b_{2}c_{3} & 0 \\
-3(b_{1} - b_{2}c_{2}) & 0 & 0
\end{bmatrix}.
\]

Comparing $(0, 0)$-entries, we get $b_{1} \equiv t_{0,1}$.

Comparing $(0, 1)$-entries, we get $3s_{0,2} \equiv 0$, which gives no information.

Comparing $(1, 0)$-entries, we get $3s_{1,1} \equiv b_{2}$, which gives us $0 \equiv b_{2}$.

Comparing $(0, 2)$-entries, we get $t_{0,1} \equiv -c_{4}b_{2}$. Since $b_{2} \equiv 0$, we get $t_{0,1} \equiv 0$.
Comparing (1,1)-entries, we get $b_2 c_3 \equiv s_{1,2}$. Since $b_2 \equiv 0$, we get $s_{1,2} \equiv 0$.

Comparing (2,0)-entries, we get $s_{2,1} \equiv c_2 b_2 - b_1$. Substituting in $b_1$ and $b_2$ we get $s_{2,1} \equiv -t_{0,1}$ and from above we know that $t_{0,1} \equiv 0$ so $s_{2,1} \equiv 0$.

We see that $\partial_2 x \in W_2$ if and only if

\begin{align*}
  s_{2,1} &\equiv 0 \quad \text{(B1)} \\
  t_{0,1} &\equiv 0 \quad \text{(B2)} \\
  s_{1,2} &\equiv 0 \quad \text{(B3)}.
\end{align*}

Since $s_{2,1} \equiv 0$, $t_{0,1} \equiv 0$, and $s_{1,2} \equiv 0$, (A1) becomes $c_4 t_{2,0} \equiv 0$, (A2) becomes $c_3 t_{2,0} \equiv 0$, and (A3) becomes $(c_2 + 1) t_{2,0} \equiv 0$.

Hence $x \in \partial^{-1} W_2$ if and only if

\begin{align*}
  c_4 t_{2,0} &\equiv 0 \quad \text{(A1)} \\
  c_3 t_{2,0} &\equiv 0 \quad \text{(A2)} \\
  (c_2 + 1) t_{2,0} &\equiv 0 \quad \text{(A3)} \\
  s_{2,1} &\equiv 0 \quad \text{(B1)} \\
  s_{1,2} &\equiv 0 \quad \text{(B2)} \\
  t_{0,1} &\equiv 0 \quad \text{(B3)}.
\end{align*}

It is convenient to consider the cases of $(c_2, c_3, c_4) \neq (2, 0, 0)$ and $(c_2, c_3, c_4) = (2, 0, 0)$. 

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4.2.1 Case 1.2.1

Let us examine the case \((c_2, c_3, c_4) \neq (2, 0, 0)\). Hence either \(c_2 \neq 2\) or \(c_3 \neq 0\) or \(c_4 \neq 0\).

We now argue that \(t_{2,0} = 0\). If \(c_2 \neq 2\), then by congruence (A3), \(t_{2,0} = 0\). If \(c_3 \neq 0\), then by congruence (A2), \(t_{2,0} = 0\). If \(c_4 \neq 0\), then by congruence (A1), \(t_{2,0} = 0\). Since \(c_4 \neq 0\), then from \(c_4 t_{2,0} \equiv 0\), \(t_{2,0} \equiv 0\).

Hence \(x \in \partial^{-1}W_2\) if and only if

\[
\begin{align*}
s_{2,1} &= 0 \\
s_{1,2} &= 0 \\
t_{0,1} &= 0 \\
t_{2,0} &= 0.
\end{align*}
\]

We regard \(t_{1,0}\) as the free variable. Taking \(t_{1,0} \equiv 1\), and the matrix \(x\) becomes

\[
v_1 = \begin{bmatrix}
0 & 0 & 0 \\
1 & 0 & 0 \\
0 & 0 & 0
\end{bmatrix}.
\]

Therefore, \(\partial^{-1}W_2 = < \partial^{-1}W_0, v_1 >\). So \(\partial^{-1}W_2 = \partial^{-1}W_1\). Thus \(\text{rank}(\partial^{-1}W_2/W_2) = \text{rank}(\partial^{-1}W_1/W_2)\) and \(W_2\) is terminal and \(W_2 \notin \hat{L}_2\).

4.2.2 Case 1.2.2

Let us examine the case \((c_2, c_3, c_4) = (2, 0, 0)\). Since \(c_2 = 2\), then the congruence (A3) is automatically satisfied. Since \(c_3 = 0\), then the congruence (A2) is automatically satisfied. Since \(c_4 = 0\) then the congruence (A1) is automatically satisfied.
Hence $x \in \partial^{-1}W_2$ if and only if

\begin{align*}
s_{2,1} &\equiv 0 \\
s_{1,2} &\equiv 0 \\
t_{0,1} &\equiv 0.
\end{align*}

Then our expression for $m_2$ becomes

\[
\begin{bmatrix}
0 & 0 & 0 \\
1 & 0 & 0 \\
0 & 0 & 0
\end{bmatrix}.
\]

We regard $t_{1,0}$ and $t_{2,0}$ as free variables. Taking $t_{1,0} \equiv 1$ and $t_{2,0} = 0$, the matrix $x$ becomes

\[
v_1 = \begin{bmatrix}
0 & 0 & 0 \\
1 & 0 & 0 \\
0 & 0 & 0
\end{bmatrix}.
\]

Taking $t_{2,0} \equiv 1$ and $t_{1,0} = 0$, the matrix $x$ becomes

\[
v_2 = \begin{bmatrix}
0 & 0 & 0 \\
0 & 0 & 0 \\
1 & 0 & 0
\end{bmatrix}.
\]

Recall that

\[
\partial^{-1}W_1 = \begin{bmatrix}
2 & 1 & 1 \\
2 & 1 & 0 \\
1 & 0 & 0
\end{bmatrix}.
\]
We see that neither $v_2$ nor $3v_2$ is contained in $W_2$ and that $\partial^{-1}W_2 = \langle \partial^{-1}W_0, v_1, v_2 \rangle$. We know that $v_2 \in \partial^{-1}W_2$ and since $3v_2 \notin W_2$ then $v_2 + W_2$ is an element of the group $\partial^{-1}W_2/W_2$ whose order is larger than 3. Recall that $|\partial^{-1}W_2/W_2| = 3^4$ and that $\partial^{-1}W_1/W_2$ has rank 3. Since $\partial^{-1}W_2$ has an element of order larger than 3 then $\text{rank}(\partial^{-1}W_2/W_2)$ is not greater than 3 since $\partial^{-1}W_2/W_2 \cong Z_9 \times Z_3 \times Z_3$ and therefore $W_2$ is terminal and $W_2 \notin \hat{\mathcal{L}}_2$.

In Case 1 we found 2 subgroups $W_1 \in \mathcal{L}_1$, both satisfying $|W_1| = 3^4$. Exactly 1 of these 2 members of $\mathcal{L}_1$ is nonterminal. We found that 1 nonterminal member of $\mathcal{L}_1$ is contained in 27 members of $\mathcal{L}_2$. Thus we found 27 subgroups $W_2 \in \mathcal{L}_2$, each of which satisfies $|W_2| = 3^5$. All 27 members of $\mathcal{L}_2$ are terminal. So in Case 1 we found a total of $2 + 27 = 29$ subgroups.
CHAPTER V

CASE 2

We pick arbitrary values $c_1, c_2 \in \{0, 1, 2\}$ and $(c_1, c_2) \neq (0, 0)$. Let $m_1 = y_2 + c_1 y_3 + c_2 y_4$. Thus

$$m_1 = \begin{bmatrix} c_2 & 0 & 3(c_1 - c_2) \\ 0 & 3 & 0 \\ -3c_2 & 0 & 0 \end{bmatrix}.$$

Let $W_1 = \langle W_0, m_1 \rangle \in \mathcal{L}_1$. The number of subgroups of this type is 8. Note that $|W_1| = 3^4$ and $\partial^{-1}W_0/W_1 \cong \mathbb{Z}_3 \times \mathbb{Z}_3 \times \mathbb{Z}_3$ and rank $(\partial^{-1}W_0/W_1) = 3$. We now calculate the pullback $\partial^{-1}W_1$. The subgroup $W_1$ is contained in the pattern subgroup

$$\begin{bmatrix} 2 & 1 & 1 \\ 1 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}.$$

Thus the pullback $\partial^{-1}W_1$ is contained in the pattern subgroup

$$\begin{bmatrix} 2 & 2 & 2 \\ 2 & 1 & 1 \\ 1 & 1 & 0 \end{bmatrix} \cap \begin{bmatrix} 2 & 2 & 1 \\ 2 & 1 & 1 \\ 2 & 1 & 0 \end{bmatrix} = \begin{bmatrix} 2 & 2 & 1 \\ 2 & 1 & 1 \\ 1 & 1 & 0 \end{bmatrix}.$$
Let
\[ x = \begin{bmatrix}
3s_{0,0} + t_{0,0} & 3s_{0,1} + t_{0,1} & 3s_{0,2} \\
3s_{1,0} + t_{1,0} & 3s_{1,1} & 3s_{1,2} \\
3s_{2,0} & 3s_{2,1} & 0
\end{bmatrix} \in \begin{bmatrix}
2 & 2 & 1 \\
2 & 1 & 1 \\
1 & 1 & 0
\end{bmatrix}. \]

Thus
\[ \partial_1 x = \begin{bmatrix}
3s_{1,0} + t_{1,0} & 3s_{1,1} & 3s_{1,2} \\
3s_{2,0} & 3s_{2,1} & 0 \\
-3t_{1,0} & 0 & 0
\end{bmatrix} \quad \text{and} \quad \partial_2 x = \begin{bmatrix}
3s_{0,1} + t_{0,1} & 3s_{0,2} & -3t_{0,1} \\
3s_{1,1} & 3s_{1,2} & 0 \\
3s_{2,1} & 0 & 0
\end{bmatrix}. \]

The variables in play are those appearing in the matrix
\[ \begin{bmatrix}
0 & t_{0,1} & 0 \\
t_{1,0} & 0 & 3s_{1,2} \\
0 & 3s_{2,1} & 0
\end{bmatrix}. \]

We want to identify a value \( a_1 \in \mathbb{Z}_9 \) such that \( \partial_1 x \equiv a_1 m_1 (\text{mod } I) \). A formal expression for \( a_1 m_1 \) is
\[ a_1 \begin{bmatrix}
c_2 & 0 & 3(c_1 - c_2) \\
0 & 3 & 0 \\
-3c_2 & 0 & 0
\end{bmatrix} = \begin{bmatrix}
a_1 c_2 & 0 & 3(c_1 - c_2) a_1 \\
0 & 3a_1 & 0 \\
-3a_1 c_2 & 0 & 0
\end{bmatrix}. \]

Comparing (0,1)-entries, we get \( 3s_{1,1} \equiv 0 \), which gives no information.

Comparing (1,0)-entries, we get \( 3s_{2,0} \equiv 0 \), which gives no information.

Comparing (1,1)-entries, we get \( s_{2,1} \equiv a_1 \).

Comparing (0,0)-entries, we get \( t_{1,0} \equiv a_1 c_2 \). Since \( a_1 \equiv s_{2,1} \), we get \( t_{1,0} \equiv c_2 s_{2,1} \).
Comparing $(2,0)$-entries, we get $t_{1,0} \equiv a_1 c_2$. Since $a_1 \equiv s_{2,1}$, we get $t_{1,0} \equiv c_2 s_{2,1}$.

Comparing $(0,2)$-entries, we get $s_{1,2} \equiv (c_1 - c_2) a_1 \pmod{3}$. Since $a_1 \equiv s_{2,1}$, we get $s_{1,2} \equiv (c_1 - c_2) s_{2,1}$.

We see that $\partial_1 x \in W_1$ if and only if

$$t_{1,0} \equiv c_2 s_{2,1} \quad (A1)$$

$$s_{1,2} \equiv (c_1 - c_2) s_{2,1} \quad (A2).$$

We want to identify a value $b_1 \in \mathbb{Z}_9$ such that $\partial_2 x \equiv b_1 m_1 \pmod{I}$. A formal expression for $b_1 m_1$ is

$$b_1 \begin{bmatrix} c_2 & 0 & 3(c_1 - c_2) \\ 0 & 3 & 0 \\ -3c_2 & 0 & 0 \end{bmatrix} = \begin{bmatrix} b_1 c_2 & 0 & 3(c_1 - c_2) b_1 \\ 0 & 3b_1 & 0 \\ -3b_1 c_2 & 0 & 0 \end{bmatrix}.$$

Comparing $(0,1)$-entries, we get $3s_{0,2} \equiv 0$, which gives no information.

Comparing $(1,0)$-entries, we get $3s_{1,1} \equiv 0$, which gives no information.

Comparing $(1,1)$-entries, we get $s_{1,2} \equiv b_1$.

Comparing $(0,0)$-entries, we get $t_{0,1} \equiv b_1 c_2$. Since $b_1 \equiv s_{1,2}$, then $t_{0,1} \equiv c_2 s_{1,2}$.

Comparing $(2,0)$-entries, we get $s_{2,1} \equiv -b_1 c_2$. Since $b_1 \equiv s_{1,2}$, we get $s_{2,1} \equiv -c_2 s_{1,2}$.

Comparing $(0,2)$-entries, we get $t_{0,1} \equiv b_1 (c_2 - c_1)$. Since $b_1 \equiv s_{1,2}$, we get $(c_2 - c_1) s_{1,2} \equiv t_{0,1}$. Since $t_{0,1} \equiv c_2 s_{1,2}$, we get $c_1 s_{1,2} \equiv 0$. 

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We see that $\partial_2 x \in W_1$ if and only if

\[ c_2 s_{1,2} \equiv t_{0,1} \quad \text{(B1)} \]
\[ -c_2 s_{1,2} \equiv s_{2,1} \quad \text{(B2)} \]
\[ c_1 s_{1,2} \equiv 0 \quad \text{(B3)}. \]

Hence $x \in \partial^{-1}W_1$ if and only if

\[ t_{1,0} \equiv c_2 s_{2,1} \quad \text{(A1)} \]
\[ (c_1 - c_2) s_{2,1} \equiv s_{1,2} \quad \text{(A2)} \]
\[ c_2 s_{1,2} \equiv t_{0,1} \quad \text{(B1)} \]
\[ -c_2 s_{1,2} \equiv s_{2,1} \quad \text{(B2)} \]
\[ c_1 s_{1,2} \equiv 0 \quad \text{(B3)}. \]

It is convenient to consider the cases $c_1 \neq 0$ and $c_1 = 0$.

5.1 Case 2.1

We consider the case $c_1 \neq 0$. Then by (B3) we have $s_{1,2} \equiv 0$. By (B1) and (B2) we get $t_{0,1} \equiv 0$ and $s_{2,1} \equiv 0$. Then by (A1) we get $t_{1,0} \equiv 0$. So $\partial^{-1}W_1 = \partial^{-1}W_0$. Thus $\text{rank}(\partial^{-1}W_1/W_1) = \text{rank}(\partial^{-1}W_0/W_1)$ and $W_1$ is terminal and $W_1 \not\in \hat{L}_1$. 

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5.2 Case 2.2

Now we consider the case $c_1 = 0$. Since $(c_1, c_2) \neq (0, 0)$, it follows that $c_2 \neq 0$. Our expression for $m_1$ becomes

$$m_1 = \begin{bmatrix}
c_2 & 0 & -3c_2 \\
0 & 3 & 0 \\
-3c_2 & 0 & 0
\end{bmatrix}.$$

Since $c_1 = 0$, (B3) is automatically satisfied (and thus may be ignored) and (A2) becomes $-c_2 s_{2,1} \equiv s_{1,2}$, which is equivalent to (B2) because $c_2^2 \equiv 1$. Hence (B2) may be ignored. Using $c_2^2 \equiv 1$ we may rewrite (A1) as $s_{2,1} \equiv c_2 t_{1,0}$. Using this last congruence to substitute for $s_{2,1}$ in (A2) and using $c_2^2 \equiv 1$, we obtain $s_{1,2} \equiv -t_{1,0}$. Using this last congruence to substitute for $s_{1,2}$ in (B1), we obtain $t_{0,1} \equiv -c_2 t_{1,0}$.

Hence $x \in \partial^{-1} W_1$ if and only if

$$s_{2,1} \equiv c_2 t_{1,0} \quad \text{(A1)}$$
$$s_{1,2} \equiv -t_{1,0} \quad \text{(A2)}$$
$$t_{0,1} \equiv -c_2 t_{1,0} \quad \text{(A3)}$$

We regard $t_{1,0}$ as the free variable. Taking $t_{1,0} = 1$, the matrix $x$ becomes

$$v_1 = \begin{bmatrix}
0 & -c_2 & 0 \\
1 & 0 & -3 \\
0 & 3c_2 & 0
\end{bmatrix}.$$
Recall that
\[
\partial^{-1}W_0 = \begin{bmatrix}
2 & 1 & 1 \\
1 & 1 & 0 \\
1 & 0 & 0
\end{bmatrix},
y_1 = \begin{bmatrix}
0 \\
0 \\
3
\end{bmatrix},
y_2 = \begin{bmatrix}
0 \\
0 \\
0
\end{bmatrix},
y_3 = \begin{bmatrix}
0 \\
0 \\
0
\end{bmatrix},
y_4 = \begin{bmatrix}
1 & 0 & -3 \\
1 & 0 & -3 \\
1 & 0 & -3
\end{bmatrix}.
\]

We see that \(\partial^{-1}W_1 = \langle \partial^{-1}W_0, v_1 \rangle\). Since \(v_1 \notin \partial^{-1}W_0\) and \(3v_1 \in \partial^{-1}W_0\), we know \(|\partial^{-1}W_1/\partial^{-1}W_0| = 3\). Since \(|\partial^{-1}W_0| = 3^7\) then \(|\partial^{-1}W_1| = 3^8\). Because \(|W_1| = 3^4\) then \(|\partial^{-1}W_1/W_1| = 3^4\). Also, \(v_1, y_1, y_2, y_3 \in \partial^{-1}W_1\) but are not contained in \(W_1\). Each of \(3v_1, 3y_1, 3y_2, 3y_3\) is contained in \(W_1\). So \(v_1 + W_1, y_1 + W_1, y_2 + W_1, y_3 + W_1, y_4 + W_1\) are elements of order 3 in the group \(\partial^{-1}W_1/W_1\). These four elements form a generating set for the group \(\partial^{-1}W_1/W_1\). Recalling that \(|\partial^{-1}W_1/W_1| = 3^4\) we obtain \(\partial^{-1}W_1/W_1 \cong \mathbb{Z}_3 \times \mathbb{Z}_3 \times \mathbb{Z}_3 \times \mathbb{Z}_3\). Since \(\text{rank}(\partial^{-1}W_1/W_1) = 4\) and \(\text{rank}(\partial^{-1}W_0/W_1) = 3\), then \(\text{rank}(\partial^{-1}W_0/W_1) < \text{rank}(\partial^{-1}W_1/W_1)\). Hence \(W_1\) is nonterminal and \(W_1 \in \hat{\mathcal{L}}_1\). A basis for \(\partial^{-1}W_1/W_1\) is \(v_1 + W_1, y_1 + W_1, y_2 + W_1, y_3 + W_1\). A basis for \(\partial^{-1}W_0/W_1\) is \(y_1 + W_1, y_2 + W_1, y_3 + W_1\).

We fix arbitrary values \(c_3, c_4, c_5 \in \{0, 1, 2\}\). There are \(3^3 = 27\) ways to choose these values. Let \(m_2 = c_3y_1 + c_4y_2 + c_5y_3 + v_1\). Thus
\[
m_2 = \begin{bmatrix}
0 & -c_2 & 3c_5 \\
1 & 3c_4 & -3 \\
3c_3 & 3c_2 & 0
\end{bmatrix}.
\]
Let $W_2 = \langle W_1, m_2 \rangle$. There are 27 subgroups of $W_2$ of this type. Since $m_2 \notin W_1$ and $3m_2 \in W_1$, then $|W_2/W_1| = 3$. Also, since $|W_1| = 3^4$ then $|W_2| = 3^5$. Recall $|\partial^{-1}W_1| = 3^8$, then $\partial^{-1}W_1/W_2 \cong Z_3 \times Z_3 \times Z_3$ and rank$(\partial^{-1}W_1/W_2) = 3$.

The subgroup $W_2$ is contained in the pattern subgroup

\[
\begin{bmatrix}
2 & 2 & 1 \\
2 & 1 & 1 \\
1 & 1 & 0
\end{bmatrix}.
\]

The pullback $\partial^{-1}W_2$ is contained in the pattern subgroup

\[
\begin{bmatrix}
2 & 2 & 2 \\
2 & 2 & 1 \\
2 & 1 & 1
\end{bmatrix} \cap \begin{bmatrix}
2 & 2 & 2 \\
2 & 1 & 1 \\
2 & 1 & 1
\end{bmatrix} = \begin{bmatrix}
2 & 2 & 2 \\
2 & 1 & 1
\end{bmatrix}.
\]

Let

\[
x = \begin{bmatrix}
3s_{0,0} + t_{0,0} & 3s_{0,1} + t_{0,1} & 3s_{0,2} + t_{0,2} \\
3s_{1,0} + t_{1,0} & 3s_{1,1} + t_{1,1} & 3s_{1,2} \\
3s_{2,0} + t_{2,0} & 3s_{2,1} & 3s_{2,2}
\end{bmatrix} \in \begin{bmatrix}
2 & 2 & 2 \\
2 & 1 & 1
\end{bmatrix}.
\]

Thus

\[
\partial_1 x = \begin{bmatrix}
3s_{1,0} + t_{1,0} & 3s_{1,1} + t_{1,1} & 3s_{1,2} \\
3s_{2,0} + t_{2,0} & 3s_{2,1} & 3s_{2,2} \\
-3(t_{1,0} + t_{2,0}) & -3t_{1,1} & 0
\end{bmatrix}
\]

and $\partial_2 x = \begin{bmatrix}
3s_{0,1} + t_{0,1} & 3s_{0,2} + t_{0,2} & -3(t_{0,1} + t_{0,2}) \\
3s_{1,1} + t_{1,1} & 3s_{1,2} & -3t_{1,1} \\
3s_{2,1} & 3s_{2,2} & 0
\end{bmatrix}$. 

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The variables that are in play are those appearing in the matrix
\[
\begin{bmatrix}
0 & t_{0,1} & t_{0,2} \\
t_{1,0} & t_{1,1} & 3s_{1,2} \\
t_{2,0} & 3s_{2,1} & 3s_{2,2}
\end{bmatrix}.
\]

We want to identify a value \( a_1, a_2 \in \mathbb{Z}_9 \) such that \( \partial_1 x \equiv a_1 m_1 + a_2 m_2 \pmod{I} \). A formal expression for \( a_1 m_1 + a_2 m_2 \) is
\[
a_1 \begin{bmatrix}
c_2 & 0 & -3c_2 \\
0 & 3 & 0 \\
-3c_2 & 0 & 0
\end{bmatrix} + a_2 \begin{bmatrix}
0 & -c_2 & 3c_5 \\
1 & 3c_4 & -3 \\
3c_3 & 3c_2 & 0
\end{bmatrix} = \begin{bmatrix}
c_2a_1 & -c_2a_2 & 3(c_5a_2 - c_2a_1) \\
a_2 & 3(a_1 + c_4a_2) & -3a_2 \\
3(c_3a_2 - c_2a_1) & 3c_2a_2 & 0
\end{bmatrix}.
\]

Comparing \((1,0)\)-entries, we get \( a_2 \equiv t_{2,0} \).

Comparing \((0,1)\)-entries, we get \( t_{1,1} \equiv -c_2a_2 \) which becomes \( t_{1,1} \equiv -c_2 t_{2,0} \).

Comparing \((2,1)\)-entries, we get \( t_{1,1} \equiv -c_2a_2 \) which again says that \( t_{1,1} \equiv -c_2 t_{2,0} \).

Comparing \((1,2)\)-entries, we get \( s_{2,2} \equiv -a_2 \) which becomes \( s_{2,2} \equiv -t_{2,0} \).

Comparing \((1,1)\)-entries, we get \( s_{2,1} \equiv a_1 + c_4a_2 \). Since \( a_2 \equiv t_{2,0} \) this becomes \( a_1 \equiv s_{2,1} - c_4 t_{2,0} \).

Comparing \((0,0)\)-entries, we get \( t_{1,0} \equiv c_2a_1 \) which becomes \( t_{1,0} \equiv c_2 s_{2,1} - c_2 c_4 t_{2,0} \).

Since \( c_2^2 \equiv 1 \) we may rewrite this as \( s_{2,1} \equiv c_2 t_{1,0} + c_4 t_{2,0} \).

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Comparing $(0, 2)$-entries, we get $s_{1,2} \equiv c_5 a_2 - c_2 a_1$ which becomes $s_{1,2} \equiv -c_2 s_{2,1} + (c_2 c_4 + c_5) t_{2,0}$. Since $s_{2,1} \equiv c_2 t_{1,0} + c_4 t_{2,0}$ we get $s_{1,2} \equiv -t_{1,0} + c_5 t_{2,0}$.

Comparing $(2, 0)$-entries, we get $t_{1,0} + t_{2,0} \equiv c_2 a_1 - c_3 a_2$. Recalling that $t_{1,0} \equiv c_2 a_1$, we rewrite this as $t_{2,0} \equiv -c_3 a_2$. Since $a_2 \equiv t_{2,0}$ this becomes $t_{2,0} \equiv -c_3 t_{2,0}$, which says $(c_3 + 1) t_{2,0} \equiv 0$.

We see that $\partial_1 x \in W_2$ if and only if

$$t_{1,1} \equiv -c_2 t_{2,0} \quad (A1)$$

$$s_{2,2} \equiv -t_{2,0} \quad (A2)$$

$$s_{2,1} \equiv c_2 t_{1,0} + c_4 t_{2,0} \quad (A3)$$

$$s_{1,2} \equiv -t_{1,0} + c_5 t_{2,0} \quad (A4)$$

$$(c_3 + 1) t_{2,0} \equiv 0 \quad (A5).$$

We want to identify a value $b_1, b_1 \in \mathbb{Z}_9$ such that $\partial_2 x \equiv b_1 m_1 + b_2 m_2 (\text{mod } I)$.

A formal expression for $b_1 m_1 + b_2 m_2$ is

$$b_1 \begin{bmatrix} c_2 & 0 & -3c_2 \\ 0 & 3 & 0 \\ -3c_2 & 0 & 0 \end{bmatrix} + b_2 \begin{bmatrix} 0 & -c_2 & 3c_5 \\ 1 & 3c_4 & -3 \\ 3c_3 & 3c_2 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} c_2 b_1 & -c_2 b_2 & 3(c_5 b_2 - c_2 b_1) \\ b_2 & 3(b_1 + c_4 b_2) & -3b_2 \\ 3(c_3 b_2 - c_2 b_1) & 3c_2 b_2 & 0 \end{bmatrix}.$$

Comparing $(1, 0)$-entries, we get $b_2 \equiv t_{1,1}$. 38
Comparing $(0,1)$-entries, we get $t_{0,2} \equiv -c_2 b_2$ which becomes $t_{0,2} \equiv -c_2 t_{1,1}$.

Comparing $(2,1)$-entries, we get $s_{2,2} \equiv c_2 b_2$ which becomes $s_{2,2} \equiv c_2 t_{1,1}$.

Comparing $(1,2)$-entries, we get $b_2 \equiv t_{1,1}$ which we obtained earlier.

Comparing $(1,1)$-entries, we get $s_{1,2} \equiv b_1 + c_4 b_2$ and so $b_1 \equiv s_{1,2} - c_4 t_{1,1}$.

Comparing $(0,0)$-entries, we get $t_{0,1} \equiv c_2 b_1$ and so $t_{0,1} \equiv c_2 s_{1,2} - c_2 c_4 t_{1,1}$.

Comparing $(0,2)$-entries, we get $t_{0,1} + t_{0,2} \equiv c_2 b_1 - c_5 b_2$. Since $t_{0,1} \equiv c_2 b_1$ this becomes $t_{0,2} \equiv -c_5 t_{1,1}$.

Comparing $(2,0)$-entries, we get $s_{2,1} \equiv c_3 b_2 - c_2 b_1$. Since $c_2 b_1 \equiv t_{0,1}$ this becomes $s_{2,1} \equiv c_3 t_{1,1} - t_{0,1}$.

We see that $\partial_2 x \in W_2$ if and only if

\[
\begin{align*}
t_{0,2} & \equiv -c_2 t_{1,1} \quad (B1) \\
s_{2,2} & \equiv c_2 t_{1,1} \quad (B2) \\
t_{0,1} & \equiv c_2 s_{1,2} - c_2 c_4 t_{1,1} \quad (B3) \\
t_{0,2} & \equiv -c_5 t_{1,1} \quad (B4) \\
s_{2,1} & \equiv c_3 t_{1,1} - t_{0,1} \quad (B5).
\end{align*}
\]

Using the congruences $(A1)$, $(A3)$, and $(A4)$, we are able to rewrite our congruences $(B1)$, $(B2)$, $(B3)$, $(B4)$, and $(B5)$. The congruence $(B1)$ becomes $t_{0,2} \equiv -c_2 (-c_2 t_{2,0})$ which says $t_{0,2} \equiv t_{2,0}$. $(B2)$ becomes $s_{2,2} \equiv c_2 (-c_2 t_{2,0})$ which says $s_{2,2} \equiv -t_{2,0}$. This is identical to $(A2)$, so we will ignore $(B2)$. $(B3)$ becomes $t_{0,1} \equiv c_2 (-t_{1,0} + c_5 t_{2,0}) - c_2 c_4 (-c_2 t_{2,0})$. This can be written as $t_{0,1} \equiv -c_2 t_{1,0} + (c_2 c_5 + c_4) t_{2,0}$.

$(B4)$ becomes $t_{0,2} \equiv -c_5 (-c_2 t_{2,0}) \equiv c_2 c_5 t_{2,0}$. Recall $t_{0,2} \equiv t_{2,0}$. Thus we get $(c_2 c_5 -$
1) \( t_{2,0} \equiv 0 \). (B5) becomes \( c_2 t_{1,0} + c_4 t_{2,0} \equiv c_3 (-c_2 t_{2,0}) - (-c_2 t_{1,0} + (c_2 c_5 + c_4) t_{2,0}) \). This may be written as \( (c_2 c_3 + c_2 c_5 - c_4) t_{2,0} \equiv 0 \).

Hence \( x \in \partial^{-1} W_2 \) if and only if

\[
\begin{align*}
t_{1,1} &\equiv -c_2 t_{2,0} \quad \text{(A1)} \\
s_{2,2} &\equiv -t_{2,0} \quad \text{(A2)} \\
s_{2,1} &\equiv c_2 t_{1,0} + c_4 t_{2,0} \quad \text{(A3)} \\
s_{1,2} &\equiv -t_{1,0} + c_5 t_{2,0} \quad \text{(A4)} \\
(c_3 + 1) t_{2,0} &\equiv 0 \quad \text{(A5)} \\
t_{0,2} &\equiv t_{2,0} \quad \text{(B1)} \\
t_{0,1} &\equiv -c_2 t_{1,0} + (c_2 c_5 + c_4) t_{2,0} \quad \text{(B3)} \\
(c_2 c_5 - 1) t_{2,0} &\equiv 0 \quad \text{(B4)} \\
(c_2 c_3 + c_2 c_5 - c_4) t_{2,0} &\equiv 0 \quad \text{(B5)}.
\end{align*}
\]

Congruences (A5), (B4), and (B5) suggest that it is convenient to consider the case \( c_3 \neq 2 \) or \( c_5 \neq c_2 \) or \( c_4 \neq 1 - c_2 \) and the case \( c_3 = 2 \) and \( c_5 = c_2 \) and \( c_4 \equiv 1 - c_2 \) separately.

5.2.1 Case 2.2.1

First we argue that if \( t_{2,0} \equiv 0 \), then \( W_2 \) is terminal. The congruences (A5), (B4), and (B5) automatically hold. Since \( t_{2,0} \equiv 0 \) then the congruences (A1), (A2), and (B1) tell us \( t_{1,1} \equiv s_{2,2} \equiv t_{0,2} \equiv 0 \). The congruences (A3), (A4), and (B3) become \( s_{2,1} \equiv c_2 t_{1,0} \), \( s_{1,2} \equiv -t_{1,0} \), and \( t_{0,1} \equiv -c_2 t_{1,0} \), respectively.
Hence $x \in \partial^{-1}W_2$ if and only if

$$s_{2,1} \equiv c_2 t_{1,0}$$

$$s_{1,2} \equiv -t_{1,0}$$

$$t_{0,1} \equiv -c_2 t_{1,0}$$

We regard $t_{1,0}$ as the free variable. Taking $t_{1,0} = 1$ the matrix $x$ becomes

$$v_1 = \begin{bmatrix} 0 & -c_2 & 0 \\ 1 & 0 & -3 \\ 0 & 3c_2 & 0 \end{bmatrix}.$$ 

Thus $\partial^{-1}W_2 =< \partial^{-1}W_0, v_1 >= \partial^{-1}W_1$. Therefore, rank $(\partial^{-1}W_2/W_2) = \text{rank} \ \partial^{-1}W_1/W_2$ and $W_2$ is terminal and $W_2 \notin \hat{L}_2$. Hence we have concluded that if $t_{2,0} \equiv 0$, then $W_2$ is terminal.

We now argue that $W_2$ is terminal in case either $c_3 \neq 2$ or $c_4 \neq 1 - c_2$ or $c_5 \neq c_2$. Suppose $c_3 \neq 2$. Then congruence tells us that $t_{2,0} \equiv 0$ and therefore $W_2$ is terminal. Now suppose $c_5 \neq c_2$. Then congruence (B4) tells us that $t_{2,0} \equiv 0$ and therefore $W_2$ is terminal. Finally, we can assume that $c_3 = -1$, $c_5 = c_2$ and $c_4 \neq 1 - c_2$. Then congruence (B5) tell us that $t_{2,0} \equiv 0$ and therefore $W_2$ is terminal.

5.2.2 Case 2.2.2

Now consider the case where $c_3 = 2$ and $c_5 = c_2$, and $c_4 \equiv 1 - c_2$. Thus

$$m_2 = \begin{bmatrix} 0 & -c_2 & 3c_2 \\ 1 & 3(1 - c_2) & -3 \\ -3 & 3c_2 & 0 \end{bmatrix}.$$ 

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Recall

\[
m_1 = \begin{bmatrix}
  c_2 & 0 & -3c_2 \\
  0 & 3 & 0 \\
-3c_2 & 0 & 0
\end{bmatrix}.
\]

The three congruences (A5), (B4), and (B5), are all automatically satisfied. Because \( c_4 \equiv 1 - c_2 \) and \( c_5 = c_2 \) we have \( c_2 c_5 + c_4 \equiv c_2^2 + (1 - c_2) \equiv -1 - c_2 \), and so the congruence (B3) becomes \( t_{0,1} \equiv -c_2 t_{1,0} - (c_2 + 1)t_{2,0} \).

Hence \( x \in \partial^{-1}W_2 \) if and only if

\[
t_{1,1} \equiv -c_2 t_{2,0} \quad (A1)
\]

\[
s_{2,2} \equiv -t_{2,0} \quad (A2)
\]

\[
s_{2,1} \equiv c_2 t_{1,0} + (1 - c_2)t_{2,0} \quad (A3)
\]

\[
s_{1,2} \equiv -t_{1,0} + c_2 t_{2,0} \quad (A4)
\]

\[
t_{0,2} \equiv t_{2,0} \quad (B1)
\]

\[
t_{0,1} \equiv -c_2 t_{1,0} - (c_2 + 1)t_{2,0} \quad (B3)
\]

We regard \( t_{1,0} \) and \( t_{2,0} \) as the free variables. Taking \( t_{1,0} \equiv 1 \) and \( t_{2,0} = 0 \), the matrix \( x \) becomes

\[
v_1 = \begin{bmatrix}
  0 & -c_2 & 0 \\
  1 & 0 & -3 \\
0 & 3c_2 & 0
\end{bmatrix}.
\]
Taking $t_{1,0} = 0$, and $t_{2,0} \equiv 1$, the matrix $x$ becomes

$$v_2 = \begin{bmatrix} 0 & -c_2 - 1 & 1 \\ 0 & -c_2 & 3c_2 \\ 1 & 3(1 - c_2) & -3 \end{bmatrix}.$$ 

Recall that

$$\partial^{-1}W_0 = \begin{bmatrix} 2 & 1 & 1 \\ 1 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}, \quad \partial^{-1}W_1 = \begin{bmatrix} 2 & 1 & 1 \\ 2 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}.$$ 

We see that neither $v_2$ nor $3v_2$ is contained in $W_2$ and that

$\partial^{-1}W_2 = < \partial^{-1}W_0, v_1, v_2 >$. We know that $v_2 \in \partial^{-1}W_2$ and since $3v_2 \notin W_2$ then $v_2 + W_2$ is an element of the group $\partial^{-1}W_2/W_2$ whose order is larger than 3. Recall that $|\partial^{-1}W_2/W_2| = 3^4$ and that $\partial^{-1}W_1/W_2$ has rank 3. Since $\partial^{-1}W_2$ has an element of order larger than 3 then rank($\partial^{-1}W_2/W_2$) is not greater than 3 since $\partial^{-1}W_2/W_2 \cong Z_9 \times Z_3 \times Z_3$ and therefore $W_2$ is terminal and $W_2 \notin \hat{\mathcal{L}}_2$. 

In Case 2 we found 8 subgroups $W_1 \in \mathcal{L}_1$, all satisfying $|W_1| = 3^4$. Exactly 2 of these 8 members are nonterminal. Each of these 2 nonterminal members of $\mathcal{L}_1$ is contained in 27 members of $\mathcal{L}_2$. Thus we found 54 subgroups $W_2$ of $\mathcal{L}_2$, all satisfying $|W_2| = 3^5$. Every member of $\mathcal{L}_2$ is terminal. So in Case 2 we found a total of $8 + 54 = 62$ subgroups.
CHAPTER VI

CASE 3

We pick arbitrary values $c_1, c_2, c_3 \in \{0, 1, 2\}$ and $(c_1, c_2, c_3) \neq (0, 0, 0)$. Let $m_1 = y_1 + c_1y_2 + c_2y_3 + c_3y_4$. Thus

$$
m_1 = \begin{bmatrix}
c_3 & 0 & 3(c_2 - c_3) \\
0 & 3c_1 & 0 \\
3(1 - c_3) & 0 & 0
\end{bmatrix}.
$$

Let $W_1 = < W_0, m_1 > \in \mathcal{L}_1$. The number of subgroups of this type is 26. Note that $|W_1| = 3^4$ and $\partial^{-1}W_0/W_1 \cong \mathbb{Z}_3 \times \mathbb{Z}_3 \times \mathbb{Z}_3$ and $\text{rank}(\partial^{-1}W_0/W_1) = 3$.

We now calculate the pullback $\partial^{-1}W_1$. The subgroup $W_1$ is contained in the pattern subgroup

$$
\begin{bmatrix}
2 & 1 & 1 \\
1 & 1 & 0 \\
1 & 0 & 0
\end{bmatrix}.
$$

Thus the pullback $\partial^{-1}W_1$ is contained in the pattern subgroup

$$
\begin{bmatrix}
2 & 2 & 2 \\
2 & 1 & 1 \\
1 & 1 & 0
\end{bmatrix} \cap \begin{bmatrix}
2 & 2 & 1 \\
2 & 1 & 1 \\
2 & 1 & 0
\end{bmatrix} = \begin{bmatrix}
2 & 2 & 1 \\
2 & 1 & 1 \\
1 & 1 & 0
\end{bmatrix}.
$$
Let
\[ x = \begin{bmatrix}
3s_{0,0} + t_{0,0} & 3s_{0,1} + t_{0,1} & 3s_{0,2} \\
3s_{1,0} + t_{1,0} & 3s_{1,1} & 3s_{1,2} \\
3s_{2,0} & 3s_{2,1} & 0
\end{bmatrix} \in \begin{bmatrix}
2 & 2 & 1 \\
2 & 1 & 1 \\
1 & 1 & 0
\end{bmatrix}. \]

Thus
\[ \partial_1 x = \begin{bmatrix}
3s_{1,0} + t_{1,0} & 3s_{1,1} & 3s_{1,2} \\
3s_{2,0} & 3s_{2,1} & 0 \\
-3t_{1,0} & 0 & 0
\end{bmatrix} \quad \text{and} \quad \partial_2 x = \begin{bmatrix}
3s_{0,1} + t_{0,1} & 3s_{0,2} & -3t_{0,1} \\
3s_{1,1} & 3s_{1,2} & 0 \\
3s_{2,1} & 0 & 0
\end{bmatrix}. \]

The variables in play are those appearing in the matrix
\[\begin{bmatrix}
0 & t_{0,1} & 0 \\
t_{1,0} & 0 & 3s_{1,2} \\
0 & 3s_{2,1} & 0
\end{bmatrix} .\]

We want to identify a value \( a_1 \in \mathbb{Z}_9 \) such that \( \partial_1 x \equiv a_1 m_1 (\mod I) \). A formal expression for \( a_1 m_1 \) is
\[ a_1 \begin{bmatrix}
c_3 & 0 & 3(c_2 - c_3) \\
0 & 3c_1 & 0 \\
3(1-c_3) & 0 & 0
\end{bmatrix} = \begin{bmatrix}
a_1 c_3 & 0 & 3a_1(c_2 - c_3) \\
0 & 3a_1 c_1 & 0 \\
3a_1(1-c_3) & 0 & 0
\end{bmatrix}. \]

Comparing (0,1)-entries, we get \( 3s_{1,1} \equiv 0 \) which gives no information.

Comparing (1,0)-entries, we get \( 3s_{2,0} \equiv 0 \) which gives no information.

Comparing (0,0)-entries, we get \( t_{1,0} \equiv a_1 c_3 \).

Comparing (0,2)-entries, we get \( s_{1,2} \equiv a_1(c_2 - c_3) \).
Comparing (1, 1)-entries, we get $s_{2,1} \equiv a_1 c_1$.

Comparing (2, 0)-entries, we get $t_{1,0} \equiv a_1 (c_3 - 1)$.

We see that $\partial_1 x \in W_1$ if and only if

$$t_{1,0} \equiv c_3 a_1 \quad \text{(A1)}$$

$$s_{1,2} \equiv (c_2 - c_3) a_1 \quad \text{(A2)}$$

$$s_{2,1} \equiv c_1 a_1 \quad \text{(A3)}$$

$$t_{1,0} \equiv (c_3 - 1) a_1 \quad \text{(A4)}.$$

We want to identify a $b_1 \in \mathbb{Z}_9$ such that $\partial_2 x \equiv b_1 m_1 \pmod{I}$. A formal expression for $b_1 m_1$ is

$$b_1 \begin{bmatrix} c_3 & 0 & 3(c_2 - c_3) \\ 0 & 3c_1 & 0 \\ 3(1 - c_3) & 0 & 0 \end{bmatrix} = \begin{bmatrix} b_1 c_3 & 0 & 3b_1 (c_2 - c_3) \\ 0 & 3b_1 c_1 & 0 \\ 3b_1 (1 - c_3) & 0 & 0 \end{bmatrix}.$$ 

Comparing (0, 1)-entries, we get $3s_{0,2} \equiv 0$ which gives no information.

Comparing (1, 0)-entries, we get $3s_{1,1} \equiv 0$ which gives no information.

Comparing (0, 0)-entries, we get $t_{0,1} \equiv b_1 c_3$.

Comparing (0, 2)-entries, we get $t_{0,1} \equiv b_1 (c_3 - c_2)$.

Comparing (1, 1)-entries, we get $s_{1,2} \equiv b_1 c_1$.

Comparing (2, 0)-entries, we get $s_{2,1} \equiv b_1 (1 - c_3)$.
We see that $\partial_2 x \in W_1$ if and only if

\begin{align*}
t_{0,1} &\equiv c_3 b_1 \quad \text{(B1)} \\
s_{1,2} &\equiv c_1 b_1 \quad \text{(B2)} \\
t_{0,1} &\equiv (c_3 - c_2) b_1 \quad \text{(B3)} \\
s_{2,1} &\equiv (1 - c_3) b_1 \quad \text{(B4)}. \\
\end{align*}

Hence $x \in \partial^{-1}W_1$ if and only if

\begin{align*}
t_{1,0} &\equiv c_3 a_1 \quad \text{(A1)} \\
s_{1,2} &\equiv (c_2 - c_3) a_1 \quad \text{(A2)} \\
s_{2,1} &\equiv c_1 a_1 \quad \text{(A3)} \\
t_{1,0} &\equiv (c_3 - 1) a_1 \quad \text{(A4)} \\
t_{0,1} &\equiv c_3 b_1 \quad \text{(B1)} \\
s_{1,2} &\equiv c_1 b_1 \quad \text{(B2)} \\
t_{0,1} &\equiv (c_3 - c_2) b_1 \quad \text{(B3)} \\
s_{2,1} &\equiv (1 - c_3) b_1 \quad \text{(B4)}. \\
\end{align*}

It is convenient to consider the cases of $c_3 = 0$ and $c_3 \neq 0$ separately.

6.1 Case 3.1

Let us examine the case $c_3 = 0$. Now (A1) and (B1) become $t_{1,0} \equiv 0$ and $t_{0,1} \equiv 0$ respectively. Also, (A4) becomes $a_1 \equiv 0$ since $t_{1,0} \equiv 0$. Thus, (A2) and (A3)
become $s_{1,2} \equiv 0$ and $s_{2,1} \equiv 0$ respectively. Because $s_{2,1} \equiv 0$, the congruence (B4) becomes $b_1 \equiv 0$. Since $b_1 \equiv 0$, while $s_{1,2} \equiv t_{0,1} \equiv 0$, the congruences (B2) and (B3) are automatically satisfied. Hence $x \in \partial^{-1}W_1$ if and only if

\[
\begin{align*}
    t_{0,1} &\equiv 0 \\
    t_{1,0} &\equiv 0 \\
    s_{1,2} &\equiv 0 \\
    s_{2,1} &\equiv 0.
\end{align*}
\]

So $\partial^{-1}W_1 = \partial^{-1}W_0$. Thus \(\text{rank}(\partial^{-1}W_1/W_1) = \text{rank}(\partial^{-1}W_0/W_1)\) and $W_1$ is terminal and $W_1 \not\in \hat{L}_1$.

6.2 Case 3.2

Let us examine the case $c_3 \neq 0$. Since $c_3 \in \{1, 2\}$, it follows that $c_3^2 \equiv 1$. Since $c_3 \neq 0$ then (A1) becomes $a_1 \equiv c_3 t_{1,0}$ and (B1) becomes $b_1 \equiv c_3 t_{0,1}$.
We see that \( x \in \partial^{-1}W_1 \) if and only if

\[
\begin{align*}
a_1 & \equiv c_3 t_{1,0} \quad (A1) \\
s_{1,2} & \equiv (c_2 c_3 - 1)t_{1,0} \quad (A2) \\
s_{2,1} & \equiv c_1 c_3 t_{1,0} \quad (A3) \\
0 & \equiv c_3 t_{1,0} \quad (A4) \\
b_1 & \equiv c_3 t_{0,1} \quad (B1) \\
s_{1,2} & \equiv c_1 c_3 t_{0,1} \quad (B2) \\
0 & \equiv c_2 c_3 t_{0,1} \quad (B3) \\
s_{2,1} & \equiv (c_3 - 1)t_{0,1} \quad (B4).
\end{align*}
\]

Since \( c_3 \neq 0 \), then from (A4) we know that \( t_{1,0} \equiv 0 \). Using \( t_{1,0} \equiv 0 \), we see from (A1) that \( a_1 \equiv 0 \). We also get \( s_{1,2} \equiv 0 \) and \( s_{2,1} \equiv 0 \) from (A2) and (A3) respectively. Since \( s_{1,2} \equiv 0 \) and \( s_{2,1} \equiv 0 \), (B2) becomes \( 0 \equiv c_2 c_3 t_{0,1} \) and \( 0 \equiv (c_3 - 1)t_{0,1} \).

Hence \( x \in \partial^{-1}W_1 \) if and only if

\[
\begin{align*}
s_{1,2} & \equiv 0 \quad (A2) \\
s_{2,1} & \equiv 0 \quad (A3) \\
t_{1,0} & \equiv 0 \quad (A4) \\
c_1 c_3 t_{0,1} & \equiv 0 \quad (B2) \\
c_2 c_3 t_{0,1} & \equiv 0 \quad (B3) \\
(c_3 - 1)t_{0,1} & \equiv 0 \quad (B4).
\end{align*}
\]

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Recall \( c_3 \in \{1, 2\} \). Consider the cases \((c_1, c_2, c_3) \neq (0, 0, 1)\) and \((c_1, c_2, c_3) = (0, 0, 1)\) separately.

First suppose \((c_1, c_2, c_3) \neq (0, 0, 1)\). Since \( c_3 \in \{1, 2\} \), then \( c_3 = 2 \). Since \( c_3 = 2 \), then from (B4) we get \( t_{0,1} \equiv 0 \). Also, since \( t_{0,1} \equiv 0 \), then the congruences (B2) and (B3) are automatically satisfied.

Hence \( x \in \partial^{-1}W_1 \) if and only if

\[
\begin{align*}
t_{0,1} &\equiv 0 \\
t_{1,0} &\equiv 0 \\
s_{1,2} &\equiv 0 \\
s_{2,1} &\equiv 0.
\end{align*}
\]

So \( \partial^{-1}W_1 = \partial^{-1}W_0 \). Thus \( \text{rank}(\partial^{-1}W_1/W_1) = \text{rank}(\partial^{-1}W_0/W_1) \) and \( W_1 \) is terminal and \( W_1 \notin \hat{\mathcal{L}}_1 \).

Now suppose \((c_1, c_2, c_3) = (0, 0, 1)\). Then the congruences (B2), (B3), and (B4) are automatically satisfied. Hence \( x \in \partial^{-1}W_1 \) if and only if

\[
\begin{align*}
s_{1,2} &\equiv 0 \quad \text{(A2)} \\
s_{2,1} &\equiv 0 \quad \text{(A3)} \\
t_{1,0} &\equiv 0 \quad \text{(A4)} \\
c_1c_3t_{0,1} &\equiv 0 \quad \text{(B2)} \\
c_2c_3t_{0,1} &\equiv 0 \quad \text{(B3)} \\
(c_3 - 1)t_{0,1} &\equiv 0 \quad \text{(B4)}.
\end{align*}
\]
We regard \( t_{0,1} \) as the free variable. Taking \( t_{0,1} \equiv 1 \), the matrix \( x \) becomes

\[
v_1 = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}.
\]

Also, the matrix \( m_1 \) becomes

\[
m_1 = \begin{bmatrix} 1 & 0 & -3 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}.
\]

Recall that in Case 1.2 our expression for \( m_1 \) became

\[
m_1 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ -3 & 0 & 0 \end{bmatrix}.
\]

Therefore, our \( m_1 \) in Case 3.2 is the transpose of our \( m_1 \) in Case 1.2.

Also, recall that

\[
\partial^{-1} W_0 = \begin{bmatrix} 2 & 1 & 1 \\ 1 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}, \quad y_1 = \begin{bmatrix} 0 & 0 & 0 \end{bmatrix}, \quad y_2 = \begin{bmatrix} 0 & 3 & 0 \end{bmatrix}, \quad y_3 = \begin{bmatrix} 0 & 0 & 0 \end{bmatrix}.
\]

We see that \( \partial^{-1} W_1 = \langle \partial^{-1} W_0, v_1 \rangle \). Since \( v_1 \not\in \partial^{-1} W_0 \) and \( 3v_1 \in \partial^{-1} W_0 \), we know \( |\partial^{-1} W_1/\partial^{-1} W_0| = 3 \). Since \( |\partial^{-1} W_0| = 3^7 \) then \( |\partial^{-1} W_1| = 3^8 \). Because \( |W_1| = 3^4 \) then \( |\partial^{-1} W_1/W_1| = 3^4 \). Also, \( v_1, y_1, y_2, y_3 \in \partial^{-1} W_1 \) but are not contained in \( W_1 \). Each of \( 3v_1, 3y_1, 3y_2, 3y_3 \) is contained in \( W_1 \). So \( v_1+W_1, y_1+W_1, y_2+W_1, y_3+W_1 \) are elements
of order 3 in the group $\partial^{-1}W_{1}/W_{1}$. These four elements form a generating set for the group $\partial^{-1}W_{1}/W_{1}$. Recall that $|\partial^{-1}W_{1}/W_{1}| = 3^4$, then $\partial^{-1}W_{1}/W_{1} \cong \mathbb{Z}_3 \times \mathbb{Z}_3 \times \mathbb{Z}_3 \times \mathbb{Z}_3$.

Since $\text{rank}(\partial^{-1}W_{1}/W_{1}) = 4$ and $\text{rank}(\partial^{-1}W_{0}/W_{1}) = 3$, then $\text{rank}(\partial^{-1}W_{0}/W_{1}) < \text{rank}(\partial^{-1}W_{1}/W_{1})$. Hence $W_{1}$ is nonterminal and $W_{1} \in \hat{\mathcal{L}}_{1}$. A basis for $\partial^{-1}W_{1}/W_{1}$ is $v_{1} + W_{1}, y_{1} + W_{1}, y_{2} + W_{1}, y_{3} + W_{1}$. A basis for $\partial^{-1}W_{0}/W_{1}$ is $y_{1} + W_{1}, y_{2} + W_{1}, y_{3} + W_{1}$.

We fix arbitrary values $c_{4}, c_{5},$ and $c_{6} \in \{0, 1, 2\}$. There are $3^3 = 27$ ways to choose these values. Let $m_{2} = c_{4}y_{1} + c_{5}y_{2} + c_{6}y_{3} + v_{1}$. Thus

$$m_{2} = \begin{bmatrix} 0 & 1 & 3c_{6} \\ 0 & 3c_{5} & 0 \\ 3c_{4} & 0 & 0 \end{bmatrix}.$$

Let $W_{2} = < W_{1}, m_{2} >$. There are 27 subgroups of $W_{2}$ of this type. Since $m_{2} \notin W_{1}$ and $3m_{2} \in W_{1}$ then $|W_{2}/W_{1}| = 3$. Also, since $|W_{1}| = 3^4$ then $|W_{2}| = 3^5$. Recall $|\partial^{-1}W_{1}| = 3^8$, then $\partial^{-1}W_{1}/W_{2} \cong \mathbb{Z}_3 \times \mathbb{Z}_3 \times \mathbb{Z}_3$ and $\text{rank}(\partial^{-1}W_{1}/W_{2}) = 3$.

The subgroup $W_{2}$ is contained in the pattern subgroup

$$\begin{bmatrix} 2 & 2 & 1 \\ 1 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}.$$  

The pullback $\partial^{-1}W_{2}$ is contained in the pattern subgroup

$$\begin{bmatrix} 2 & 2 & 2 \\ 2 & 2 & 1 \\ 1 & 1 & 0 \end{bmatrix} \cap \begin{bmatrix} 2 & 2 & 2 \\ 2 & 1 & 1 \\ 2 & 1 & 0 \end{bmatrix} = \begin{bmatrix} 2 & 2 \\ 1 & 1 \\ 1 & 0 \end{bmatrix}.$$
Let
\[ x = \begin{bmatrix} 3s_{0,0} + t_{0,0} & 3s_{0,1} + t_{0,1} & 3s_{0,2} + t_{0,2} \\ 3s_{1,0} + t_{1,0} & 3s_{1,1} & 3s_{1,2} \\ 3s_{2,0} & 3s_{2,1} & 0 \end{bmatrix} \in \begin{bmatrix} 2 & 2 & 2 \\ 2 & 1 & 1 \\ 1 & 1 & 0 \end{bmatrix}. \]

Thus
\[ \partial_1 x = \begin{bmatrix} 3s_{1,0} + t_{1,0} & 3s_{1,1} & 3s_{1,2} \\ 3s_{2,0} & 3s_{2,1} & 0 \\ -3t_{1,0} & 0 & 0 \end{bmatrix} \text{ and } \partial_2 x = \begin{bmatrix} 3s_{0,1} + t_{0,1} & 3s_{0,2} + t_{0,2} & -3(t_{0,1} + t_{0,2}) \\ 3s_{1,1} & 3s_{1,2} & 0 \\ 3s_{2,1} & 0 & 0 \end{bmatrix}. \]

The variables that are in play are those appearing in the matrix
\[ \begin{bmatrix} 0 & t_{0,1} & t_{0,2} \\ t_{1,0} & 0 & 3s_{1,2} \\ 0 & 3s_{2,1} & 0 \end{bmatrix} \]

We want to identify a value \( a_1, a_2 \in \mathbb{Z}_9 \) such that \( \partial_1 x \equiv a_1 m_1 + a_2 m_2 (\text{mod } I) \). A formal expression for \( a_1 m_1 + a_2 m_2 \) is
\[ \begin{bmatrix} 1 & 0 & -3 \\ 0 & 0 & 0 \end{bmatrix} + a_2 \begin{bmatrix} 0 & 1 & 3c_6 \\ 0 & 3c_5 & 0 \end{bmatrix} = \begin{bmatrix} a_1 & a_2 & 3(a_2c_6 - a_1) \\ a_1 & a_2 & 3a_2c_5 \end{bmatrix}. \]

Comparing \((0, 1)\)-entries, we get \( a_2 \equiv 0 \).
Comparing (1, 0)-entries, we get $3s_{2,0} \equiv 0$, which gives no information.

Comparing (0, 0)-entries, we get $a_1 \equiv t_{1,0}$.

Comparing (1, 1)-entries, we get $a_2 c_5 \equiv s_{2,1}$. Since $a_2 \equiv 0$, we get $s_{2,1} \equiv 0$.

Comparing (2, 0)-entries, we get $-t_{1,0} \equiv a_2 c_4$. Since $a_2 \equiv 0$, we have $t_{1,0} \equiv 0$.

Comparing (0, 2)-entries, we get $a_2 c_6 - a_1 \equiv s_{1,2}$ Since $a_1 \equiv t_{1,0} \equiv 0$, and $a_2 \equiv 0$, and we have $s_{1,2} \equiv 0$.

We see that $\partial_1 x \in W_2$ if and only if

\[ t_{1,0} \equiv 0 \quad \text{(A1)} \]
\[ s_{2,1} \equiv 0 \quad \text{(A2)} \]
\[ s_{1,2} \equiv 0 \quad \text{(A3)}. \]

We want to identify a value $b_1, b_1 \in \mathbb{Z}_9$ such that $\partial_2 x \equiv b_1 m_1 + b_2 m_2 \pmod{I}$.

A formal expression for $b_1 m_1 + b_2 m_2$ is

\[
b_1 \begin{bmatrix} 1 & 0 & -3 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} + b_2 \begin{bmatrix} 0 & 1 & 3c_6 \\ 0 & 3c_5 & 0 \\ 3c_4 & 0 & 0 \end{bmatrix} = \begin{bmatrix} b_1 & b_2 & 3(b_2 c_6 - b_1) \\ 0 & 3b_2 c_5 & 0 \\ 3b_2 c_4 & 0 & 0 \end{bmatrix}.
\]

Comparing (0, 1)-entries, we get $t_{0,2} \equiv b_2$.

Comparing (1, 0)-entries, we get $3s_{1,1} \equiv 0$, which gives no information.

Comparing (0, 0)-entries, we get $b_1 \equiv t_{0,1}$.

Comparing (1, 1)-entries, we get $b_2 c_5 \equiv s_{1,2}$. Since $b_2 \equiv t_{0,2}$, we obtain the congruence $c_5 t_{0,2} \equiv s_{1,2}$. 54
Comparing \((0, 2)\)-entries, we get \(-t_{0,1} - t_{0,2} \equiv -b_1 + b_2 c_6\). Substituting \(b_1\) and \(b_2\) we get \(t_{0,2} \equiv -c_6 t_{0,2}\).

Comparing \((2, 0)\)-entries, we get \(s_{2,1} \equiv b_2 c_4\). Since \(b_2 \equiv t_{0,2}\), we get \(s_{2,1} \equiv c_4 t_{0,2}\).

We see that \(\partial_2 x \in W_2\) if and only if

\[
\begin{align*}
s_{1,2} & \equiv c_5 t_{0,2} \quad \text{(B1)} \\
t_{0,2} & \equiv -c_6 t_{0,2} \quad \text{(B2)} \\
s_{2,1} & \equiv c_4 t_{0,2} \quad \text{(B3)}.
\end{align*}
\]

Since \(s_{1,2} \equiv 0\) and \(s_{2,1} \equiv 0\), (B1) becomes \(c_5 t_{0,2} \equiv 0\) and (B3) becomes \(c_4 t_{0,2} \equiv 0\).

Hence \(x \in \partial^{-1} W_2\) if and only if

\[
\begin{align*}
t_{1,0} & \equiv 0 \quad \text{(A1)} \\
s_{2,1} & \equiv 0 \quad \text{(A2)} \\
s_{1,2} & \equiv 0 \quad \text{(A3)} \\
c_5 t_{0,2} & \equiv 0 \quad \text{(B1)} \\
t_{0,2} & \equiv -c_6 t_{0,2} \quad \text{(B2)} \\
c_4 t_{0,2} & \equiv 0 \quad \text{(B3)}.
\end{align*}
\]

It is convenient to consider the cases of \((c_4, c_5, c_6) \neq (0, 0, 0)\) and \((c_4, c_5, c_6) = (0, 0, 0)\).
6.2.1 Case 3.2.1

Let us examine the case \((c_4, c_5, c_6) \neq (0, 0, 0)\). Since \(c_5 \neq 0\), then from (B1) \(t_{0,2}\) must be 0. Since \(t_{0,2}\), then (B2) and (B3) hold.

Hence \(x \in \partial^{-1}W_2\) if and only if

\[
\begin{align*}
s_{1,2} &\equiv 0 \\
s_{2,1} &\equiv 0 \\
t_{1,0} &\equiv 0 \\
t_{0,2} &\equiv 0.
\end{align*}
\]

So \(\partial^{-1}W_2 = \partial^{-1}W_1\). Thus \(\text{rank}(\partial^{-1}W_2/W_2) = \text{rank}(\partial^{-1}W_1/W_2)\) and \(W_2\) is terminal and \(W_2 \notin \hat{\mathcal{L}}_2\).

6.2.2 Case 3.2.2

Let us examine the case \((c_4, c_5, c_6) = (0, 0, 0)\). Then each of the congruences (B1), (B2), and (B3) is automatically satisfied.

Hence \(x \in \partial^{-1}W_2\) if and only if

\[
\begin{align*}
t_{1,0} &\equiv 0 \\
s_{2,1} &\equiv 0 \\
s_{1,2} &\equiv 0.
\end{align*}
\]
Then our expression for $m_2$ becomes
\[
\begin{pmatrix}
0 & 1 & 0 \\
1 & 0 & 0 \\
0 & 0 & 0
\end{pmatrix}.
\]
We regard $t_{0,1}$ and $t_{0,2}$ as free variables. Taking $t_{0,1} \equiv 1$ and $t_{0,2} = 0$, the matrix $x$ becomes
\[
v_1 = \begin{pmatrix}
0 & 1 & 0 \\
0 & 0 & 0 \\
0 & 0 & 0
\end{pmatrix}.
\]
Taking $t_{0,2} \equiv 1$ and $t_{0,1} = 0$, the matrix $x$ becomes
\[
v_2 = \begin{pmatrix}
0 & 0 & 1 \\
0 & 0 & 0 \\
0 & 0 & 0
\end{pmatrix}.
\]
Recall that
\[
\partial^{-1}W_1 = \begin{pmatrix}
2 & 2 & 1 \\
2 & 1 & 1 \\
1 & 1 & 0
\end{pmatrix}.
\]
We see that neither $v_2$ nor $3v_2$ is contained in $W_2$ and
\[
\partial^{-1}W_2 = < \partial^{-1}W_0, v_1, v_2 >.
\]
We know that $v_2 \in \partial^{-1}W_2$ and since $3v_2 \notin W_2$ then $v_2 + W_2$ is an element of the group $\partial^{-1}W_2/W_2$ whose order is larger than 3. Recall that $|\partial^{-1}W_2/W_2| = 3^4$ and that $\partial^{-1}W_1/W_2$ has rank 3. Since $\partial^{-1}W_2$ has an element of order larger than 3 then rank($\partial^{-1}W_2/W_2$) is not greater than 3 since $\partial^{-1}W_2/W_2 \cong \mathbb{Z}_9 \times \mathbb{Z}_3 \times \mathbb{Z}_3$ and therefore $W_2$ is terminal and $W_2 \notin \hat{\mathcal{L}}_2$. 57
In Case 3 we found 26 subgroups $W_1 \in \mathcal{L}_1$, all satisfying $|W_1| = 3^4$. Exactly 1 of these 26 subgroups is nonterminal and is contained in 27 members of $\mathcal{L}_2$. Thus we found 27 subgroups $W_2$ of $\mathcal{L}_2$, all satisfying $|W_2| = 3^5$. Every member of $\mathcal{L}_2$ is terminal. So in Case 3 we found a total of $26 + 27 = 53$ subgroups.
We fix arbitrary values $(c_1, c_2) \in \{1, 2\}$. Let $m_1 = y_2 + c_1 y_4$ and $m_2 = y_3 + c_2 y_4$. Thus

$$m_1 = \begin{bmatrix} c_1 & 0 & -3c_1 \\ 0 & 3 & 0 \\ -3c_1 & 0 & 0 \end{bmatrix} \quad \text{and} \quad m_2 = \begin{bmatrix} c_2 & 0 & 3(1 - c_2) \\ 0 & 0 & 0 \\ -3c_2 & 0 & 0 \end{bmatrix}.$$

Let $W_1 = < W_0, m_1, m_2 > \in \mathcal{L}_1$. The number of subgroups of this type is $2^2 = 4$. Note that $|W_1| = 3^5$ and $\partial^{-1} W_0/W_1 \cong \mathbb{Z}_3 \times \mathbb{Z}_3$ and rank $(\partial^{-1} W_0/W_1) = 2$.

We now calculate the pullback $\partial^{-1} W_1$. The subgroup $W_1$ is contained in the pattern subgroup

$$\begin{bmatrix} 2 & 1 & 1 \\ 1 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}.$$

Thus the pullback $\partial^{-1} W_1$ is contained in the pattern subgroup

$$\begin{bmatrix} 2 & 2 & 2 \\ 2 & 1 & 1 \\ 1 & 1 & 0 \end{bmatrix} \cap \begin{bmatrix} 2 & 2 & 1 \\ 2 & 1 & 1 \\ 2 & 1 & 0 \end{bmatrix} = \begin{bmatrix} 2 & 2 & 1 \\ 2 & 1 & 1 \\ 1 & 1 & 0 \end{bmatrix}.$$
Let
\[
x = \begin{bmatrix}
3s_{0,0} + t_{0,0} & 3s_{0,1} + t_{0,1} & 3s_{0,2} \\
3s_{1,0} + t_{1,0} & 3s_{1,1} & 3s_{1,2} \\
3s_{2,0} & 3s_{2,1} & 0
\end{bmatrix}
\in \begin{bmatrix}
2 & 2 & 1 \\
2 & 1 & 1 \\
1 & 1 & 0
\end{bmatrix}.
\]

Thus
\[
\partial_1 x = \begin{bmatrix}
3s_{1,0} + t_{1,0} & 3s_{1,1} & 3s_{1,2} \\
3s_{2,0} & 3s_{2,1} & 0 \\
-3t_{1,0} & 0 & 0
\end{bmatrix}
\quad \text{and} \quad
\partial_2 x = \begin{bmatrix}
3s_{0,1} + t_{0,1} & 3s_{0,2} & -3t_{0,1} \\
3s_{1,1} & 3s_{1,2} & 0 \\
3s_{2,1} & 0 & 0
\end{bmatrix}.
\]

The variables that are in play are those appearing in the matrix
\[
\begin{bmatrix}
0 & t_{0,1} & 0 \\
t_{1,0} & 0 & 3s_{1,2} \\
0 & 3s_{2,1} & 0
\end{bmatrix}.
\]

We want to identify a value \(a_1, a_2 \in \mathbb{Z}_9\) such that \(\partial_1 x \equiv a_1 m_1 + a_2 m_2 (\mod I)\). A formal expression for \(a_1 m_1 + a_2 m_2\) is
\[
a_1 \begin{bmatrix}
c_1 & 0 & -3c_1 \\
0 & 3 & 0 \\
-3c_1 & 0 & 0
\end{bmatrix}
+ a_2 \begin{bmatrix}
c_2 & 0 & 3(1 - c_2) \\
0 & 0 & 0 \\
-3c_2 & 0 & 0
\end{bmatrix}
= \begin{bmatrix}
a_1 c_1 + a_2 c_2 & 0 & 3(a_2 (1 - c_2) - a_1 c_1) \\
0 & 3a_1 & 0 \\
-3(a_1 c_1 + a_2 c_2) & 0 & 0
\end{bmatrix}.
\]

Comparing \((0, 1)\)-entries, we get \(3s_{1,1} \equiv 0\) which gives no information.
Comparing (1,0)-entries, we get $3s_{2,0} \equiv 0$ which gives no information.

Comparing (1,1)-entries, we get $a_1 \equiv s_{2,1}$.

Comparing (0,0)-entries, we get $c_1a_1 + c_2a_2 \equiv t_{1,0}$. Since $a_1 \equiv s_{2,1}$, we get $c_1s_{2,1} + c_2a_2 \equiv t_{1,0}$. So $c_2a_2 \equiv t_{1,0} - c_1s_{2,1}$ and since $c_2 \in \{1, 2\}$, $c_2^2 \equiv 1$ so $a_2 \equiv c_2t_{1,0} - c_1c_2s_{2,1}$.

Comparing (0,2)-entries, we get $a_2(1 - c_2) - c_1a_1 \equiv s_{1,2}$. Since $a_1 \equiv s_{2,1}$ and $a_2 \equiv c_2t_{1,0} - c_1c_2s_{2,1}$, we get $(c_2 - 1)t_{1,0} - c_1c_2s_{2,1} \equiv s_{1,2}$.

Comparing (2,0)-entries, we get $-(c_1a_1 + c_2a_2) \equiv -t_{1,0}$. Since $a_1 \equiv s_{2,1}$ and $a_2 \equiv c_2t_{1,0} - c_1c_2s_{2,1}$, this is equivalent to $t_{1,0} \equiv t_{1,0}$ which is always automatically true.

We see that $\partial_1 x \in W_1$ if and only if

$$(c_2 - 1)t_{1,0} - c_1c_2s_{2,1} \equiv s_{1,2} \quad \text{(A1)}.$$ 

We want to identify a value $b_1, b_2 \in \mathbb{Z}_9$ such that $\partial_2 x \equiv b_1m_1 + b_2m_2 (\text{mod} \ I)$.

A formal expression for $b_1m_1 + b_2m_2$ is

$$b_1 \begin{bmatrix} c_1 & 0 & -3c_1 \\ 0 & 3 & 0 \\ -3c_1 & 0 & 0 \end{bmatrix} + b_2 \begin{bmatrix} c_2 & 0 & 3(1 - c_2) \\ 0 & 0 & 0 \\ -3c_2 & 0 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} b_1c_1 + b_2c_2 & 0 & 3(b_2(1 - c_2) - b_1c_1) \\ 0 & 3b_1 & 0 \\ -3(b_1c_1 + b_2c_2) & 0 & 0 \end{bmatrix}.$$ 

Comparing (0,1)-entries, we get $3s_{0,2} \equiv 0$ which gives no information.

Comparing (1,0)-entries, we get $3s_{1,1} \equiv 0$ which gives no information.
Comparing \((1,1)\)-entries, we get \(b_1 \equiv s_{1,2}\).

Comparing \((0,0)\)-entries, we get \(c_1 b_1 + c_2 b_2 \equiv t_{0,1}\). Since \(b_1 \equiv s_{1,2}\), we get \(c_1 s_{1,2} + c_2 b_2 \equiv t_{0,1}\). So \(c_2 b_2 \equiv t_{0,1} - b_1 c_2\) and since \(c_2 \in \{1, 2\}\), \(c_2^2 \equiv 1\), we obtain \(b_2 \equiv c_2 t_{0,1} - c_1 c_2 s_{1,2}\).

Comparing \((0,2)\)-entries, we get \(b_2 (1 - c_2) - c_1 b_1 \equiv -t_{0,1}\). Since \(b_1 \equiv s_{1,2}\) and \(b_2 \equiv c_2 t_{0,1} - c_1 c_2 s_{1,2}\), we get \(c_2 t_{0,1} - c_1 c_2 s_{1,2} \equiv 0\). Since \(c_2^2 \equiv 1\), multiplying by \(c_2\) gives us \(s_{1,2} \equiv c_1 t_{0,1}\).

Comparing \((2,0)\)-entries, we get \(- (c_1 b_1 + c_2 b_2) \equiv s_{2,1}\). Since \(b_1 \equiv s_{1,2}\) and \(b_2 \equiv c_2 t_{0,1} - c_1 c_2 s_{1,2}\), we get \(s_{2,1} \equiv -t_{0,1}\).

We see that \(\partial_2 x \in W_1\) if and only if

\[
\begin{align*}
  s_{1,2} &\equiv c_1 t_{0,1} \quad \text{(B1)} \\
  s_{2,1} &\equiv -t_{0,1} \quad \text{(B2)}.
\end{align*}
\]

Hence \(x \in \partial^{-1} W_1\) if and only if

\[
\begin{align*}
  (c_2 - 1) t_{1,0} - c_1 c_2 s_{2,1} &\equiv s_{1,2} \quad \text{(A1)} \\
  s_{1,2} &\equiv c_1 t_{0,1} \quad \text{(B1)} \\
  s_{2,1} &\equiv -t_{0,1} \quad \text{(B2)}.
\end{align*}
\]

It is convenient to consider the cases of \(c_2 = 2\) and \(c_2 \neq 2\) separately.
7.1 Case 4.1

Let us examine the case $c_2 = 2$. Our expressions for $m_1$ and $m_2$ become

$$m_1 = \begin{bmatrix}
  c_1 & 0 & -3c_1 \\
  0 & 3 & 0 \\
  -3c_1 & 0 & 0
\end{bmatrix}, \quad m_2 = \begin{bmatrix}
  2 & 0 & -3 \\
  0 & 0 & 0 \\
  3 & 0 & 0
\end{bmatrix}.$$ 

Congruence (A1) becomes $t_{1,0} \equiv s_{1,2} - c_1 s_{2,1}$. Substituting for $s_{1,2}$ and $s_{2,1}$, using (B1) and (B2), we obtain $t_{1,0} \equiv c_1 t_{0,1} - c_1 (-t_{0,1})$. This says that $t_{1,0} \equiv -c_1 t_{0,1}$ which can rewritten as $t_{0,1} \equiv -c_1 t_{1,0}$. Next substituting $t_{0,1}$ into (B1) we obtain $-s_{1,2} \equiv t_{1,0}$. Also, since $c_1^2 \equiv 1$ we can multiply (B2) to obtain $c_1 s_{2,1} \equiv t_{1,0}$.

Hence $x \in \partial^{-1}W_1$ if and only if

$$t_{0,1} \equiv -c_1 t_{1,0}$$

$$s_{1,2} \equiv -t_{1,0}$$

$$s_{2,1} \equiv c_1 t_{1,0}.$$ 

We regard $t_{1,0}$ as the free variable. Taking $t_{1,0} \equiv 1$, the matrix $x$ becomes

$$v_1 = \begin{bmatrix}
  0 & -c_1 & 0 \\
  1 & 0 & -3 \\
  0 & 3c_1 & 0
\end{bmatrix}.$$ 

Recall that

$$\partial^{-1}W_0 = \begin{bmatrix}
  2 & 1 & 1 \\
  1 & 1 & 0 \\
  1 & 0 & 0
\end{bmatrix}, \quad y_1 = \begin{bmatrix}
  0 & 0 & 0 \\
  0 & 0 & 0 \\
  3 & 0 & 0
\end{bmatrix}, \quad y_3 = \begin{bmatrix}
  0 & 0 & 3 \\
  0 & 0 & 0 \\
  0 & 0 & 0
\end{bmatrix}.$$
We see that $\partial^{-1}W_1 = \langle \partial^{-1}W_0, v_1 \rangle$. Since $v_1 \notin \partial^{-1}W_0$ and $3v_1 \in \partial^{-1}W_0$, we know $|\partial^{-1}W_1/\partial^{-1}W_0| = 3$. Since $|\partial^{-1}W_0| = 3^7$ then $|\partial^{-1}W_1| = 3^8$. Because $|W_1| = 3^5$ then $|\partial^{-1}W_1/W_1| = 3^3$. Also, $v_1, y_1, y_3 \in \partial^{-1}W_1$ but are not contained in $W_1$. Each of $3v_1, 3y_1, 3y_3$ is contained in $W_1$. So $v_1 + W_1, y_1 + W_1, y_3 + W_1$ are elements of order 3 in the group $\partial^{-1}W_1/W_1$. These three elements form a generating set for the group $\partial^{-1}W_1/W_1$. Recall that $|\partial^{-1}W_1/W_1| = 3^3$ we obtain $\partial^{-1}W_1/W_1 \cong \mathbb{Z}_3 \times \mathbb{Z}_3 \times \mathbb{Z}_3$. Since rank($\partial^{-1}W_1/W_1$) = 3 and rank($\partial^{-1}W_0/W_1$) = 2, then rank($\partial^{-1}W_0/W_1$) < rank($\partial^{-1}W_1/W_1$). Hence $W_1$ is nonterminal and $W_1 \in \hat{L}_1$. A basis for $\partial^{-1}W_1/W_1$ is $v_1 + W_1, y_1 + W_1, y_3 + W_1$. A basis for $\partial^{-1}W_0/W_1$ is $y_1 + W_1, y_3 + W_1$.

We fix arbitrary values $c_3, c_4 \in \{0, 1, 2\}$. There are $3^2 = 9$ ways to choose these values. Let $m_3 = c_3y_1 + c_4y_3 + v_1$. Thus

$$m_3 = \begin{bmatrix} 0 & -c_1 & 3c_4 \\ 1 & 0 & -3 \\ 3c_3 & 3c_1 & 0 \end{bmatrix}.$$ 

Let $W_2 = \langle W_1, m_3 \rangle$. There are 9 subgroups of $W_2$ of this type. Since $m_3 \notin W_1$ and $3m_3 \in W_1$, then $|W_2/W_1| = 3$. Also, since $|W_1| = 3^5$ then $|W_2| = 3^6$. Recall $|\partial^{-1}W_1| = 3^8$, then $\partial^{-1}W_1/W_2 \cong \mathbb{Z}_3 \times \mathbb{Z}_3$ and rank($\partial^{-1}W_1/W_2$) = 2.

The subgroup $W_2$ is contained in the pattern subgroup

$$\begin{bmatrix} 2 & 2 & 1 \\ 2 & 1 & 1 \\ 1 & 1 & 0 \end{bmatrix}.$$

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The pullback $\partial^{-1} W_2$ is contained in the pattern subgroup

$$\begin{bmatrix} 2 & 2 & 2 \\ 2 & 2 & 1 \\ 2 & 1 & 1 \end{bmatrix} \cap \begin{bmatrix} 2 & 2 & 2 \\ 2 & 2 & 1 \\ 2 & 1 & 1 \end{bmatrix} = \begin{bmatrix} 2 & 2 & 2 \\ 2 & 2 & 1 \\ 2 & 1 & 1 \end{bmatrix}.$$ 

Let

$$x = \begin{bmatrix} 3s_{0,0} + t_{0,0} & 3s_{0,1} + t_{0,1} & 3s_{0,2} + t_{0,2} \\ 3s_{1,0} + t_{1,0} & 3s_{1,1} + t_{1,1} & 3s_{1,2} \\ 3s_{2,0} + t_{2,0} & 3s_{2,1} & 3s_{2,2} \end{bmatrix} \in \begin{bmatrix} 2 & 2 & 2 \\ 2 & 2 & 1 \\ 2 & 1 & 1 \end{bmatrix}.$$ 

Thus

$$\partial_1 x = \begin{bmatrix} 3s_{1,0} + t_{1,0} & 3s_{1,1} + t_{1,1} & 3s_{1,2} \\ 3s_{2,0} + t_{2,0} & 3s_{2,1} & 3s_{2,2} \\ -3(t_{1,0} + t_{2,0}) & -3t_{1,1} & 0 \end{bmatrix}$$

and

$$\partial_2 x = \begin{bmatrix} 3s_{0,1} + t_{0,1} & 3s_{0,2} + t_{0,2} & -3(t_{0,1} + t_{0,2}) \\ 3s_{1,1} + t_{1,1} & 3s_{1,2} & -3t_{1,1} \\ 3s_{2,1} & 3s_{2,2} & 0 \end{bmatrix}.$$ 

The variables that are in play are those appearing in the matrix

$$\begin{bmatrix} 0 & t_{0,1} & t_{0,2} \\ t_{1,0} & t_{1,1} & 3s_{1,2} \\ t_{2,0} & 3s_{2,1} & 3s_{2,2} \end{bmatrix}.$$
We want to identify a value \( a_1, a_2, a_3 \in \mathbb{Z}_9 \) such that \( \partial_1 x \equiv a_1 m_1 + a_2 m_2 + a_3 m_3 \) (mod \( I \)). A formal expression for \( a_1 m_1 + a_2 m_2 + a_3 m_3 \) is

\[
\begin{bmatrix}
  c_1 & 0 & -3c_1 \\
  0 & 3 & 0 \\
  -3c_1 & 0 & 0
\end{bmatrix} + a_2
\begin{bmatrix}
  2 & 0 & -3 \\
  0 & 0 & 0 \\
  3 & 0 & 0
\end{bmatrix} + a_3
\begin{bmatrix}
  0 & -c_1 & 3c_4 \\
  1 & 0 & -3 \\
  3c_3 & 3c_1 & 0
\end{bmatrix}
\]

Comparing \((1,0)\)-entries, we get \( a_3 \equiv t_{2,0} \).

Comparing \((1,2)\)-entries, we get \( a_3 \equiv -s_{2,2} \) and \( s_{2,2} \equiv -t_{2,0} \).

Comparing \((1,1)\)-entries, we get \( a_1 \equiv s_{2,1} \).

Comparing \((0,0)\)-entries, we get \( a_1 c_1 + 2a_2 \equiv t_{1,0} \). Substituting \( a_1 \equiv s_{2,1} \) we obtain \( a_2 \equiv c_1 s_{2,1} - t_{1,0} \).

Comparing \((0,1)\)-entries, we get \(-a_3 c_1 \equiv t_{1,1} \). Substituting \( a_3 \equiv t_{2,0} \) we get \( t_{1,1} \equiv -c_1 t_{2,0} \).

Comparing \((2,1)\)-entries, we get \( a_3 c_1 \equiv -t_{1,1} \) which becomes \(-c_1 t_{2,0} \equiv t_{1,1} \) and gives us no new information.

Comparing \((0,2)\)-entries, we get \( a_3 c_4 - a_1 c_1 - a_2 \equiv s_{1,2} \). Substituting for \( a_3, a_1, \) and \( a_2 \) we get \( s_{1,2} \equiv c_1 s_{2,1} + t_{1,0} + c_4 t_{2,0} \).

Comparing \((2,0)\)-entries, we get \( t_{1,0} + t_{2,0} \equiv a_1 c_1 - a_2 - a_3 c_3 \). Substituting \( a_2, a_3 \) and \( a_1 \) we obtain \( (c_3 + 1) t_{2,0} \equiv 0 \).
We see that $\partial_1 x \in W_2$ if and only if

\[ s_{2,2} \equiv -t_{2,0} \quad \text{(A1)} \]
\[ t_{1,1} \equiv -c_1 t_{2,0} \quad \text{(A2)} \]
\[ s_{1,2} \equiv c_1 s_{2,1} + t_{1,0} + c_4 t_{2,0} \quad \text{(A3)} \]
\[ (c_3 + 1)t_{2,0} \equiv 0 \quad \text{(A4)}. \]

We want to identify a value $b_1, b_2, b_3 \in \mathbb{Z}_9$ such that $\partial_1 x \equiv b_1 m_1 + b_2 m_2 + b_3 m_3 \pmod{I}$. A formal expression for $b_1 m_1 + b_2 m_2 = b_3 m_3$ is

\[
\begin{bmatrix}
  c_1 & 0 & -3c_1 \\
  0 & 3 & 0 \\
  -3c_1 & 0 & 0
\end{bmatrix} + b_2
\begin{bmatrix}
  2 & 0 & -3 \\
  0 & 0 & 0 \\
  3 & 0 & 0
\end{bmatrix} + b_3
\begin{bmatrix}
  0 & -c_1 & 3c_4 \\
  1 & 0 & -3 \\
  3c_3 & 3c_1 & 0
\end{bmatrix}
\]

\[
= \begin{bmatrix}
  b_1 c_1 + 2b_2 & -b_3 c_1 & 3(b_3 c_4 - b_1 c_1 - b_2) \\
  b_3 & 3b_1 & -3b_3 \\
  3(b_2 + b_3 c_3 - b_1 c_1) & 3b_3 c_1 & 0
\end{bmatrix}.
\]

Comparing (1,0)-entries, we get $b_3 \equiv t_{1,1}$.

Comparing (1,2)-entries, we get $b_3 \equiv t_{1,1}$, which gives no new information.

Comparing (1,1)-entries, we get $b_1 \equiv s_{1,2}$.

Comparing (0,0)-entries, we get $b_1 c_1 + 2b_2 \equiv t_{0,1}$. Substituting $b_1$ we get $b_2 \equiv c_1 s_{1,2} - t_{0,1}$.

Comparing (0,1)-entries, we get $-b_3 c_1 \equiv t_{0,2}$. Substituting $b_3$ the congruence becomes $t_{1,1} \equiv -c_1 t_{0,2}$. 

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Comparing (2, 1)-entries, we get \(b_3c_1 \equiv s_{2,2}\). Substituting \(b_3\) we obtain \(c_1 t_{1,1} \equiv s_{2,2}\).

Combining this with the (0, 1) entry we obtain \(s_{2,2} \equiv -t_{0,2}\).

Comparing (0, 2)-entries, we get \(t_{0,1} + t_{0,2} \equiv b_1c_1 + b_2 - b_3c_4\). Substituting \(b_3, b_2\) and \(b_1\) we obtain \(t_{0,1} + t_{0,2} \equiv c_1s_{1,2} + c_1s_{1,2} - t_{0,1} - c_4t_{1,1}\) Recalling that \(t_{1,1} \equiv -c_1t_{0,2}\) and \(c_1^2 \equiv 1\) we obtain \(s_{1,2} \equiv c_1t_{0,1} + (c_4 - c_1)t_{0,2}\).

Comparing (2, 0)-entries, we get \(b_2 + b_3c_3 - b_1c_1 \equiv s_{2,1}\). Substituting \(b_1, b_2, b_3,\) and \(t_{1,1}\) we obtain \(s_{2,1} \equiv -t_{0,1} - c_1c_3t_{0,2}\).

We see that \(\partial_1x \in W_2\) if and only if

\[
t_{1,1} \equiv -c_1t_{0,2} \quad \text{(B1)}
\]

\[
s_{2,2} \equiv -t_{0,2} \quad \text{(B2)}
\]

\[
s_{1,2} \equiv c_1t_{0,1} + (c_4 - c_1)t_{0,2} \quad \text{(B3)}
\]

\[
s_{2,1} \equiv -t_{0,1} - c_1c_3t_{0,2} \quad \text{(B4)}.
\]

Substituting (A1) into (B2) we obtain \(t_{0,2} \equiv t_{2,0}\). Using \(t_{0,2} \equiv t_{2,0}\), (B1) becomes \(t_{1,1} \equiv -c_1t_{2,0}\), (B3) becomes \(s_{1,2} \equiv c_1t_{0,1} + (c_4 - c_1)t_{2,0}\) and (B4) becomes \(s_{2,1} \equiv -t_{0,1} - c_1c_3t_{2,0}\). Substituting (B3) and (B4) into (A3) we obtain \(t_{0,1} \equiv (c_4c_3 - 1)t_{2,0} - c_1t_{1,0}\). Next substituting our new (A3) into (B3) and (B4) we obtain \(s_{1,2} \equiv (c_1 + c_3 + c_4)t_{2,0} - t_{1,0}\) and \(s_{2,1} \equiv (c_1c_3 + 1)t_{2,0} + c_1t_{1,0}\) respectively.
Hence $x \in \partial^{-1}W_2$ if and only if

$$s_{2,2} \equiv -t_{2,0} \quad \text{(A1)}$$

$$t_{1,1} \equiv -c_1t_{2,0} \quad \text{(A2)}$$

$$t_{0,1} \equiv (c_1c_3 - 1)t_{2,0} - c_1t_{1,0} \quad \text{(A3)}$$

$$(c_3 + 1)t_{2,0} \equiv 0 \quad \text{(A4)}$$

$$t_{1,1} \equiv -c_1t_{2,0} \quad \text{(B1)}$$

$$t_{0,2} \equiv t_{2,0} \quad \text{(B2)}$$

$$s_{1,2} \equiv (c_1 + c_3 + c_4)t_{2,0} - t_{1,0} \quad \text{(B3)}$$

$$s_{2,1} \equiv (c_1c_3 + 1)t_{2,0} + c_1t_{1,0} \quad \text{(B4)}.$$ 

It is convenient to consider the cases $c_3 \neq 2$ and $c_3 \equiv 2$.

7.1.1 Case 4.1.1

First we consider the case $c_3 \neq 2$. Then congruence (A4) tells us that $t_{2,0} \equiv 0$. 
Hence $x \in \partial^{-1}W_2$ if and only if

\[
\begin{align*}
t_{2,0} &\equiv 0 \\
t_{0,2} &\equiv 0 \\
s_{2,2} &\equiv 0 \\
t_{1,1} &\equiv 0 \\
s_{2,1} &\equiv c_1 t_{1,0} \\
s_{1,2} &\equiv -t_{1,0} \\
t_{0,1} &\equiv -c_1 t_{1,0}.
\end{align*}
\]

We regard $t_{1,0}$ as the free variable. Letting $t_{1,0} \equiv 1$ the matrix $x$ becomes

\[
v_2 = \begin{bmatrix} 0 & -c_1 & 0 \\ 1 & 0 & -3 \\ 0 & 3c_1 & 0 \end{bmatrix}.
\]

So $\partial^{-1}W_2 = \partial^{-1}W_1$. Thus $\text{rank}(\partial^{-1}W_2/W_2) = \text{rank}(\partial^{-1}W_1/W_2)$ and $W_2$ is terminal and $W_2 \not\in \hat{\mathcal{L}}_2$. 

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7.1.2 Case 4.1.2

Now we consider the case $c_3 \equiv 2$. The congruence (A4) holds automatically.

Hence $x \in \partial^{-1}W_2$ if and only if

\[
\begin{align*}
    s_{2,2} &\equiv -t_{2,0} \quad (A1) \\
    t_{1,1} &\equiv -c_1 t_{2,0} \quad (A2) \\
    t_{0,1} &\equiv (2c_1 - 1)t_{2,0} - c_1 t_{1,0} \quad (A3) \\
    t_{1,1} &\equiv -c_1 t_{2,0} \quad (B1) \\
    t_{0,2} &\equiv t_{2,0} \quad (B2) \\
    s_{1,2} &\equiv (c_1 + 2 + c_4)t_{2,0} - t_{1,0} \quad (B3) \\
    s_{2,1} &\equiv (2c_1 + 1)t_{2,0} + c_1 t_{1,0} \quad (B4). \\
\end{align*}
\]

We regard $t_{1,0}$ and $t_{2,0}$ as free variables. Letting $t_{1,0} \equiv 1$ and $t_{2,0} \equiv 0$, the matrix $x$ becomes

\[
v_1 = \begin{bmatrix} 0 & -c_1 & 0 \\ 1 & 0 & -3 \\ 0 & 3c_1 & 0 \end{bmatrix}.
\]

Letting $t_{1,0} \equiv 0$ and $t_{2,0} \equiv 1$, the matrix $x$ becomes

\[
v_2 = \begin{bmatrix} 0 & 2c_1 - 1 & 1 \\ 0 & -c_1 & 3(c_1 + 2 + c_4) \\ 1 & 3(2c_1 + 1) & -3 \end{bmatrix}.
\]

We see that neither $v_2$ nor $3v_2$ is contained in $W_2$ and that $\partial^{-1}W_2 =< \partial^{-1}W_0, v_1, v_2 >$. We know that $v_2 \in \partial^{-1}W_2$ and since $3v_2 \notin W_2$ then
$v_2 + W_2$ is an element of the group $\partial^{-1}W_2/W_2$ whose order is larger than 3. Recall that $|\partial^{-1}W_2/W_2| = 3^4$ and that $\partial^{-1}W_1/W_2$ has rank 3. Since $\partial^{-1}W_2$ has an element of order larger than 3 then $\text{rank}(\partial^{-1}W_2/W_2)$ is not greater than 3 since $\partial^{-1}W_2/W_2 \cong \mathbb{Z}_9 \times \mathbb{Z}_3 \times \mathbb{Z}_3$ and therefore $W_2$ is terminal and $W_2 \notin \mathcal{L}_2$.

7.2 Case 4.2

Now consider the case $c_2 \neq 2$. Therefore $c_2 = 1$.

The matrices $m_1$ and $m_2$ become

$$m_1 = \begin{bmatrix} c_1 & 0 & -3c_1 \\ 0 & 3 & 0 \\ -3c_1 & 0 & 0 \end{bmatrix}, \quad m_2 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ -3 & 0 & 0 \end{bmatrix}.$$ 

The congruence (A1) becomes $-c_1s_{1,2} \equiv s_{1,2}$. Substituting for $s_{2,1}, s_{1,2}$ using (B1) and (B2) gives us a congruence that holds automatically. Therefore (A1) is redundant.

Hence $x \in \partial^{-1}W_2$ if and only if

$$s_{1,2} \equiv c_1t_{0,1}$$

$$s_{2,1} \equiv -t_{0,1}$$

We regard $t_{0,1}$ and $t_{1,0}$ as the free variables. Taking $t_{0,1} \equiv 1$ and $t_{1,0} = 0$, the matrix $x$ becomes

$$v_1 = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 3c_1 \\ 0 & -3 & 0 \end{bmatrix}.$$
Taking \( t_{1,0} \equiv 1 \) and \( t_{0,1} = 0 \), the matrix \( x \) becomes

\[
v_2 = \begin{bmatrix}
0 & 0 & 0 \\
1 & 0 & 0 \\
0 & 0 & 0
\end{bmatrix}.
\]

We see that \( \partial^{-1}W_1 = < \partial^{-1}W_0, v_1, v_2 > \). Since \( v_1 \notin \partial^{-1}W_0, v_2 \notin \partial^{-1}W_0, v_1 >, 3v_1 \in \partial^{-1}W_0, \) and \( 3v_2 \in < \partial^{-1}W_0, v_1 > \) we know \( |\partial^{-1}W_1/\partial^{-1}W_0| = 3^2 \). Since \( |\partial^{-1}W_0| = 3^7 \) then \( |\partial^{-1}W_1| = 3^9 \). Because \( |W_1| = 3^5 \) then \( |\partial^{-1}W_1/W_1| = 3^4 \).

Also, \( v_1, v_2, y_1, y_3 \in \partial^{-1}W_1 \) but are not contained in \( W_1 \). Each of \( 3v_1, 3v_2, 3y_1, 3y_3 \) is contained in \( W_1 \). So \( v_1 + W_1, v_2 + W_1, y_1 + W_1, y_3 + W_1 \) are elements of order 3 in the group \( \partial^{-1}W_1/W_1 \). These four elements form a generating set for the group \( \partial^{-1}W_1/W_1 \). Recall that \( |\partial^{-1}W_1/W_1| = 3^4 \), we obtain \( \partial^{-1}W_1/W_1 \cong \mathbb{Z}_3 \times \mathbb{Z}_3 \times \mathbb{Z}_3 \times \mathbb{Z}_3 \).

Since \( \text{rank}(\partial^{-1}W_1/W_1) = 4 \) and \( \text{rank}(\partial^{-1}W_0/W_1) = 2 \), then \( \text{rank}(\partial^{-1}W_0/W_1) < \text{rank}(\partial^{-1}W_1/W_1) \). Hence \( W_1 \) is nonterminal and \( W_1 \in \hat{\mathcal{L}}_1 \). A basis for \( \partial^{-1}W_1/W_1 \) is \( v_1 + W_1, v_2 + W_1, y_1 + W_1, y_3 + W_1 \). A basis for \( \partial^{-1}W_0/W_1 \) is \( y_1 + W_1, y_3 + W_1 \).

The subgroups \( W_2 \) belonging to \( \mathcal{L}(W_1) \) correspond to the nontrivial proper subspace \( W_2/W_1 \) of \( \partial^{-1}W_1/W_1 \) for which the intersection \( W_2/W_1 \cap \partial^{-1}W_0/W_1 \) is trivial. Since \( \partial^{-1}W_1/W_1 \) has dimension 4 while its subspace \( \partial^{-1}W_0/W_1 \) has dimension 2, every such subspace \( W_2/W_1 \) has dimension either 1 or 2.

To help us define the subgroups \( W_2 \) belonging to \( \mathcal{L}_2(W_1) \), it will be convenient to identify each element of the vector space \( \partial^{-1}W_1/W_1 \) with its coordinate vector with respect to the ordered basis \( y_1 + W_1, y_3 + W_1, v_1 + W_1, v_2 + W_1 \). In this way, we identify
\(\partial^{-1}W_1/W_1\) with the vector space \(Z_3 \times Z_3 \times Z_3 \times Z_3\) consisting of row vectors. Under this identification, the elements \(y_1 + W_1, y_3 + W_1, v_1 + W_1, v_2 + W_1\) are associated with the so-called standard basis vectors \([1, 0, 0, 0], [0, 1, 0, 0], [0, 0, 1, 0], [0, 0, 0, 1]\) in \(Z_3 \times Z_3 \times Z_3 \times Z_3\).

The subgroups \(W_2\) belonging to \(\mathcal{L}(W_1)\) are in one-to-one correspondence with the nontrivial proper subgroup \(W_2/W_1\) of \(\partial^{-1}W_1/W_1\) for which the intersection \(W_2/W_1 \cap \partial^{-1}W_0/W_1\) is trivial. Note that \(\partial^{-1}W_0W_1\) is the 2-dimensional subspace generated by the pair of elements \(y_1 + W_1\) and \(y_3 + W_1\). Under our identification, each such subspace \(W_2/W_1\) is associated with a subspace \(S\) of \(Z_3 \times Z_3 \times Z_3 \times Z_3\) that contains no nonzero vector that is a linear combination of \([1, 0, 0, 0]\) and \([0, 1, 0, 0]\).

Let \(m\) denote the unique matrix in reduced row-echelon form that is row equivalent to the matrix whose rows are the members of an arbitrarily-chosen basis of such a subspace \(S\).

In Case 4.2.1 we consider the 1-dimensional subspaces \(W_2/W_1\). There are four possible forms for the matrix \(m\). The first form is

\[
m = [0, 0, 0, 1]
\]

(1 possibility), which is considered in Case 4.2.1.1. The second form is

\[
m = [0, 0, 1, c_3]\quad \text{for } c_3 \in \{0, 1, 2\}
\]

(3 possibilities), which is considered in Case 4.2.1.2. The third form is

\[
m = [0, 1, c_3, c_4]\quad \text{for } c_3, c_4 \in \{0, 1, 2\}, (c_3, c_4) \neq (0, 0)
\]

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(8 possibilities), which is considered in Case 4.2.1.3. The fourth form is

\[ m = [1, c_3, c_4, c_5] \quad \text{for } c_3, c_4, c_5 \in \{0, 1, 2\} \ (c_4, c_5) \neq (0, 0) \]

(24 possibilities), which is considered in Case 4.2.1.4.

In Case 4.2.2 we consider the 2-dimensional subspaces \( W_2/W_1 \). There are six possible forms for the matrix \( m \). The first form is

\[
m = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}
\]

(1 possibility), which is considered in Case 4.2.2.1. The second form is

\[
m = \begin{bmatrix} 0 & 1 & c_3 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \quad \text{for } c_3 \in \{1, 2\}
\]

(2 possibilities), which is considered in Case 4.2.2.2. The third form is

\[
m = \begin{bmatrix} 0 & 1 & 0 & c_3 \\ 0 & 0 & 1 & c_4 \end{bmatrix} \quad \text{for } c_3 \in \{1, 2\} \quad \text{and } c_4 \in \{0, 1, 2\}
\]

(6 possibilities), which is considered in Case 4.2.2.3. The fourth form is

\[
m = \begin{bmatrix} 1 & c_3 & c_4 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \quad \text{for } c_3 \in \{0, 1, 2\} \quad \text{and } c_4 \in \{1, 2\}
\]

(6 possibilities), which is considered in Case 4.2.2.4. The fifth form is

\[
m = \begin{bmatrix} 1 & c_3 & 0 & c_4 \\ 0 & 0 & 1 & c_5 \end{bmatrix} \quad \text{for } c_3, c_5 \in \{0, 1, 2\} \quad \text{and } c_4 \in \{1, 2\}
\]
(18 possibilities), which is considered in Case 4.2.2.5. The sixth form is

\[
m = \begin{bmatrix} 1 & 0 & c_3 & c_4 \\ 0 & 1 & c_5 & c_6 \end{bmatrix}
\]

where \{(c_3, c_4), (c_5, c_6)\} is an ordered pair of linearly independent vectors in \(\mathbb{Z}_3 \times \mathbb{Z}_3\) (48 possibilities) which is considered in Case 4.2.2.6.

7.2.1 Case 4.2.1

We consider the 1-dimensional subspaces \(W_2/W_1\). Let \(d_1, d_2, d_3, d_4\) be unspecified variables. Let \(m_3 = d_1 y_1 + d_2 y_3 + d_3 v_1 + d_4 v_2\). A formal expression for \(m_3\) is

\[
m_3 = d_1 \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 3 & 0 & 0 \end{bmatrix} + d_2 \begin{bmatrix} 0 & 0 & 3 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} + d_3 \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 3 c_1 \end{bmatrix} + d_4 \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}
\]

\[
= \begin{bmatrix} 0 & d_3 & 3d_2 \\ d_4 & 0 & 3c_1 d_3 \\ 3d_1 & -3d_3 & 0 \end{bmatrix}.
\]

Let \(W_2 = \langle W_1, m_3 \rangle \in \mathcal{L}_2\). We now calculate the pullback \(\partial^{-1}W_2\). The subgroup \(W_2\) is contained in the pattern subgroup

\[
\begin{bmatrix} 2 & 2 & 1 \\ 2 & 1 & 1 \\ 1 & 1 & 0 \end{bmatrix}.
\]
Thus the pullback $\partial^{-1}W_2$ is contained in the pattern subgroup

$$\begin{bmatrix} 2 & 2 & 2 \\ 2 & 2 & 1 \\ 2 & 1 & 1 \end{bmatrix} \cap \begin{bmatrix} 2 & 2 & 2 \\ 2 & 2 & 1 \\ 2 & 1 & 1 \end{bmatrix} = \begin{bmatrix} 2 & 2 & 2 \\ 2 & 2 & 1 \\ 2 & 1 & 1 \end{bmatrix} .$$

Let

$$x = \begin{bmatrix} 3s_{0,0} + t_{0,0} & 3s_{0,1} + t_{0,1} & 3s_{0,2} + t_{0,2} \\ 3s_{1,0} + t_{1,0} & 3s_{1,1} + t_{1,1} & 3s_{1,2} \\ 3s_{2,0} + t_{2,0} & 3s_{2,1} & 3s_{2,2} \end{bmatrix} \in \begin{bmatrix} 2 & 2 & 2 \\ 2 & 2 & 1 \\ 2 & 1 & 1 \end{bmatrix} .$$

Thus

$$\partial_1 x = \begin{bmatrix} 3s_{0,0} + t_{0,0} & 3s_{0,1} + t_{0,1} & 3s_{0,2} + t_{0,2} & 3s_{1,0} + t_{1,0} & 3s_{1,1} + t_{1,1} & 3s_{1,2} \\ 3s_{2,0} + t_{2,0} & 3s_{2,1} & 3s_{2,2} & -3(t_{1,0} + t_{2,0}) & -3t_{1,1} & 0 \end{bmatrix}$$

and

$$\partial_2 x = \begin{bmatrix} 3s_{0,0} + t_{0,0} & 3s_{0,1} + t_{0,1} & 3s_{0,2} + t_{0,2} & -3(t_{0,1} + t_{0,2}) \\ 3s_{1,1} + t_{1,1} & 3s_{1,2} & -3t_{1,1} \\ 3s_{2,1} & 3s_{2,2} & 0 \end{bmatrix} .$$

The variables that are in play are those appearing in the matrix

$$\begin{bmatrix} 0 & t_{0,1} & t_{0,2} \\ t_{1,0} & t_{1,1} & 3s_{1,2} \\ t_{2,0} & 3s_{2,1} & 3s_{2,2} \end{bmatrix} .$$
We wish to identify values \(a_1, a_2, a_3 \in \mathbb{Z}_9\) such that \(\partial_1 x \equiv a_1 m_1 + a_2 m_2 + a_3 m_3 \pmod{I}\). A formal expression for \(a_1 m_1 + a_2 m_2 + a_3 m_3\) is

\[
\begin{bmatrix}
    c_1 & 0 & -3c_1 \\
    0 & 3 & 0 \\
    -3c_1 & 0 & 0
\end{bmatrix}
+ a_2
\begin{bmatrix}
    1 & 0 & 0 \\
    0 & 0 & 0 \\
    -3 & 0 & 0
\end{bmatrix}
+ a_3
\begin{bmatrix}
    0 & d_3 & 3d_2 \\
    d_4 & 0 & 3c_1 d_3 \\
    3d_1 & -3d_3 & 0
\end{bmatrix}
\]

Comparing \((0, 1)\)-entries, we get \(t_{1, 1} \equiv d_3 a_3\).

Comparing \((2, 1)\)-entries, we get \(t_{1, 1} \equiv d_3 a_3\) which gives no new information.

Comparing \((1, 0)\)-entries, we get \(t_{2, 0} \equiv d_4 a_3\).

Comparing \((1, 1)\)-entries, we get \(a_1 \equiv s_{2, 1}\).

Comparing \((0, 0)\)-entries, we get \(c_1 a_1 + a_2 \equiv t_{1, 0}\). Substituting \(a_1\) we obtain \(a_2 \equiv t_{1, 0} - c_1 s_{2, 1}\).

Comparing \((1, 2)\)-entries, we get \(d_3 a_3 c_1 \equiv s_{2, 2}\). Substituting \(d_3 a_3\) we obtain \(c_1 t_{1, 1} \equiv s_{2, 2}\).

Comparing \((0, 2)\)-entries, we get \(d_2 a_3 - a_1 c_1 \equiv s_{1, 2}\). Substituting \(a_1\) we obtain \(s_{1, 2} \equiv a_3 d_2 - c_1 s_{2, 1}\).

Comparing \((2, 0)\)-entries, we get \(t_{1, 0} + t_{2, 0} \equiv c_1 a_1 + a_2 - d_1 a_3\). Substituting \(a_1\) and \(a_2\) we obtain \(t_{2, 0} \equiv -d_1 a_3\).
We see that $\partial_1x \in W_2$ if and only if

$$t_{1,1} \equiv d_3a_3 \quad (A1)$$

$$t_{2,0} \equiv d_4a_3 \quad (A2)$$

$$s_{2,2} \equiv c_1t_{1,1} \quad (A3)$$

$$s_{1,2} \equiv d_2a_3 - c_1s_{2,1} \quad (A4)$$

$$t_{2,0} \equiv -d_1a_3 \quad (A5).$$

We wish to identify values $b_1, b_2, b_3 \in \mathbb{Z}_9$ such that $\partial_2x \equiv b_1m_1 + b_2m_2 + b_3m_3 (\text{mod } I)$. A formal expression for $b_1m_1 + b_2m_2 + b_3m_3$ is

$$b_1 \begin{bmatrix} c_1 & 0 & -3c_1 \\ 0 & 3 & 0 \\ -3c_1 & 0 & 0 \end{bmatrix} + b_2 \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ -3 & 0 & 0 \end{bmatrix} + b_3 \begin{bmatrix} 0 & d_3 & 3d_2 \\ d_4 & 0 & 3c_1d_3 \\ 3d_1 & -3d_3 & 0 \end{bmatrix} = \begin{bmatrix} c_1b_1 + b_2 & d_3b_3 & 3(d_2b_3 - c_1b_1) \\ d_4b_3 & 3b_1 & 3c_1d_3b_3 \\ 3(d_1b_3 - c_1b_1 - b_2) & -3d_3b_3 & 0 \end{bmatrix}.$$

Comparing $(0,1)$-entries, we get $d_3b_3 \equiv t_{0,2}$.

Comparing $(2,1)$-entries, we get $-d_3b_3 \equiv s_{2,2}$. Since $d_3b_3 \equiv t_{0,2}$ then we obtain $t_{0,2} \equiv -s_{2,2}$.

Comparing $(1,0)$-entries, we get $d_4b_3 \equiv t_{1,1}$.

Comparing $(1,1)$-entries, we get $b_1 \equiv s_{1,2}$. 

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Comparing (0,0)-entries, we get $c_1 b_1 + b_2 \equiv t_{0,1}$. Substituting $b_1$ we obtain $b_2 \equiv t_{0,1} - c_1 s_{1,2}$.

Comparing (1,2)-entries, we get $d_3 b_3 c_1 \equiv -t_{1,1}$. Substituting $d_3 b_3$ we obtain $t_{0,2} \equiv -c_1 t_{1,1}$.

Comparing (0,2)-entries, we get $t_{0,1} + t_{0,2} \equiv c_1 b_1 - d_2 b_3$. Substituting $b_1$ we obtain $t_{0,1} \equiv c_1 s_{1,2} - d_2 b_3 - t_{0,2}$. Since $c_1^2 \equiv 1$ this becomes $s_{1,2} \equiv c_1 t_{0,1} + c_1 t_{0,2} + c_1 d_2 b_3$.

Comparing (2,0)-entries, we get $d_1 b_3 - c_1 b_1 - b_2 \equiv s_{2,1}$. Substituting $b_1$ and $b_2$ we obtain $d_1 b_3 - t_{0,1} \equiv s_{2,1}$.

We see that $\partial_2 x \in W_2$ if and only if

$$d_3 b_3 \equiv t_{0,2} \quad (B1)$$
$$t_{0,2} \equiv -s_{2,2} \quad (B2)$$
$$d_4 b_3 \equiv t_{1,1} \quad (B3)$$
$$t_{0,2} \equiv -c_1 t_{1,1} \quad (B4)$$
$$s_{1,2} \equiv c_1 t_{0,1} + c_1 t_{0,2} + c_1 d_2 b_3 \quad (B5)$$
$$d_1 b_3 - t_{0,1} \equiv s_{2,1} \quad (B6)$$

Case 4.2.1.1

Let $m_3 = v_2$. Thus

$$m_3 = \begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$
Let $W_2 = < W_1, m_3 > \in L_2$. The number of subgroups $W_2$ of this type is 1. Since $m_3 \notin W_1$ and $3m_1 \in W_1$ we have $|W_2/W_1| = 3$. Since $|W_1| = 3^5$ it follows that $|W_2| = 3^6$. Note that $\partial^{-1}W_1/W_2 \cong \mathbb{Z}_3 \times \mathbb{Z}_3 \times \mathbb{Z}_3$ and rank($\partial^{-1}W_1/W_2$) = 3. We now calculate the pullback of $\partial^{-1}W_2$. We observed in Case 4.2.1 that $\partial^{-1}W_2$ is contained in the pattern subgroup

$$\begin{bmatrix}
2 & 2 & 2 \\
2 & 2 & 1 \\
2 & 1 & 1
\end{bmatrix}.$$

Let

$$x = \begin{bmatrix}
3s_{0,0} + t_{0,0} & 3s_{0,1} + t_{0,1} & 3s_{0,2} + t_{0,2} \\
3s_{1,0} + t_{1,0} & 3s_{1,1} + t_{1,1} & 3s_{1,2} \\
3s_{2,0} + t_{2,0} & 3s_{2,1} & 3s_{2,2}
\end{bmatrix} \in \begin{bmatrix}
2 & 2 & 2 \\
2 & 2 & 1 \\
2 & 1 & 1
\end{bmatrix}.$$

The variables that are in play are those appearing in the matrix

$$\begin{bmatrix}
0 & t_{0,1} & t_{0,2} \\
t_{1,0} & t_{1,1} & 3s_{1,2} \\
t_{2,0} & 3s_{2,1} & 3s_{2,2}
\end{bmatrix}.$$

In the notation of Case 4.2.1, we are taking $d_1 = 0$, $d_2 = 0$, $d_3 = 0$, and $d_4 = 1$. We wish to identify values $a_1, a_2, a_3 \in \mathbb{Z}_9$ such that $\partial_1 x \equiv a_1 m_1 + a_2 m_2 + a_3 m_3 \pmod{I}$.
We see that \( \partial_1 x \in W_2 \) if and only if

\[
\begin{align*}
t_{1,1} & \equiv 0 \quad \text{(A1)} \\
t_{2,0} & \equiv a_3 \quad \text{(A2)} \\
s_{2,2} & \equiv c_1 t_{1,1} \quad \text{(A3)} \\
s_{1,2} & \equiv -c_1 s_{2,1} \quad \text{(A4)} \\
t_{2,0} & \equiv 0 \quad \text{(A5)}.
\end{align*}
\]

We wish to identify values \( b_1, b_2, b_3 \in \mathbb{Z}_9 \) such that \( \partial_1 x \equiv b_1 m_1 + b_2 m_2 + b_3 m_3 \pmod{I} \). We see that \( \partial_2 x \in W_2 \) if and only if

\[
\begin{align*}
0 & \equiv t_{0,2} \quad \text{(B1)} \\
0 & \equiv s_{2,2} \quad \text{(B2)} \\
b_3 & \equiv t_{1,1} \quad \text{(B3)} \\
t_{0,2} & \equiv -c_1 t_{1,1} \quad \text{(B4)} \\
s_{1,2} & \equiv c_1 t_{0,1} + c_1 t_{0,2} \quad \text{(B5)} \\
s_{2,1} & \equiv -t_{0,1} \quad \text{(B6)}.
\end{align*}
\]

Substituting (A1) into congruences (A3), (B3), and (B4) we obtain \( s_{2,2} \equiv 0, b_3 \equiv 0, \) and \( t_{0,2} \equiv 0 \) respectively. Substituting (A5) into (A2) we obtain \( a_3 \equiv 0 \). Substituting (B6), congruence (A4) becomes \( s_{1,2} \equiv c_1 t_{0,1} \). Also, substituting (B4) into (B5) we obtain \( s_{1,2} \equiv c_1 t_{0,1} \) which is redundant with (A4).
Hence $x \in \partial^{-1}W_2$ if and only if

$$
t_{1,1} \equiv 0
$$

$$
t_{2,0} \equiv 0
$$

$$
t_{0,2} \equiv 0
$$

$$
s_{2,1} \equiv 0
$$

$$
s_{1,2} \equiv c_1t_{0,1}
$$

$$
s_{2,1} \equiv -t_{0,1}.
$$

These are the same congruences as those in Case 4.2 therefore we regard $t_{0,1}$ and $t_{1,0}$ as the free variables. Taking $t_{0,1} \equiv 1$ and $t_{1,0} = 0$, the matrix $x$ becomes

$$
v_1 = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 3c_1 \\ 0 & -3 & 0 \end{bmatrix}.
$$

Taking $t_{1,0} \equiv 1$ and $t_{0,1} = 0$, the matrix $x$ becomes

$$
v_2 = \begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}.
$$

So $\partial^{-1}W_2 = \partial^{-1}W_1$. Thus $\text{rank}(\partial^{-1}W_2/W_2) = \text{rank}(\partial^{-1}W_1/W_2)$, $W_2$ is terminal, and $W_2 \notin \hat{L}_2$. 

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Case 4.2.1.2

We fix arbitrary value $c_3 \in \{0, 1, 2\}$. There are 3 ways to choose the value $c_3$. Let $m_3 = v_1 + c_3 v_2$. Thus

$$m_3 = \begin{bmatrix} 0 & 1 & 0 & \cr c_3 & 0 & 3c_1 & \cr 0 & -3 & 0 & \end{bmatrix}.$$ 

Let $W_2 = < W_1, m_3 > \in \mathcal{L}_2$. The number of subgroups $W_2$ of this type is 3. Since $m_3 \notin W_1$ and $3m_1 \in W_1$ we have $|W_2/W_1| = 3$. Since $|W_1| = 3^5$ it follows that $|W_2| = 3^6$. Note that $\partial^{-1}W_1/W_2 \cong \mathbb{Z}_3 \times \mathbb{Z}_3 \times \mathbb{Z}_3$ and $\text{rank}(\partial^{-1}W_1/W_2) = 3$. We now calculate the pullback of $\partial^{-1}W_2$. We observed in case 4.2.1 that $\partial^{-1}W_2$ is contained in the pattern subgroup

$$\begin{bmatrix} 2 & 2 & 2 \\ 2 & 2 & 1 \\ 2 & 1 & 1 \end{bmatrix}.$$ 

Let

$$x = \begin{bmatrix} 3s_{0,0} + t_{0,0} & 3s_{0,1} + t_{0,1} & 3s_{0,2} + t_{0,2} \\ 3s_{1,0} + t_{1,0} & 3s_{1,1} + t_{1,1} & 3s_{1,2} \\ 3s_{2,0} + t_{2,0} & 3s_{2,1} & 3s_{2,2} \end{bmatrix} \in \begin{bmatrix} 2 & 2 & 2 \\ 2 & 2 & 1 \\ 2 & 1 & 1 \end{bmatrix}.$$ 

The variables that are in play are those appearing in the matrix

$$\begin{bmatrix} 0 & t_{0,1} & t_{0,2} \\ t_{1,0} & t_{1,1} & 3s_{1,2} \\ t_{2,0} & 3s_{2,1} & 3s_{2,2} \end{bmatrix}.$$ 

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In the notation of Case 4.2.1, we are taking \( d_1 = 0, d_2 = 0, d_3 = 1 \), and 
\( d_4 = c_3 \). We wish to identify values \( a_1, a_2, a_3 \in \mathbb{Z}_9 \) such that 
\( \partial_1 x \equiv a_1 m_1 + a_2 m_2 + a_3 m_3 \) \((\text{mod} \ I)\).

We see that \( \partial_1 x \in W_2 \) if and only if

\[
\begin{align*}
    t_{1,1} &\equiv a_3 \quad \text{(A1)} \\
    t_{2,0} &\equiv c_3 a_3 \quad \text{(A2)} \\
    s_{2,2} &\equiv c_1 t_{1,1} \quad \text{(A3)} \\
    s_{1,2} &\equiv -c_1 s_{2,1} \quad \text{(A4)} \\
    t_{2,0} &\equiv 0 \quad \text{(A5)}.
\end{align*}
\]

We wish to identify values \( b_1, b_2, b_3 \in \mathbb{Z}_9 \) such that 
\( \partial_2 x \equiv b_1 m_1 + b_2 m_2 + b_3 m_3 \) \((\text{mod} \ I)\).

We see that \( \partial_2 x \in W_2 \) if and only if

\[
\begin{align*}
    b_3 &\equiv t_{0,2} \quad \text{(B1)} \\
    t_{0,2} &\equiv -s_{2,2} \quad \text{(B2)} \\
    c_3 b_3 &\equiv t_{1,1} \quad \text{(B3)} \\
    t_{0,2} &\equiv -c_1 t_{1,1} \quad \text{(B4)} \\
    s_{1,2} &\equiv c_1 t_{0,1} + c_1 t_{0,2} \quad \text{(B5)} \\
    s_{2,1} &\equiv -t_{0,1} \quad \text{(B6)}.
\end{align*}
\]

Substituting (A1) into (A2) we obtain \( t_{2,0} \equiv c_3 t_{1,1} \). Since \( t_{2,0} \equiv 0 \) from (A5) 
then (A2) becomes \( c_3 t_{1,1} \equiv 0 \). Combining (B1) and (B3), (B3) becomes \( c_3 t_{0,2} \equiv t_{1,1} \).
Hence \( x \in \partial^{-1}W_2 \) if and only if

\[
\begin{align*}
c_3t_{1,1} &\equiv 0 \quad (A2) \\
s_{2,2} &\equiv c_1t_{1,1} \quad (A3) \\
s_{1,2} &\equiv -c_1s_{2,1} \quad (A4) \\
t_{2,0} &\equiv 0 \quad (A5) \\
t_{0,2} &\equiv -s_{2,2} \quad (B2) \\
c_3t_{0,2} &\equiv t_{1,1} \quad (B3) \\
t_{0,2} &\equiv -c_1t_{1,1} \quad (B4) \\
s_{1,2} &\equiv c_1t_{0,1} + c_1t_{0,2} \quad (B5) \\
s_{2,1} &\equiv -t_{0,1} \quad (B6).
\end{align*}
\]

It is convenient to consider the cases \( c_3 \neq 0 \) and \( c_3 = 0 \) separately.

**Case 4.2.1.2.1** First we consider the case \( c_3 \neq 0 \). Then (A2) implies that \( t_{1,1} \equiv 0 \).

Therefore, \( t_{1,1} \equiv s_{2,2} \equiv t_{2,0} \equiv t_{0,2} \equiv 0 \). Substituting (B6) into (A4) we obtain \( s_{1,2} \equiv c_1t_{0,1} \) and (B5) becomes redundant.
Hence \( x \in \partial^{-1}W_2 \) if and only if

\[
\begin{align*}
    t_{1,1} &\equiv 0 \\
    t_{2,0} &\equiv 0 \\
    t_{0,2} &\equiv 0 \\
    s_{2,2} &\equiv 0 \\
    s_{2,1} &\equiv -t_{0,1} \\
    s_{1,2} &\equiv c_1t_{0,1}.
\end{align*}
\]

These are the same congruences as those in Case 4.2 and therefore we conclude that \( \partial^{-1}W_2 = \partial^{-1}W_1 \). Thus \( \text{rank}(\partial^{-1}W_2/W_2) = \text{rank}(\partial^{-1}W_1/W_2) \), \( W_2 \) is terminal, and \( W_2 \not\in \hat{L}_2 \).

**Case 4.2.1.2.2** Now we consider the case \( c_3 = 0 \). Then the congruence (A2) holds automatically. Congruence (B3) becomes \( t_{1,1} \equiv 0 \). Therefore, as shown in Case 4.2.1.2.1 above, we find that \( \partial^{-1}W_2 = \partial^{-1}W_1 \). Thus \( \text{rank}(\partial^{-1}W_2/W_2) = \text{rank}(\partial^{-1}W_1/W_2) \), \( W_2 \) is terminal, and \( W_2 \not\in \hat{L}_2 \).
Case 4.2.1.3

We fix arbitrary values $c_3, c_4 \in \{0, 1, 2\}, (c_3, c_4) \neq (0, 0)$. There are 8 ways to choose the values $c_3, c_4$. Let $m_3 = y_1 + c_3 v_1 + c_4 v_2$. Thus

$$m_3 = \begin{bmatrix}
0 & c_3 & 3 \\
-3c_3 & 0 & 3c_1 c_3 \\
0 & c_4 & 0 \\
\end{bmatrix}.$$ 

Let $W_2 = \langle W_1, m_3 \rangle \in \mathcal{L}_2$. The number of subgroups $W_2$ of this type is 8. Since $m_3 \not\in W_1$ and $3m_1 \in W_1$ we have $|W_2/W_1| = 3$. Since $|W_1| = 3^5$ it follows that $|W_2| = 3^6$. Note that $\partial^{-1} W_1/W_2 \cong \mathbb{Z}_3 \times \mathbb{Z}_3 \times \mathbb{Z}_3$ and $\text{rank}(\partial^{-1} W_1/W_2) = 3$. We now calculate the pullback of $\partial^{-1} W_2$. We observed in case 4.2.1 that $\partial^{-1} W_2$ is contained in the pattern subgroup

$$\begin{bmatrix}
2 & 2 & 2 \\
2 & 1 & 1 \\
2 & 1 & 1
\end{bmatrix}.$$ 

Let

$$x = \begin{bmatrix}
3s_{0,0} + t_{0,0} & 3s_{0,1} + t_{0,1} & 3s_{0,2} + t_{0,2} \\
3s_{1,0} + t_{1,0} & 3s_{1,1} + t_{1,1} & 3s_{1,2} \\
3s_{2,0} + t_{2,0} & 3s_{2,1} & 3s_{2,2}
\end{bmatrix} \in \begin{bmatrix}
2 & 2 & 2 \\
2 & 2 & 1 \\
2 & 1 & 1
\end{bmatrix}.$$ 

The variables that are in play are those appearing in the matrix

$$\begin{bmatrix}
0 & t_{0,1} & t_{0,2} \\
t_{1,0} & t_{1,1} & 3s_{1,2} \\
t_{2,0} & 3s_{2,1} & 3s_{2,2}
\end{bmatrix}.$$
In the notation of Case 4.2.1, we are taking $d_1 = 0$, $d_2 = 1$, $d_3 = c_3$, and $d_4 = c_4$. We wish to identify values $a_1, a_2, a_3 \in \mathbb{Z}_9$ such that $\partial_1 x \equiv a_1 m_1 + a_2 m_2 + a_3 m_3 \pmod{I}$.

We see that $\partial_1 x \in W_2$ if and only if

\begin{align*}
t_{1,1} &\equiv c_3 a_3 \quad \text{(A1)} \\
t_{2,0} &\equiv c_4 a_3 \quad \text{(A2)} \\
s_{2,2} &\equiv c_1 t_{1,1} \quad \text{(A3)} \\
s_{1,2} &\equiv a_3 - c_1 s_{2,1} \quad \text{(A4)} \\
t_{2,0} &\equiv 0 \quad \text{(A5)}. \end{align*}

We wish to identify values $b_1, b_2, b_3 \in \mathbb{Z}_9$ such that $\partial_2 x \equiv b_1 m_1 + b_2 m_2 + b_3 m_3 \pmod{I}$.

We see that $\partial_2 x \in W_2$ if and only if

\begin{align*}
c_3 b_3 &\equiv t_{0,2} \quad \text{(B1)} \\
t_{0,2} &\equiv -s_{2,2} \quad \text{(B2)} \\
c_4 b_3 &\equiv t_{1,1} \quad \text{(B3)} \\
t_{0,2} &\equiv -c_1 t_{1,1} \quad \text{(B4)} \\
s_{1,2} &\equiv c_1 t_{0,1} + c_1 t_{0,2} + c_1 b_3 \quad \text{(B5)} \\
s_{2,1} &\equiv -t_{0,1} \quad \text{(B6)}. \end{align*}

Substituting (A5) into (A2) we obtain $c_4 a_3 \equiv 0$. 

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Hence $x \in \partial^{-1}W_2$ if and only if

\[ t_{1,1} \equiv c_3a_3 \quad \text{(A1)} \]
\[ c_4a_3 \equiv 0 \quad \text{(A2)} \]
\[ s_{2,2} \equiv c_1t_{1,1} \quad \text{(A3)} \]
\[ s_{1,2} \equiv a_3 - c_1s_{2,1} \quad \text{(A4)} \]
\[ t_{2,0} \equiv 0 \quad \text{(A5)} \]
\[ c_3b_3 \equiv t_{0,2} \quad \text{(B1)} \]
\[ t_{0,2} \equiv -s_{2,2} \quad \text{(B2)} \]
\[ c_4b_3 \equiv t_{1,1} \quad \text{(B3)} \]
\[ t_{0,2} \equiv -c_1t_{1,1} \quad \text{(B4)} \]
\[ s_{1,2} \equiv c_1t_{0,1} + c_1t_{0,2} + c_1b_3 \quad \text{(B5)} \]
\[ s_{2,1} \equiv -t_{0,1} \quad \text{(B6)}. \]

It is convenient to consider the cases $c_4 = 0$ and $c_4 \neq 0$ separately.

**Case 4.2.1.3.1** We consider the case $c_4 = 0$. Then $c_3 \neq 0$ and (A2) automatically holds. The congruences (B3) and (A1) become $t_{1,1} \equiv 0$ and $a_3 \equiv 0$ respectively. Using $t_{1,1} \equiv 0$, the congruences (A3) and (B2) become $s_{2,2} \equiv 0$ and $t_{0,2} \equiv 0$ respectively and (B4) holds. Using $t_{0,2} \equiv 0$, (B1) becomes $b_3 \equiv 0$. Using $a_3 \equiv 0$ (A4) becomes $s_{1,2} \equiv -c_1s_{2,1}$. Substituting $b_3 \equiv 0$ and $t_{0,2}$ into (B5) we obtain $s_{1,2} \equiv c_1t_{0,1}$. Substituting (B6) into the new (A4) we obtain $s_{1,2} \equiv c_1t_{0,1}$ and therefore (B5) is redundant.
Hence \( x \in \partial^{-1}W_2 \) if and only if

\[
\begin{align*}
t_{1,1} & \equiv 0 \\
t_{2,0} & \equiv 0 \\
t_{0,2} & \equiv 0 \\
s_{2,2} & \equiv 0 \\
s_{2,1} & \equiv -t_{0,1} \\
s_{1,2} & \equiv c_1 t_{0,1}.
\end{align*}
\]

These are the same congruences as those in Case 4.2 and therefore we conclude that \( \partial^{-1}W_2 = \partial^{-1}W_1 \). Thus \( \text{rank}(\partial^{-1}W_2/W_2) = \text{rank}(\partial^{-1}W_1/W_2) \), \( W_2 \) is terminal, and \( W_2 \not\in \hat{L}_2 \).

**Case 4.2.1.3.2** Now we consider the case \( c_4 \neq 0 \). Then from (A2) we obtain \( a_3 = 0 \). Substituting \( a_3 = 0 \) into (A1) we obtain \( t_{1,1} = 0 \). Using \( t_{1,1} \equiv 0 \) and \( a_3 = 0 \) we obtain \( s_{2,2} \equiv t_{2,0} \equiv t_{0,2} \equiv b_3 \equiv 0 \). Congruences (A4) and (B5) become \( s_{1,2} \equiv -c_1 s_{2,1} \) and \( s_{1,2} \equiv c_1 t_{0,1} \) respectively. Substituting (B6) into (A4) we obtain \( s_{1,2} \equiv c_1 t_{0,1} \) and therefore (B5) is redundant.
Hence $x \in \partial^{-1}W_2$ if and only if

\begin{align*}
    t_{1,1} &\equiv 0 \\
    t_{2,0} &\equiv 0 \\
    t_{0,2} &\equiv 0 \\
    s_{2,2} &\equiv 0 \\
    s_{2,1} &\equiv -t_{0,1} \\
    s_{1,2} &\equiv c_1 t_{0,1}.
\end{align*}

These are the same congruences as those in Case 4.2 and therefore we conclude that $\partial^{-1}W_2 = \partial^{-1}W_1$. Thus $\text{rank}(\partial^{-1}W_2/W_2) = \text{rank}(\partial^{-1}W_1/W_2)$, $W_2$ is terminal, and $W_2 \not\in \hat{L}_2$.

Case 4.2.1.4

We fix arbitrary values $c_3, c_4, c_5 \in \{0, 1, 2\}, (c_4, c_5) \neq (0, 0)$. There are 24 ways to choose the values $c_3, c_4, c_5$. Let $m_3 = y_1 + c_3 y_3 + c_4 v_1 + c_5 v_2$. Thus

$$m_3 = \begin{bmatrix}
    0 & c_4 & 3c_3 \\
    c_5 & 0 & 3c_1 c_4 \\
    3 & -3c_4 & 0
\end{bmatrix}.$$

Let $W_2 = \langle W_1, m_3 \rangle \in \mathcal{L}_2$. The number of subgroups $W_2$ of this type is 24. Since $m_3 \not\in W_1$ and $3m_1 \in W_1$ we have $|W_2/W_1| = 3$. Since $|W_1| = 3^5$ it follows that $|W_2| = 3^6$. Note that $\partial^{-1}W_1/W_2 \cong \mathbb{Z}_3 \times \mathbb{Z}_3 \times \mathbb{Z}_3$ and $\text{rank}(\partial^{-1}W_1/W_2) = 3$. We now calculate the pullback of $\partial^{-1}W_2$. We observed in case 4.2.1 that $\partial^{-1}W_2$ is contained
in the pattern subgroup

\[
\begin{bmatrix}
2 & 2 & 2 \\
2 & 2 & 1 \\
2 & 1 & 1
\end{bmatrix}.
\]

Let

\[
x = \begin{bmatrix}
3s_{0,0} + t_{0,0} & 3s_{0,1} + t_{0,1} & 3s_{0,2} + t_{0,2} \\
3s_{1,0} + t_{1,0} & 3s_{1,1} + t_{1,1} & 3s_{1,2} \\
3s_{2,0} + t_{2,0} & 3s_{2,1} & 3s_{2,2}
\end{bmatrix}
\in
\begin{bmatrix}
2 & 2 & 2 \\
2 & 2 & 1 \\
2 & 1 & 1
\end{bmatrix}.
\]

The variables that are in play are those appearing in the matrix

\[
\begin{bmatrix}
0 & t_{0,1} & t_{0,2} \\
t_{1,0} & t_{1,1} & 3s_{1,2} \\
t_{2,0} & 3s_{2,1} & 3s_{2,2}
\end{bmatrix}.
\]

In the notation of Case 4.2.1, we are taking \(d_1 = 1, d_2 = c_3, d_3 = c_4,\) and \(d_4 = c_5.\) We wish to identify values \(a_1, a_2, a_3 \in \mathbb{Z}_9\) such that \(\partial_1 x \equiv a_1 m_1 + a_2 m_2 + a_3 m_3 \pmod{I}\).

We see that \(\partial_1 x \in W_2\) if and only if

\[
\begin{align*}
t_{1,1} &\equiv c_4 a_3 \quad \text{(A1)} \\
t_{2,0} &\equiv c_5 a_3 \quad \text{(A2)} \\
s_{2,2} &\equiv c_1 t_{1,1} \quad \text{(A3)} \\
s_{1,2} &\equiv c_3 a_3 - c_1 s_{2,1} \quad \text{(A4)} \\
a_3 &\equiv -t_{2,0} \quad \text{(A5)}.
\end{align*}
\]
We wish to identify values $b_1, b_2, b_3 \in \mathbb{Z}_9$ such that $\partial_2 x \equiv b_1 m_1 + b_2 m_2 + b_3 m_3 \pmod{I}$.

We see that $\partial_2 x \in W_2$ if and only if

\[ c_4 b_3 \equiv t_{0,2} \quad \text{(B1)} \]
\[ t_{0,2} \equiv -s_{2,2} \quad \text{(B2)} \]
\[ c_5 b_3 \equiv t_{1,1} \quad \text{(B3)} \]
\[ t_{0,2} \equiv -c_1 t_{1,1} \quad \text{(B4)} \]
\[ s_{1,2} \equiv c_1 t_{0,1} + c_1 t_{0,2} + c_1 c_3 b_3 \quad \text{(B5)} \]
\[ b_3 \equiv s_{2,1} + t_{0,1} \quad \text{(B6)}. \]

Substituting $a_3$ and $b_3$ we see that $x \in \partial^{-1} W_2$ if and only if

\[ t_{1,1} \equiv -c_4 t_{2,0} \quad \text{(A1)} \]
\[ (c_5 + 1) t_{2,0} \equiv 0 \quad \text{(A2)} \]
\[ s_{2,2} \equiv c_1 t_{1,1} \quad \text{(A3)} \]
\[ s_{1,2} \equiv -c_3 t_{2,0} - c_1 s_{2,1} \quad \text{(A4)} \]
\[ t_{0,2} \equiv c_4 t_{0,1} + c_4 s_{2,1} \quad \text{(B1)} \]
\[ t_{0,2} \equiv -s_{2,2} \quad \text{(B2)} \]
\[ t_{1,1} \equiv c_5 t_{0,1} + c_5 s_{2,1} \quad \text{(B3)} \]
\[ t_{0,2} \equiv -c_1 t_{1,1} \quad \text{(B4)} \]
\[ s_{1,2} \equiv c_1 (c_3 + 1) t_{0,1} + c_1 t_{0,2} + c_1 c_3 s_{2,1} \quad \text{(B5)}. \]
We can substitute (A1) into (A3) to obtain \( s_{2,2} \equiv -c_1c_4t_{2,0} \) and we can substitute \( s_{2,2} \) into (B2) to obtain \( t_{0,2} \equiv c_1c_4t_{2,0} \). Substituting \( s_{2,2}, t_{1,1}, \) and \( t_{0,2} \) into the congruences (B1), (B3), and (B5) we obtain \( c_1c_4t_{2,0} \equiv c_4t_{0,1} + c_4s_{2,1}, -c_4t_{2,0} \equiv c_5t_{0,1} + c_5s_{2,1}, \) and \( s_{1,2} \equiv c_1(c_3 + 1)t_{0,1} + c_4t_{2,0} + c_1c_3s_{2,1} \), respectively. Substituting \( t_{1,1} \) into (B4) gives \( t_{0,2} \equiv c_1c_4t_{2,0} \) which is redundant with (B2) so we may ignore it.

We see that \( x \in \partial^{-1}W_2 \) if and only if

\[
t_{1,1} \equiv -c_4t_{2,0} \quad \text{(A1)}
\]
\[
(c_5 + 1)t_{2,0} \equiv 0 \quad \text{(A2)}
\]
\[
s_{2,2} \equiv -c_1c_4t_{2,0} \quad \text{(A3)}
\]
\[
s_{1,2} \equiv -c_3t_{2,0} - c_1s_{2,1} \quad \text{(A4)}
\]
\[
c_1c_4t_{2,0} \equiv c_4t_{0,1} + c_4s_{2,1} \quad \text{(B1)}
\]
\[
t_{0,2} \equiv c_1c_4t_{2,0} \quad \text{(B2)}
\]
\[
-c_4t_{2,0} \equiv c_5t_{0,1} + c_5s_{2,1} \quad \text{(B3)}
\]
\[
s_{1,2} \equiv c_1(c_3 + 1)t_{0,1} + c_4t_{2,0} + c_1c_3s_{2,1} \quad \text{(B5)}
\]

It is convenient to consider the cases \( c_5 \neq 2 \) and \( c_5 = 2 \) separately.

**Case 4.2.1.4.1** First we consider the case \( c_5 \neq 2 \). Then by congruence (A2) \( t_{2,0} \equiv 0 \). It follows that \( t_{1,1} \equiv s_{2,2} \equiv t_{0,2} \equiv t_{2,0} \equiv 0 \). Congruences (B1) and (B3) become \( c_4(t_{0,1} + s_{2,1}) \equiv 0 \) and \( c_5(t_{0,1} + s_{2,1}) \equiv 0 \). Since \((c_4, c_5) \neq (0, 0)\) we conclude that \( s_{2,1} \equiv -t_{0,1} \). Substituting \( s_{2,1} \) into (A4) we obtain \( s_{1,2} \equiv c_1t_{0,1} \). Also, substituting \( s_{2,1} \) into (B5) we obtain \( s_{1,2} \equiv c_1t_{0,1} \) which is redundant with (A4).
Hence $x \in \partial^{-1}W_2$ if and only if

\[
\begin{align*}
  t_{1,1} &\equiv 0 \\
  t_{2,0} &\equiv 0 \\
  t_{0,2} &\equiv 0 \\
  s_{2,2} &\equiv 0 \\
  s_{2,1} &\equiv -t_{0,1} \\
  s_{1,2} &\equiv c_1 t_{0,1}.
\end{align*}
\]

These are the same congruences as those in Case 4.2 and therefore we conclude that $\partial^{-1}W_2 = \partial^{-1}W_1$. Thus $\text{rank}(\partial^{-1}W_2/W_2) = \text{rank}(\partial^{-1}W_1/W_2)$, $W_2$ is terminal, and $W_2 \not\in \hat{\mathcal{L}}_2$.

**Case 4.2.1.4.2** Now we consider the case $c_5 = 2$. Congruence (A2) is automatically satisfied and therefore may be ignored. Congruence (B3) becomes $s_{2,1} \equiv c_4 t_{2,0} - t_{0,1}$.

Substituting $s_{2,1}$ into (A4) we obtain $s_{1,2} \equiv (-c_3 - c_1 c_4) t_{2,0} + c_1 t_{0,1}$. Substituting $s_{2,1}$ into (B1) we obtain $c_1 c_4 t_{2,0} \equiv c_4^2 t_{2,0}$ which can be rewritten as $c_4 (c_1 - c_4) t_{2,0} \equiv 0$.

Substituting $s_{1,2}$ and $s_{2,1}$ into (B5) we obtain $0 \equiv (c_3 + c_1 c_4 + c_1 c_3 c_4) t_{2,0} + c_4 t_{0,2}$.
Hence $x \in \partial^{-1}W_2$ if and only if

\[
t_{1,1} \equiv -c_4t_{2,0} \quad \text{(A1)}
\]
\[
s_{2,2} \equiv -c_1c_4t_{2,0} \quad \text{(A3)}
\]
\[
s_{1,2} \equiv (c_3 - c_1c_4)t_{2,0} + c_1t_{0,1} \quad \text{(A4)}
\]
\[
c_4(c_1 - c_4)t_{2,0} \equiv 0 \quad \text{(B1)}
\]
\[
t_{0,2} \equiv c_1c_4t_{2,0} \quad \text{(B2)}
\]
\[
s_{2,1} \equiv c_4t_{2,0} - t_{0,1} \quad \text{(B3)}
\]
\[
0 \equiv (c_3 + c_1c_4 + c_1c_3c_4)t_{2,0} + c_4t_{0,2} \quad \text{(B5)}.
\]

Now we consider the case where $c_4 \neq 0$ and $c_1 \neq c_4$. Then $t_{2,0} \equiv 0$ which is the same as the Case 4.2.1.4.1. Therefore we conclude that $\partial^{-1}W_2 = \partial^{-1}W_1$. Thus $\text{rank}(\partial^{-1}W_2/W_2) = \text{rank}(\partial^{-1}W_1/W_2)$, $W_2$ is terminal, and $W_2 \not\in \hat{L}_2$.

Hence we may assume that either $c_4 = 0$ or $c_1 = c_4$. If $c_4 = 0$ then $t_{2,0}$ is a free variable. By congruences (A1), (A3), and (B2) we obtain $t_{1,1} \equiv s_{2,2} \equiv t_{0,2} \equiv 0$. The congruence (B3) becomes $s_{2,1} \equiv -t_{0,1}$ and (B5) becomes $0 \equiv c_3t_{2,0}$. The congruence (A4) becomes $s_{1,2} \equiv -c_3t_{2,0} + c_1t_{0,1}$. Substituting (B5) into the new (A4) we obtain $s_{1,2} \equiv c_1t_{0,1}$. 

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Hence \( x \in \partial^{-1}W_2 \) if and only if

\[
\begin{align*}
 t_{1,1} & \equiv 0 \\
 t_{2,0} & \equiv 0 \\
 t_{0,2} & \equiv 0 \\
 s_{2,2} & \equiv 0 \\
 s_{2,1} & \equiv -t_{0,1} \\
 s_{1,2} & \equiv c_1 t_{0,1}.
\end{align*}
\]

We regard \( t_{0,1}, t_{1,0}, t_{2,0} \) as the free variables. Taking \( t_{0,1} \equiv 1, t_{1,0} = 0, \) and \( t_{2,0} = 0, \)
the matrix \( x \) becomes

\[
 v_1 = \begin{bmatrix}
 0 & 1 & 0 \\
 0 & 0 & 3c_1 \\
 0 & -3 & 0
\end{bmatrix}.
\]

Taking \( t_{1,0} \equiv 1, t_{0,1} = 0, \) and \( t_{2,0} = 0, \) the matrix \( x \) becomes

\[
 v_2 = \begin{bmatrix}
 0 & 0 & 0 \\
 1 & 0 & 0 \\
 0 & 0 & 0
\end{bmatrix}.
\]

Taking \( t_{2,0} \equiv 1, t_{1,0} = 0, \) and \( t_{0,1} = 0, \) the matrix \( x \) becomes

\[
 v_3 = \begin{bmatrix}
 0 & 0 & 0 \\
 0 & 0 & 0 \\
 1 & 0 & 0
\end{bmatrix}.
\]

We see that neither \( v_3 \) nor \( 3v_3 \) is contained in \( W_2 \) and that

\( \partial^{-1}W_2 =< \partial^{-1}W_0, v_1, v_2, v_3 >. \) We know that \( v_3 \in \partial^{-1}W_2 \) and since \( 3v_3 \notin W_2 \)
then $v_3 + W_2$ is an element of the group $\partial^{-1}W_2/W_2$ whose order is larger than 3.

Recall that $|\partial^{-1}W_2/W_2| = 3^4$ and that $\partial^{-1}W_1/W_2$ has rank 3. Since $\partial^{-1}W_2$ has an element of order larger than 3 then $\text{rank}(\partial^{-1}W_2/W_2)$ is not greater than 3 since $\partial^{-1}W_2/W_2 \cong \mathbb{Z}_9 \times \mathbb{Z}_3 \times \mathbb{Z}_3$ and therefore $W_2$ is terminal and $W_2 \not\in \hat{L}_2$.

Now we consider the case $c_1 = c_4$. Hence $x \in \partial^{-1}W_2$ if and only if

$$t_{1,1} \equiv -c_1 t_{2,0} \quad (A1)$$
$$s_{2,2} \equiv -t_{2,0} \quad (A3)$$
$$s_{1,2} \equiv (-c_3 - 1)t_{2,0} + c_1 t_{0,1} \quad (A4)$$
$$t_{0,2} \equiv t_{2,0} \quad (B2)$$
$$s_{2,1} \equiv c_1 t_{2,0} - t_{0,1} \quad (B3)$$
$$0 \equiv (1 - c_3)t_{2,0} + c_1 t_{0,2} \quad (B5)$$

Substituting $t_{0,2} \equiv t_{2,0}$ into (B5) we obtain $0 \equiv (1 - c_3 + c_1)t_{2,0}$. We will consider the cases $(1 - c_3 + c_1) \neq 0$ and $(1 - c_3 + c_1) = 0$ separately. If $(1 - c_3 + c_1) \neq 0$ then $t_{2,0} \equiv 0$. This implies that $t_{0,2} \equiv s_{2,2} \equiv t_{1,1} \equiv t_{2,0} \equiv 0$. $(A4)$ becomes $s_{1,2} \equiv c_1 t_{0,1}$ and (B3) becomes $s_{2,1} \equiv -t_{0,1}$. 

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Hence \( x \in \partial^{-1}W_2 \) if and only if

\[
\begin{align*}
  t_{1,1} & \equiv 0 \\
  t_{2,0} & \equiv 0 \\
  t_{0,2} & \equiv 0 \\
  s_{2,2} & \equiv 0 \\
  s_{2,1} & \equiv -t_{0,1} \\
  s_{1,2} & \equiv c_1 t_{0,1}.
\end{align*}
\]

These are the same congruences as those in Case 4.2 and therefore we conclude that \( \partial^{-1}W_2 = \partial^{-1}W_1 \). Thus \( \text{rank}(\partial^{-1}W_2/W_2) = \text{rank}(\partial^{-1}W_1/W_2) \), \( W_2 \) is terminal, and \( W_2 \not\in \hat{L}_2 \).

Thus we assume that \((1 - c_3 + c_1) = 0\). Thus \( c_3 \equiv 1 + c_1 \) and \( t_{2,0} \) is a free variable.

Hence \( x \in \partial^{-1}W_2 \) if and only if

\[
\begin{align*}
  t_{1,1} & \equiv -c_1 t_{2,0} \quad \text{(A1)} \\
  s_{2,2} & \equiv -t_{2,0} \quad \text{(A3)} \\
  s_{1,2} & \equiv (1 + c_1) t_{2,0} + c_1 t_{0,1} \quad \text{(A4)} \\
  t_{0,2} & \equiv t_{2,0} \quad \text{(B2)} \\
  s_{2,1} & \equiv c_1 t_{2,0} - t_{0,1} \quad \text{(B3)}.
\end{align*}
\]
We regard $t_{0,1}$, $t_{1,0}$, $t_{2,0}$ as the free variables. Taking $t_{0,1} \equiv 1$, $t_{1,0} =$, and $t_{2,0} = 0$, the matrix $x$ becomes

$$v_1 = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 3c_1 \\ 0 & -3 & 0 \end{bmatrix}.$$ 

Taking $t_{1,0} \equiv 1$, $t_{0,1} = 0$, and $t_{2,0} = 0$, the matrix $x$ becomes

$$v_2 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$ 

Taking $t_{2,0} \equiv 1$, $t_{1,0} = 0$, and $t_{0,1} = 0$, the matrix $x$ becomes

$$v_3 = \begin{bmatrix} 0 & 0 & 1 \\ 0 & -c_1 & 3(1 + c_1) \\ 1 & -3c_1 & -3 \end{bmatrix}.$$ 

We see that neither $v_3$ nor $3v_3$ is contained in $W_2$ and that

$$\partial^{-1}W_2 = < \partial^{-1}W_0, v_1, v_2, v_3 >.$$ 

We know that $v_3 \in \partial^{-1}W_2$ and since $3v_3 \notin W_2$ then $v_3 + W_2$ is an element of the group $\partial^{-1}W_2/W_2$ whose order is larger than 3.

Recall that $|\partial^{-1}W_2/W_2| = 3^4$ and that $\partial^{-1}W_1/W_2$ has rank 3. Since $\partial^{-1}W_2$ has an element of order larger than 3 then rank$(\partial^{-1}W_2/W_2)$ is not greater than 3 since $\partial^{-1}W_2/W_2 \cong \mathbb{Z}_9 \times \mathbb{Z}_3 \times \mathbb{Z}_3$ and therefore $W_2$ is terminal and $W_2 \notin \hat{\mathcal{L}}_2$.

7.2.2 Case 4.2.2

We consider the 2-dimensional subspaces $W_2/W_1$. Let $d_1, d_2, d_3, d_4, e_1, e_2, e_3, e_4$ be unspecified variables. Let $m_3 = d_1y_1 + d_2y_3 + d_3v_1 + d_4v_2$ and $m_4 = e_1y_1 + e_2y_3 + 101$
\[ e_3v_1 + e_4v_2. \] In all the cases we consider the value of \( e_1 = 0 \) therefore we may exclude it from our expression of \( m_4 \). A formal expression for \( m_3 \) is

\[
m_3 = d_1 \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 3 & 0 & 0 \end{bmatrix} + d_2 \begin{bmatrix} 0 & 0 & 3 \\ 0 & 0 & 0 \\ 0 & -3 & 0 \end{bmatrix} + d_3 \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 3c_1 \\ 0 & -3 & 0 \end{bmatrix} + d_4 \begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}
\]

\[
= \begin{bmatrix} 0 & d_3 & 3d_2 \\ d_4 & 0 & 3c_1d_3 \\ 3d_1 & -3d_3 & 0 \end{bmatrix}.
\]

A formal expression for \( m_4 \) is

\[
m_4 = e_2 \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 3c_1 \\ 0 & -3 & 0 \end{bmatrix} + e_3 \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 3c_1 \\ 0 & -3 & 0 \end{bmatrix} + e_4 \begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}
\]

\[
= \begin{bmatrix} 0 & e_3 & 3e_2 \\ e_4 & 0 & 3c_1e_3 \\ 0 & -3e_3 & 0 \end{bmatrix}.
\]

Let \( W_2 = \langle W_1, m_3, m_4 \rangle \in \mathcal{L}_2 \). We now calculate the pullback \( \partial^{-1}W_2 \). The subgroup \( W_2 \) is contained in the pattern subgroup

\[
\begin{bmatrix} 2 & 2 & 1 \\ 2 & 1 & 1 \\ 1 & 1 & 0 \end{bmatrix}.
\]
Thus the pullback $\partial^{-1}W_2$ is contained in the pattern subgroup
\[
\begin{bmatrix}
2 & 2 & 2 \\
2 & 2 & 1 \\
2 & 1 & 1
\end{bmatrix} \cap \begin{bmatrix}
2 & 2 & 2 \\
2 & 2 & 1 \\
2 & 1 & 1
\end{bmatrix} = \begin{bmatrix}
2 & 2 & 1 \\
2 & 1 & 1
\end{bmatrix}.
\]

Let
\[
x = \begin{bmatrix}
3s_{0,0} + t_{0,0} & 3s_{0,1} + t_{0,1} & 3s_{0,2} + t_{0,2} \\
3s_{1,0} + t_{1,0} & 3s_{1,1} + t_{1,1} & 3s_{1,2} \\
3s_{2,0} + t_{2,0} & 3s_{2,1} & 3s_{2,2}
\end{bmatrix} \in \begin{bmatrix}
2 & 2 & 2 \\
2 & 1 & 1
\end{bmatrix}.
\]

Thus
\[
\partial_1 x = \begin{bmatrix}
3s_{1,0} + t_{1,0} & 3s_{1,1} + t_{1,1} & 3s_{1,2} \\
3s_{2,0} + t_{2,0} & 3s_{2,1} & 3s_{2,2} \\
-3(t_{1,0} + t_{2,0}) & -3t_{1,1} & 0
\end{bmatrix}
\quad \text{and}
\]
\[
\partial_2 x = \begin{bmatrix}
3s_{0,1} + t_{0,1} & 3s_{0,2} + t_{0,2} & -3(t_{0,1} + t_{0,2}) \\
3s_{1,1} + t_{1,1} & 3s_{1,2} & -3t_{1,1} \\
3s_{2,1} & 3s_{2,2} & 0
\end{bmatrix}.
\]

The variables that are in play are those appearing in the matrix
\[
\begin{bmatrix}
0 & t_{0,1} & t_{0,2} \\
t_{1,0} & t_{1,1} & 3s_{1,2} \\
t_{2,0} & 3s_{2,1} & 3s_{2,2}
\end{bmatrix}.
\]
We wish to identify values $a_1, a_2, a_3 \in \mathbb{Z}_9$ such that $\partial_1 x \equiv a_1 m_1 + a_2 m_2 + a_3 m_3 \pmod{I}$. A formal expression for $a_1 m_1 + a_2 m_2 + a_3 m_3 + a_4 m_4$ is

$$a_1 \begin{bmatrix} c_1 & 0 & -3c_1 \\ 0 & 3 & 0 \\ -3c_1 & 0 & 0 \end{bmatrix} + a_2 \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ -3 & 0 & 0 \end{bmatrix} + a_3 \begin{bmatrix} d_3 & 3d_2 \\ 0 & 0 \\ 3d_1 & -3d_3 \end{bmatrix} + a_4 \begin{bmatrix} 0 & e_3 & 3e_2 \\ e_4 & 0 & 3c_1 e_3 \\ 0 & -3c_3 & 0 \end{bmatrix} = \begin{bmatrix} c_1 a_1 + a_2 & d_3 a_3 + e_3 a_4 & 3(d_2 a_3 + e_2 a_4 - c_1 a_1) \\ d_4 a_3 + e_4 a_4 & 3a_1 & 3(c_1 d_3 a_3 + c_1 e_3 a_4) \\ 3(d_1 a_3 - c_1 a_1 - a_2) & 3(-d_3 a_3 - e_3 a_4) & 0 \end{bmatrix}.$$  

Comparing $(0,1)$-entries, we get $t_{1,1} \equiv d_3 a_3 + e_3 a_4$.

Comparing $(2,1)$-entries, we get $t_{1,1} \equiv d_3 a_3 + e_3 a_4$ which gives no new information.

Comparing $(1,0)$-entries, we get $t_{2,0} \equiv d_4 a_3 + e_4 a_4$.

Comparing $(1,1)$-entries, we get $a_1 \equiv s_{2,1}$.

Comparing $(0,0)$-entries, we get $c_1 a_1 + a_2 \equiv t_{1,0}$. Substituting $a_1$ we obtain $a_2 \equiv t_{1,0} - c_1 s_{2,1}$.

Comparing $(1,2)$-entries, we get $c_1 (d_3 a_3 + e_3 a_4) \equiv s_{2,2}$. Substituting $d_3 a_3 + e_3 a_4$ we obtain $c_1 t_{1,1} \equiv s_{2,2}$.

Comparing $(0,2)$-entries, we get $d_2 a_3 + e_2 a_4 - a_1 c_1 \equiv s_{1,2}$. Substituting $a_1$ we obtain $s_{1,2} \equiv a_3 d_2 + e_2 a_4 - c_1 s_{2,1}$.

Comparing $(2,0)$-entries, we get $t_{1,0} + t_{2,0} \equiv c_1 a_1 + a_2 - d_1 a_3$. Substituting $a_1$ and $a_2$ we obtain $t_{2,0} \equiv -d_1 a_3$.  

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We see that $\partial_1 x \in W_2$ if and only if

\begin{align*}
    t_{1,1} &\equiv d_3 a_3 + e_3 a_4 \quad \text{(A1)} \\
    t_{2,0} &\equiv d_4 a_3 + e_4 a_4 \quad \text{(A2)} \\
    s_{2,2} &\equiv c_1 t_{1,1} \quad \text{(A3)} \\
    s_{1,2} &\equiv d_2 a_3 + e_2 a_4 - c_1 s_{2,1} \quad \text{(A4)} \\
    t_{2,0} &\equiv -d_1 a_3 \quad \text{(A5)}.
\end{align*}

We wish to identify values $b_1, b_2, b_3 \in \mathbb{Z}_9$ such that $\partial_2 x \equiv b_1 m_1 + b_2 m_2 + b_3 m_3 \pmod{I}$. A formal expression for $b_1 m_1 + b_2 m_2 + b_3 m_3$ is

\[
    b_1 \begin{bmatrix}
        c_1 & 0 & -3c_1 \\
        0 & 3 & 0 \\
        -3c_1 & 0 & 0
    \end{bmatrix}
    + b_2 \begin{bmatrix}
        1 & 0 & 0 \\
        0 & 0 & 0 \\
        -3 & 0 & 0
    \end{bmatrix}
    + b_3 \begin{bmatrix}
        0 & d_3 & 3d_2 \\
        d_4 & 0 & 3c_1 d_3 \\
        3d_1 & -3d_3 & 0
    \end{bmatrix}
    + b_4 \begin{bmatrix}
        0 & e_3 & 3e_2 \\
        e_4 & 0 & 3c_1 e_3 \\
        0 & -3e_3 & 0
    \end{bmatrix}
\]

\[
    = \begin{bmatrix}
        c_1 b_1 + b_2 & d_3 b_3 + e_3 b_4 & 3(d_2 b_3 + e_2 b_4 - c_1 b_1) \\
        d_4 b_3 + e_4 b_4 & 3b_1 & 3(c_1 d_3 b_3 + c_1 e_3 b_4) \\
        3(d_1 b_3 - c_1 b_1 - b_2) & 3(-d_3 b_3 - e_3 b_4) & 0
    \end{bmatrix}.
\]

Comparing $(0,1)$-entries, we get $d_3 b_3 + e_3 b_4 \equiv t_{0,2}$.

Comparing $(2,1)$-entries, we get $-d_3 b_3 - e_3 b_4 \equiv s_{2,2}$. Combining this with the $(0,1)$ entry we obtain $t_{0,2} \equiv -s_{2,2}$.

Comparing $(1,0)$-entries, we get $d_4 b_3 + e_4 b_4 \equiv t_{1,1}$.

Comparing $(1,1)$-entries, we get $b_1 \equiv s_{1,2}$.
Comparing $(0,0)$-entries, we get $c_1b_1 + b_2 \equiv t_{0,1}$. Substituting $b_1$ we obtain $b_2 \equiv t_{0,1} - c_1s_{1,2}$.

Comparing $(1,2)$-entries, we get $c_1(d_3b_3 + e_3b_4) \equiv -t_{1,1}$. Substituting $d_3b_3 + e_3b_4$ we obtain $t_{0,2} \equiv -c_1t_{1,1}$.

Comparing $(0,2)$-entries, we get $t_{0,1} + t_{0,2} \equiv c_1b_1 - d_2b_3 - e_2b_4$. Substituting $b_1$ we obtain $t_{0,1} \equiv c_1s_{1,2} - d_2b_3 - e_2b_4 - t_{0,2}$. Since $c_1^2 \equiv 1$ this becomes $s_{1,2} \equiv c_1t_{0,1} + c_1t_{0,2} + c_1d_2b_3 + c_1e_2b_4$.

Comparing $(2,0)$-entries, we get $d_1b_3 - c_1b_1 - b_2 \equiv s_{2,1}$. Substituting $b_1$ and $b_2$ we obtain $d_1b_3 - t_{0,1} \equiv s_{2,1}$.

We see that $\partial_2x \in W_2$ if and only if

\begin{align*}
    d_3b_3 + e_3b_4 & \equiv t_{0,2} \quad \text{(B1)} \\
    t_{0,2} & \equiv -s_{2,2} \quad \text{(B2)} \\
    d_4b_3 + e_4b_4 & \equiv t_{1,1} \quad \text{(B3)} \\
    t_{0,2} & \equiv -c_1t_{1,1} \quad \text{(B4)} \\
    s_{1,2} & \equiv c_1t_{0,1} + c_1t_{0,2} + c_1d_2b_3 + c_1e_2b_4. \quad \text{(B5)} \\
    d_1b_3 - t_{0,1} & \equiv s_{2,1} \quad \text{(B6)}.
\end{align*}

Case 4.2.2.1

Let $m_3 = v_1$ and $m_4 = v_2$. Thus
\[
    m_3 = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 3c_1 \\ 0 & -3 & 0 \end{bmatrix} \quad \text{and} \quad m_4 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix}.
\]
Let $W_2 = <W_1, m_3, m_4> \in \mathcal{L}_2$. The number of subgroups $W_2$ of this type is 1. Since $m_3, m_4 \notin W_1$ and $3m_1, 3m_4 \in W_1$ we have $|W_2/W_1| = 3^2$. Since $|W_1| = 3^5$ it follows that $|W_2| = 3^7$. Note that $\partial^{-1}W_1/W_2 \cong \mathbb{Z}_3 \times \mathbb{Z}_3$ and rank($\partial^{-1}W_1/W_2$) = 2. We now calculate the pullback of $\partial^{-1}W_2$. We observed in case 4.2.2 that $\partial^{-1}W_2$ is contained in the pattern subgroup

$$
\begin{pmatrix}
2 & 2 & 2 \\
2 & 2 & 1 \\
2 & 1 & 1
\end{pmatrix}
$$

Let

$$
x = \begin{pmatrix}
3s_{0,0} + t_{0,0} & 3s_{0,1} + t_{0,1} & 3s_{0,2} + t_{0,2} \\
3s_{1,0} + t_{1,0} & 3s_{1,1} + t_{1,1} & 3s_{1,2} \\
3s_{2,0} + t_{2,0} & 3s_{2,1} & 3s_{2,2}
\end{pmatrix} \in \begin{pmatrix}
2 & 2 & 2 \\
2 & 2 & 1 \\
2 & 1 & 1
\end{pmatrix}.
$$

The variables that are in play are those appearing in the matrix

$$
\begin{pmatrix}
0 & t_{0,1} & t_{0,2} \\
t_{1,0} & t_{1,1} & 3s_{1,2} \\
t_{2,0} & 3s_{2,1} & 3s_{2,2}
\end{pmatrix}
$$

In the notation of Case 4.2.2, we are taking $d_1 = 0$, $d_2 = 0$, $d_3 = 1$, $d_4 = 0$, $e_2 = 0$, $e_3 = 0$, and $e_4 = 1$. We wish to identify values $a_1, a_2, a_3, a_4 \in \mathbb{Z}_9$ such that $\partial_1 x \equiv a_1 m_1 + a_2 m_2 + a_3 m_3 + a_4 m_4 \pmod{I}$.
We see that $\partial_1 x \in W_2$ if and only if

\[ s_{2,2} \equiv c_1 t_{1,1} \quad \text{(A3)} \]
\[ s_{1,2} \equiv -c_1 s_{2,1} \quad \text{(A4)} \]
\[ t_{2,0} \equiv 0 \quad \text{(A5)}. \]

We wish to identify values $b_1, b_2, b_3, b_4 \in \mathbb{Z}_9$ such that $\partial_1 x \equiv b_1 m_1 + b_2 m_2 + b_3 m_3 + b_4 m_4 \pmod{I}$. We see that $\partial_2 x \in W_2$ if and only if

\[ t_{0,2} \equiv -s_{2,2} \quad \text{(B2)} \]
\[ t_{0,2} \equiv -c_1 t_{1,1} \quad \text{(B4)} \]
\[ s_{1,2} \equiv c_1 t_{0,1} + c_1 t_{0,2} \quad \text{(B5)} \]
\[ s_{2,1} \equiv -t_{0,1} \quad \text{(B6)}. \]

Substituting (B6) into (A4) we obtain $s_{1,2} \equiv c_1 t_{0,1}$. Then setting our new (A4) equal to (B5) we get $c_1 t_{0,1} \equiv c_1 t_{0,1} + c_1 t_{0,2}$ which becomes $0 \equiv c_1 t_{0,2}$. Since $c_1 \neq 0$, then $t_{0,2} \equiv 0$. Substituting $t_{0,2} \equiv 0$ into (B2),(B4), and (B5) we obtain $s_{2,2} \equiv 0$, $t_{1,1} \equiv 0$, and $s_{1,2} \equiv c_1 t_{0,1}$. 
Hence \( x \in \partial^{-1}W_2 \) if and only if
\[
\begin{align*}
t_{0,2} &\equiv 0 \\
s_{2,2} &\equiv 0 \\
t_{1,1} &\equiv 0 \\
t_{2,0} &\equiv 0 \\
s_{1,2} &\equiv c_1 t_{0,1} \\
s_{2,1} &\equiv -t_{0,1}.
\end{align*}
\]

These are the same congruences as those in Case 4.2 and therefore we conclude that \( \partial^{-1}W_2 = \partial^{-1}W_1 \). Thus \( \text{rank}(\partial^{-1}W_2/W_2) = \text{rank}(\partial^{-1}W_1/W_2) \), \( W_2 \) is terminal, and \( W_2 \not\in \hat{\mathcal{L}}_2 \).

Case 4.2.2.2

We fix arbitrary value \( c_3 \in \{1,2\} \). There are 2 ways to choose the value \( c_3 \). Let \( m_3 = y_3 + c_3 v_2 \) and \( m_4 = v_1 \). Thus
\[
m_3 = \begin{bmatrix} 0 & c_3 & 3 \\ 0 & 0 & 3c_1c_3 \\ 0 & -3c_3 & 0 \end{bmatrix} \quad \text{and} \quad m_4 = \begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}.
\]

Let \( W_2 = <W_1, m_3, m_4> \in \mathcal{L}_2 \). The number of subgroups \( W_2 \) of this type is 2. Since \( m_3, m_4 \notin W_1 \) and \( 3m_1, 3m_4 \in W_1 \) we have \( |W_2/W_1| = 3^2 \). Since \( |W_1| = 3^5 \) it follows that \( |W_2| = 3^7 \). Note that \( \partial^{-1}W_1/W_2 \cong \mathbb{Z}_3 \times \mathbb{Z}_3 \) and \( \text{rank}(\partial^{-1}W_1/W_2) = 2 \). We now calculate the pullback of \( \partial^{-1}W_2 \). We observed in case 4.2.2 that \( \partial^{-1}W_2 \) is contained
in the pattern subgroup

\[
\begin{bmatrix}
2 & 2 & 2 \\
2 & 2 & 1 \\
2 & 1 & 1
\end{bmatrix}
\]

Let

\[
x = \begin{bmatrix}
3s_{0,0} + t_{0,0} & 3s_{0,1} + t_{0,1} & 3s_{0,2} + t_{0,2} \\
3s_{1,0} + t_{1,0} & 3s_{1,1} + t_{1,1} & 3s_{1,2} \\
3s_{2,0} + t_{2,0} & 3s_{2,1} & 3s_{2,2}
\end{bmatrix}
\in \begin{bmatrix}
2 & 2 & 2 \\
2 & 2 & 1 \\
2 & 1 & 1
\end{bmatrix}
\]

The variables that are in play are those appearing in the matrix

\[
\begin{bmatrix}
0 & t_{0,1} & t_{0,2} \\
t_{1,0} & t_{1,1} & 3s_{1,2} \\
t_{2,0} & 3s_{2,1} & 3s_{2,2}
\end{bmatrix}
\]

In the notation of Case 4.2.2, we are taking \(d_1 = 0, \, d_2 = 1, \, d_3 = c_3, \, d_4 = 0, \)
\(e_2 = 0, \, e_3 = 0, \) and \(e_4 = 1.\) We wish to identify values \(a_1, a_2, a_3, a_4 \in \mathbb{Z}_9\) such that

\[\partial_1 x \equiv a_1 m_1 + a_2 m_2 + a_3 m_3 + a_4 m_4 \pmod I.\]

We see that \(\partial_1 x \in W_2\) if and only if

\[t_{1,1} \equiv c_3 a_3 \quad \text{(A1)}\]

\[t_{2,0} \equiv a_4 \quad \text{(A2)}\]

\[s_{2,2} \equiv c_1 t_{1,1} \quad \text{(A3)}\]

\[s_{1,2} \equiv a_3 - c_1 s_{2,1} \quad \text{(A4)}\]

\[t_{2,0} \equiv 0 \quad \text{(A5)}.
\]
We wish to identify values $b_1, b_2, b_3, b_4 \in \mathbb{Z}_9$ such that $\partial_2 x \equiv b_1 m_1 + b_2 m_2 + b_3 m_3 + b_4 m_4 \pmod{I}$.

We see that $\partial_2 x \in W_2$ if and only if

\begin{align*}
c_3 b_3 & \equiv t_{0,2} \quad \text{(B1)} \\
t_{0,2} & \equiv -s_{2,2} \quad \text{(B2)} \\
t_{1,1} & \equiv b_4 \quad \text{(B3)} \\
t_{0,2} & \equiv -c_1 t_{1,1} \quad \text{(B4)} \\
s_{1,2} & \equiv c_1 t_{0,1} + c_1 t_{0,2} + c_1 b_3 \quad \text{(B5)} \\
s_{2,1} & \equiv -t_{0,1} \quad \text{(B6)}.
\end{align*}

We can write (A1) as $a_3 \equiv c_3 t_{1,1}$ and (B1) as $b_3 \equiv c_3 t_{0,2}$. Substituting $a_3$ into (A4) and $b_3$ into (B5) we obtain $s_{1,2} \equiv c_3 t_{1,1} - c_1 s_{2,1}$ and $s_{1,2} \equiv c_1 t_{0,1} + (c_1 + c_1 c_3) t_{0,2}$. Also, substituting (B6) into the new (A4) we obtain $s_{1,2} \equiv c_3 t_{1,1} + c_1 t_{0,1}$.

Hence $x \in \partial^{-1} W_2$ if and only if

\begin{align*}
s_{2,2} & \equiv c_1 t_{1,1} \quad \text{(A3)} \\
s_{1,2} & \equiv c_3 t_{1,1} + c_1 t_{0,1} \quad \text{(A4)} \\
t_{2,0} & \equiv 0 \quad \text{(A5)} \\
t_{0,2} & \equiv -s_{2,2} \quad \text{(B2)} \\
t_{0,2} & \equiv -c_1 t_{1,1} \quad \text{(B4)} \\
s_{1,2} & \equiv c_1 t_{0,1} + (c_1 + c_1 c_3) t_{0,2} \quad \text{(B5)} \\
s_{2,1} & \equiv -t_{0,1} \quad \text{(B6)}.
\end{align*}
Combining the new (A4) and (B5) we obtain $c_3 t_{1,1} \equiv (c_1 + c_1 c_3) t_{0,2}$. Substituting $t_{1,1} \equiv c_1 s_{2,2}$ and $s_{1,1} \equiv -t_{0,2}$ from (A3) and (B2) this becomes $0 \equiv (c_1 - c_1 c_3) t_{0,2}$. Multiplying by $c_1$ we obtain $0 \equiv (1 - c_3) t_{0,2}$. It is convenient to consider the cases $(1 - c_3) \neq 0$ and $(1 - c_3) = 0$ separately.

**Case 4.2.2.2.1** First we consider the case $(1 - c_3) \neq 0$. Then $t_{0,2} \equiv 0$. Thus we obtain $s_{2,2} \equiv t_{1,1} \equiv t_{0,2} \equiv 0$. The congruences (A4) and (B5) becomes $s_{1,2} \equiv c_1 t_{0,1}$. Thus (B5) is redundant with (A4).

Hence $x \in \partial^{-1} W_2$ if and only if

\[
\begin{align*}
    t_{1,1} & \equiv 0 \\
    t_{2,0} & \equiv 0 \\
    t_{0,2} & \equiv 0 \\
    s_{2,2} & \equiv 0 \\
    s_{1,2} & \equiv c_1 t_{0,1} \\
    s_{2,1} & \equiv -t_{0,1}.
\end{align*}
\]

These are the same congruences as those in Case 4.2 and therefore we conclude that $\partial^{-1} W_2 = \partial^{-1} W_1$. Thus $\text{rank}(\partial^{-1} W_2 / W_2) = \text{rank}(\partial^{-1} W_1 / W_2)$, $W_2$ is terminal, and $W_2 \not\in \hat{L}_2$.

**Case 4.2.2.2.2** Now we consider the case $(1 - c_3) = 0$. Then $c_3 \equiv 1$. 

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Hence \( x \in \partial^{-1}W_2 \) if and only if

\[
\begin{align*}
    s_{2,2} &\equiv c_1 t_{1,1} \quad \text{(A3)} \\
    s_{1,2} &\equiv t_{1,1} + c_1 t_{0,1} \quad \text{(A4)} \\
    t_{2,0} &\equiv 0 \quad \text{(A5)} \\
    t_{0,2} &\equiv -s_{2,2} \quad \text{(B2)} \\
    t_{0,2} &\equiv -c_1 t_{1,1} \quad \text{(B4)} \\
    s_{1,2} &\equiv c_1 t_{0,1} + c_1 t_{0,2} \quad \text{(B5)} \\
    s_{2,1} &\equiv -t_{0,1} \quad \text{(B6)}.
\end{align*}
\]

We regard \( t_{0,1}, t_{1,0}, \) and \( t_{0,2} \) as free variables. Note that the congruences (A3), (B2), and (B4) give us \( s_{2,2} \equiv c_1 t_{1,1} \equiv -t_{0,2} \). Taking \( t_{0,1} \equiv 1, t_{1,0} = 0, \) and \( t_{2,0} = 0 \), the matrix \( x \) becomes

\[
\begin{bmatrix}
    0 & 1 & 0 \\
    0 & 0 & 3c_1 \\
    0 & -3 & 0
\end{bmatrix}.
\]

Taking \( t_{1,0} \equiv 1, t_{0,1} = 0, \) and \( t_{2,0} = 0 \), the matrix \( x \) becomes

\[
\begin{bmatrix}
    0 & 0 & 0 \\
    1 & 0 & 0 \\
    0 & 0 & 0
\end{bmatrix}.
\]
Taking \( t_{0,2} \equiv 1, t_{0,1} = 0, \) and \( t_{1,0} = 0 \), the matrix \( x \) becomes
\[
v_3 = \begin{bmatrix}
0 & 0 & 1 \\
0 & -c_1 & -3c_1 \\
0 & 0 & -3
\end{bmatrix}.
\]

We see that neither \( v_3 \) nor \( 3v_3 \) is contained in \( W_2 \) and that
\[
\partial^{-1}W_2 = \langle \partial^{-1}W_0, v_1, v_2, v_3 \rangle.
\]
We know that \( v_3 \in \partial^{-1}W_2 \) and since \( 3v_3 \notin W_2 \) then \( v_3 + W_2 \) is an element of the group \( \partial^{-1}W_2/W_2 \) whose order is larger than 3.

Recall that \( |\partial^{-1}W_2/W_2| = 3^4 \) and that \( \partial^{-1}W_1/W_2 \) has rank \( 2 \). Since \( \partial^{-1}W_2 \) has an element of order larger than 3 then \( \text{rank}(\partial^{-1}W_2/W_2) \) is not greater than 2 since \( \partial^{-1}W_2/W_2 \cong \mathbb{Z}_9 \times \mathbb{Z}_3 \) and therefore \( W_2 \) is terminal and \( W_2 \notin \hat{\mathcal{L}}_2 \).

Case 4.2.2.3

We fix arbitrary values \( c_3 \in \{1, 2\} \) and \( c_4 \in \{0, 1, 2\} \). There are 6 ways to choose the values \( c_3 \) and \( c_4 \). Let \( m_3 = y_3 + c_3v_2 \) and \( m_4 = v_1 + c_4v_2 \). Thus
\[
m_3 = \begin{bmatrix}
0 & 0 & 3 \\
c_3 & 0 & 0 \\
0 & 0 & 0
\end{bmatrix} \quad \text{and} \quad m_4 = \begin{bmatrix}
0 & 1 & 0 \\
c_4 & 0 & 3c_1 \\
0 & -3 & 0
\end{bmatrix}.
\]

Let \( W_2 = \langle W_1, m_3, m_4 \rangle \in \mathcal{L}_2 \). The number of subgroups \( W_2 \) of this type is 6. Since \( m_3, m_4 \notin W_1 \) and \( 3m_1, 3m_4 \in W_1 \) we have \( |W_2/W_1| = 3^2 \). Since \( |W_1| = 3^5 \) it follows that \( |W_2| = 3^7 \). Note that \( \partial^{-1}W_1/W_2 \cong \mathbb{Z}_3 \times \mathbb{Z}_3 \) and \( \text{rank}(\partial^{-1}W_1/W_2) = 2 \). We now calculate the pullback of \( \partial^{-1}W_2 \). We observed in case 4.2.2 that \( \partial^{-1}W_2 \) is contained
in the pattern subgroup

\[
\begin{bmatrix}
  2 & 2 & 2 \\
  2 & 2 & 1 \\
  2 & 1 & 1
\end{bmatrix}.
\]

Let

\[
x = \begin{bmatrix}
  3s_{0,0} + t_{0,0} & 3s_{0,1} + t_{0,1} & 3s_{0,2} + t_{0,2} \\
  3s_{1,0} + t_{1,0} & 3s_{1,1} + t_{1,1} & 3s_{1,2} \\
  3s_{2,0} + t_{2,0} & 3s_{2,1} & 3s_{2,2}
\end{bmatrix} \in \begin{bmatrix}
  2 & 2 & 2 \\
  2 & 2 & 1 \\
  2 & 1 & 1
\end{bmatrix}.
\]

The variables that are in play are those appearing in the matrix

\[
\begin{bmatrix}
  0 & t_{0,1} & t_{0,2} \\
  t_{1,0} & t_{1,1} & 3s_{1,2} \\
  t_{2,0} & 3s_{2,1} & 3s_{2,2}
\end{bmatrix}.
\]

In the notation of Case 4.2.2, we are taking \(d_1 = 0, d_2 = 1, d_3 = 0, d_4 = c_3,\) \(e_2 = 0, e_3 = 1,\) and \(e_4 = c_4.\) We wish to identify values \(a_1, a_2, a_3, a_4 \in \mathbb{Z}_9\) such that \(\partial_1 x \equiv a_1m_1 + a_2m_2 + a_3m_3 + a_4m_4 \pmod{I}.\)

We see that \(\partial_1 x \in W_2\) if and only if

\[
\begin{align*}
t_{1,1} & \equiv a_4 \quad \text{(A1)} \\
t_{2,0} & \equiv c_3a_3 + c_4a_4 \quad \text{(A2)} \\
s_{2,2} & \equiv c_1t_{1,1} \quad \text{(A3)} \\
s_{1,2} & \equiv a_3 - c_1s_{2,1} \quad \text{(A4)} \\
t_{2,0} & \equiv 0 \quad \text{(A5)}.
\end{align*}
\]
We wish to identify values $b_1, b_2, b_3, b_4 \in \mathbb{Z}_9$ such that $\partial_2 x \equiv b_1 m_1 + b_2 m_2 + b_3 m_3 + b_4 m_4 (\text{mod } I)$.

We see that $\partial_2 x \in W_2$ if and only if

$\begin{align*}
  t_{0,2} &\equiv b_4 \quad \text{(B1)} \\
  t_{0,2} &\equiv -s_{2,2} \quad \text{(B2)} \\
  t_{1,1} &\equiv c_3 b_3 + c_4 b_4 \quad \text{(B3)} \\
  t_{0,2} &\equiv -c_1 t_{1,1} \quad \text{(B4)} \\
  s_{1,2} &\equiv c_1 t_{0,1} + c_1 t_{0,2} + c_1 b_3 \quad \text{(B5)} \\
  s_{2,1} &\equiv -t_{0,1} \quad \text{(B6)}.
\end{align*}$

We can rewrite (B4) as $t_{1,1} \equiv -c_1 t_{0,2}$. Substituting $a_4 \equiv t_{1,1}$ and $t_{2,0} \equiv 0$ into (A2) we obtain $a_3 \equiv -c_3 c_4 t_{1,1}$. Substituting $a_3$ and $t_{1,1}$ into (A4) we obtain $s_{1,2} \equiv c_1 c_3 c_4 t_{0,2} - c_1 s_{2,1}$. Substituting (B6) this congruence becomes $s_{1,2} \equiv c_1 c_3 c_4 t_{0,2} + c_1 t_{0,1}$.

Substituting $b_4$ into (B3) we obtain $b_3 \equiv c_3 t_{1,1} - c_3 c_4 t_{0,2}$. Now substituting $b_3$ and $t_{1,1}$ into (B5) we get $s_{1,2} \equiv c_1 t_{0,1} + c_1 t_{0,2} - c_3 t_{0,2} - c_1 c_3 c_4 t_{0,2}$. 

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Hence \( x \in \partial^{-1}W_2 \) if and only if

\[
\begin{align*}
s_{2,2} &\equiv c_1 t_{1,1} \quad \text{(A3)} \\
s_{1,2} &\equiv c_1 c_3 c_4 t_{0,2} + c_1 t_{0,1} \quad \text{(A4)} \\
t_{2,0} &\equiv 0 \quad \text{(A5)} \\
t_{0,2} &\equiv -s_{2,2} \quad \text{(B2)} \\
t_{0,2} &\equiv -c_1 t_{1,1} \quad \text{(B4)} \\
s_{1,2} &\equiv c_1 t_{0,1} + c_1 t_{0,2} - c_3 t_{0,2} - c_1 c_3 c_4 t_{0,2} \quad \text{(B5)} \\
s_{2,1} &\equiv -t_{0,1} \quad \text{(B6)}.
\end{align*}
\]

Combining our new (A4) and (B5) we obtain \( 0 \equiv (c_1 - c_3 + c_1 c_3 c_4) t_{0,2} \). Therefore it is convenient to consider the cases \((c_1 - c_3 + c_1 c_3 c_4) \neq 0\) and \((c_1 - c_3 + c_1 c_3 c_4) = 0\) separately.

**Case 4.2.2.3.1** First we consider the case \((c_1 - c_3 + c_1 c_3 c_4) \neq 0\). Then \( t_{0,2} \equiv 0 \).

This implies that \( s_{2,2} \equiv t_{1,1} \equiv t_{0,2} \equiv 0 \). Congruence (A4) becomes \( s_{1,2} \equiv c_1 t_{0,1} \). The congruence (B5) becomes \( c_1 t_{0,1} \) which is redundant with (A4).
Hence \( x \in \partial^{-1}W_2 \) if and only if

\[
\begin{align*}
t_{1,1} &\equiv 0 \\
t_{2,0} &\equiv 0 \\
t_{0,2} &\equiv 0 \\
s_{2,2} &\equiv 0 \\
s_{1,2} &\equiv c_1 t_{0,1} \\
s_{2,1} &\equiv -t_{0,1}.
\end{align*}
\]

These are the same congruences as those in Case 4.2 and therefore we conclude that \( \partial^{-1}W_2 = \partial^{-1}W_1 \). Thus \( \text{rank}(\partial^{-1}W_2/W_2) = \text{rank}(\partial^{-1}W_1/W_2) \), \( W_2 \) is terminal, and \( W_2 \notin \hat{L}_2 \).

**Case 4.2.2.3.2** Now we consider the case \( (c_1 - c_3 + c_1 c_3 c_4) = 0 \). Then \( t_{0,2} \) becomes a free variable. Hence \( x \in \partial^{-1}W_2 \) if and only if

\[
\begin{align*}
s_{2,2} &\equiv c_1 t_{1,1} \quad \text{(A3)} \\
s_{1,2} &\equiv (c_3 - c_1) t_{0,2} + c_1 t_{0,1} \quad \text{(A4)} \\
t_{2,0} &\equiv 0 \quad \text{(A5)} \\
t_{0,2} &\equiv -s_{2,2} \quad \text{(B2)} \\
t_{0,2} &\equiv -c_1 t_{1,1} \quad \text{(B4)} \\
s_{1,2} &\equiv c_1 t_{0,1} - c_1 t_{0,2} + c_3 t_{0,2} \quad \text{(B5)} \\
s_{2,1} &\equiv -t_{0,1} \quad \text{(B6)}.
\end{align*}
\]

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We regard $t_{0,1}$, $t_{1,0}$, and $t_{0,2}$ as free variables. Note that the congruences (A3), (B2), and (B4) give us $s_{2,2} \equiv c_1 t_{1,1} \equiv -t_{0,2}$. Taking $t_{0,1} \equiv 1$, $t_{1,0} = 0$, and $t_{0,2} = 0$, the matrix $x$ becomes

$$v_1 = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 3c_1 \\ 0 & -3 & 0 \end{bmatrix}.$$ 

Taking $t_{1,0} \equiv 1$, $t_{0,1} = 0$, and $t_{0,2} = 0$, the matrix $x$ becomes

$$v_2 = \begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$ 

Taking $t_{0,2} \equiv 1$, $t_{0,1} = 0$, and $t_{1,0} = 0$, the matrix $x$ becomes

$$v_3 = \begin{bmatrix} 0 & 0 & 1 \\ 0 & -c_1 & 3(c_3 - c_1) \\ 0 & 0 & -3 \end{bmatrix}.$$ 

We see that neither $v_3$ nor $3v_3$ is contained in $W_2$ and that $\partial^{-1}W_2 = \langle \partial^{-1}W_0, v_1, v_2, v_3 \rangle$. We know that $v_3 \in \partial^{-1}W_2$ and since $3v_3 \not\in W_2$ then $v_3 + W_2$ is an element of the group $\partial^{-1}W_2/W_2$ whose order is larger than 3. Recall that $|\partial^{-1}W_2/W_2| = 3^4$ and that $\partial^{-1}W_1/W_2$ has rank 2. Since $\partial^{-1}W_2$ has an element of order larger than 3 then rank($\partial^{-1}W_2/W_2$) is not greater than 2 since $\partial^{-1}W_2/W_2 \cong \mathbb{Z}_9 \times \mathbb{Z}_3$ and therefore $W_2$ is terminal and $W_2 \not\in \hat{L}_2$. 

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Case 4.2.2.4

We fix arbitrary values $c_3 \in \{0, 1, 2\}$ and $c_4 \in \{1, 2\}$. There are 6 ways to choose the values $c_3$ and $c_4$. Let $m_3 = y_1 + c_3 y_3 + c_4 v_1$ and $m_4 = v_2$. Thus

$$m_3 = \begin{bmatrix} 0 & c_4 & 3c_3 \\ 0 & 0 & 3c_1 c_4 \\ 3 & -3c_4 & 0 \end{bmatrix} \quad \text{and} \quad m_4 = \begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

Let $W_2 = \langle W_1, m_3, m_4 \rangle \in \mathcal{L}_2$. The number of subgroups $W_2$ of this type is 6. Since $m_3, m_4 \notin W_1$ and $3m_1, 3m_4 \in W_1$ we have $|W_2/W_1| = 3^2$. Since $|W_1| = 3^5$ it follows that $|W_2| = 3^7$. Note that $\partial^{-1} W_1/W_2 \cong \mathbb{Z}_3 \times \mathbb{Z}_3$ and rank($\partial^{-1} W_1/W_2$) = 2. We now calculate the pullback of $\partial^{-1} W_2$. We observed in case 4.2.2 that $\partial^{-1} W_2$ is contained in the pattern subgroup

$$\begin{bmatrix} 2 & 2 & 2 \\ 2 & 2 & 1 \\ 2 & 1 & 1 \end{bmatrix}.$$

Let

$$x = \begin{bmatrix} 3s_{0,0} + t_{0,0} & 3s_{0,1} + t_{0,1} & 3s_{0,2} + t_{0,2} \\ 3s_{1,0} + t_{1,0} & 3s_{1,1} + t_{1,1} & 3s_{1,2} \\ 3s_{2,0} + t_{2,0} & 3s_{2,1} & 3s_{2,2} \end{bmatrix} \in \begin{bmatrix} 2 & 2 & 2 \\ 2 & 2 & 1 \\ 2 & 1 & 1 \end{bmatrix}.$$

The variables that are in play are those appearing in the matrix

$$\begin{bmatrix} 0 & t_{0,1} & t_{0,2} \\ t_{1,0} & t_{1,1} & 3s_{1,2} \\ t_{2,0} & 3s_{2,1} & 3s_{2,2} \end{bmatrix}.$$
In the notation of Case 4.2.2, we are taking $d_1 = 1$, $d_2 = c_3$, $d_3 = c_4, d_4 = 0$, $e_2 = 0$, $e_3 = 0$, and $e_4 = 1$. We wish to identify values $a_1, a_2, a_3, a_4 \in \mathbb{Z}_9$ such that
\[
\partial_1 x \equiv a_1 m_1 + a_2 m_2 + a_3 m_3 + a_4 m_4 \pmod{I}.
\]

We see that $\partial_1 x \in W_2$ if and only if
\[
\begin{align*}
t_{1,1} &\equiv c_4 a_4 \quad (A1) \\
t_{2,0} &\equiv a_4 \quad (A2) \\
s_{2,2} &\equiv c_1 t_{1,1} \quad (A3) \\
s_{1,2} &\equiv c_3 a_3 - c_1 s_{2,1} \quad (A4) \\
t_{2,0} &\equiv -a_3 \quad (A5).
\end{align*}
\]

We wish to identify values $b_1, b_2, b_3, b_4 \in \mathbb{Z}_9$ such that $\partial_2 x \equiv b_1 m_1 + b_2 m_2 + b_3 m_3 + b_4 m_4 \pmod{I}$.

We see that $\partial_2 x \in W_2$ if and only if
\[
\begin{align*}
t_{0,2} &\equiv c_4 b_3 \quad (B1) \\
t_{0,2} &\equiv -s_{2,2} \quad (B2) \\
t_{1,1} &\equiv b_4 \quad (B3) \\
t_{0,2} &\equiv -c_1 t_{1,1} \quad (B4) \\
s_{1,2} &\equiv c_1 t_{0,1} + c_1 t_{0,2} + c_1 c_3 b_3 \quad (B5) \\
s_{2,1} &\equiv b_3 - t_{0,1} \quad (B6).
\end{align*}
\]

The congruences (A1) and (B1) can be rewritten as $a_3 \equiv c_4 t_{1,1}$ and $b_3 \equiv c_4 t_{0,2}$.

Substituting $b_3$ into (B6) we obtain $s_{2,1} \equiv c_4 t_{0,2} - t_{0,1}$. Substituting $a_3$ and (B6) into
(A4) we obtain $s_{1,2} \equiv -c_1c_3c_4t_{0,2} - c_1c_4t_{0,2} + c_1t_{0,1}$. Also, substituting $b_3$ into (B5) we obtain $s_{1,2} \equiv c_1t_{0,1} + c_1t_{0,2} + c_1c_3c_4t_{0,2}$.

Hence $x \in \partial^{-1}W_2$ if and only if

$$s_{2,2} \equiv c_1t_{1,1} \quad \text{(A3)}$$
$$s_{1,2} \equiv -c_1c_3c_4t_{0,2} - c_1c_4t_{0,2} + c_1t_{0,1} \quad \text{(A4)}$$
$$t_{0,2} \equiv c_4b_3 \quad \text{(B1)}$$
$$t_{0,2} \equiv -s_{2,2} \quad \text{(B2)}$$
$$t_{0,2} \equiv -c_1t_{1,1} \quad \text{(B4)}$$
$$s_{1,2} \equiv c_1t_{0,1} + c_1t_{0,2} + c_1c_3c_4t_{0,2} \quad \text{(B5)}$$
$$s_{2,1} \equiv b_3 - t_{0,1} \quad \text{(B6)}.$$

Combining our new (A4) and (B5) we obtain $0 \equiv (c_1 + c_1c_4 - c_1c_3c_4)t_{0,2}$. Multiplying by $c_1$ we obtain $0 \equiv (1 + c_4 - c_3c_4)t_{0,2}$. Therefore it is convenient to consider the cases $(1 + c_4 - c_3c_4) \neq 0$ and $(1 + c_4 - c_3c_4) = 0$ separately.

**Case 4.2.2.4.1** First we consider the case $(1 + c_4 - c_3c_4) \neq 0$. Then $t_{0,2} \equiv 0$.

From congruence (B1) we see that $b_3 \equiv 0$ since $c_4 \in \{1, 2\}$. Since $t_{0,2} \equiv 0$, then $s_{2,2} \equiv t_{1,1} \equiv t_{0,2} \equiv 0$. The congruences (A4) and (B5) become $s_{1,2} \equiv c_1t_{0,1}$. Therefore (B5) is redundant with (A4). Substituting $b_3 \equiv 0$, the congruence (B6) becomes $s_{2,1} \equiv -t_{0,1}$. 

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Hence $x \in \partial^{-1}W_2$ if and only if

\[
\begin{align*}
    t_{1,1} & \equiv 0 \\
    t_{2,0} & \equiv 0 \\
    t_{0,2} & \equiv 0 \\
    s_{2,2} & \equiv 0 \\
    s_{1,2} & \equiv c_1 t_{0,1} \\
    s_{2,1} & \equiv -t_{0,1}.
\end{align*}
\]

These are the same congruences as those in Case 4.2 and therefore we conclude that $\partial^{-1}W_2 = \partial^{-1}W_1$. Thus $\text{rank}(\partial^{-1}W_2/W_2) = \text{rank}(\partial^{-1}W_1/W_2)$, $W_2$ is terminal, and $W_2 \not\in \hat{L}_2$.

**Case 4.2.2.4.2** Now we consider the case $(1 + c_4 - c_3 c_4) = 0$. Then $c_3 \equiv c_4 + 1$ and $t_{0,2}$ is a free variable. The congruences (A4) and (B5) become $s_{1,2} \equiv -c_1 t_{0,2} + c_1 c_4 t_{0,2} + c_1 t_{0,1}$ and $s_{1,2} \equiv c_1 t_{0,1}$ respectively.
Hence \( x \in \partial^{-1}W_2 \) if and only if

\[
\begin{align*}
    s_{2,2} & \equiv c_1 t_{1,1} \quad \text{(A3)} \\
    s_{1,2} & \equiv -c_1 t_{0,2} + c_1 c_4 t_{0,2} + c_1 t_{0,1} \quad \text{(A4)} \\
    t_{0,2} & \equiv c_4 b_3 \quad \text{(B1)} \\
    t_{0,2} & \equiv -s_{2,2} \quad \text{(B2)} \\
    t_{0,2} & \equiv -c_1 t_{1,1} \quad \text{(B4)} \\
    s_{1,2} & \equiv c_1 t_{0,1} \quad \text{(B5)} \\
    s_{2,1} & \equiv b_3 - t_{0,1} \quad \text{(B6)}.
\end{align*}
\]

We regard \( t_{0,1}, t_{1,0} \), and \( t_{0,2} \) as free variables. Note that the congruences (A3), (B2), and (B4) give us \( s_{2,2} \equiv c_1 t_{1,1} \equiv -t_{0,2} \). Taking \( t_{0,1} \equiv 1, t_{1,0} = 0 \), and \( t_{0,2} = 0 \), the matrix \( x \) becomes

\[
v_1 = \begin{bmatrix}
0 & 1 & 0 \\
0 & 0 & 3c_1 \\
0 & -3 & 0
\end{bmatrix}.
\]

Taking \( t_{1,0} \equiv 1, t_{0,1} = 0 \), and \( t_{0,2} = 0 \), the matrix \( x \) becomes

\[
v_2 = \begin{bmatrix}
0 & 0 & 0 \\
1 & 0 & 0 \\
0 & 0 & 0
\end{bmatrix}.
\]

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Taking \( t_{0,2} \equiv 1, t_{0,1} = 0, \) and \( t_{1,0} = 0, \) the matrix \( x \) becomes

\[
v_3 = \begin{bmatrix}
0 & 0 & 1 \\
0 & -c_1 & 0 \\
0 & 3c_4 & -3
\end{bmatrix}.
\]

We see that neither \( v_3 \) nor \( 3v_3 \) is contained in \( W_2 \) and that

\[
\partial^{-1}W_2 = < \partial^{-1}W_0, v_1, v_2, v_3 >.
\]

We know that \( v_3 \in \partial^{-1}W_2 \) and since \( 3v_3 \notin W_2 \) then \( v_3 + W_2 \) is an element of the group \( \partial^{-1}W_2/W_2 \) whose order is larger than 3.

Recall that \(|\partial^{-1}W_2/W_2| = 3^4\) and that \( \partial^{-1}W_1/W_2 \) has rank 2. Since \( \partial^{-1}W_2 \) has an element of order larger than 3 then \( \text{rank}(\partial^{-1}W_2/W_2) \) is not greater than 2 since \( \partial^{-1}W_2/W_2 \cong \mathbb{Z}_9 \times \mathbb{Z}_3 \) and therefore \( W_2 \) is terminal and \( W_2 \notin \hat{\mathcal{L}}_2 \).

Case 4.2.2.5

We fix arbitrary values \( c_3, c_5 \in \{0, 1, 2\} \) and \( c_4 \in \{1, 2\} \). There are 18 ways to choose the values \( c_3, c_4, c_5 \). Let \( m_3 = y_1 + c_3y_3 + c_4v_2 \) and \( m_4 = v_1 + c_5v_2 \). Thus

\[
m_3 = \begin{bmatrix}
0 & 0 & 3c_4 \\
c_4 & 0 & 0 \\
3 & 0 & 0
\end{bmatrix}, \text{ and } m_4 = \begin{bmatrix}
0 & 1 & 0 \\
c_5 & 0 & 3c_4 \\
0 & -3 & 0
\end{bmatrix}.
\]

Let \( W_2 = < W_1, m_3, m_4 > \in \mathcal{L}_2 \). The number of subgroups \( W_2 \) of this type is 18.

Since \( m_3, m_4 \notin W_1 \) and \( 3m_1, 3m_4 \in W_1 \) we have \(|W_2/W_1| = 3^2\). Since \(|W_1| = 3^5\) it follows that \(|W_2| = 3^7\). Note that \( \partial^{-1}W_1/W_2 \cong \mathbb{Z}_3 \times \mathbb{Z}_3 \) and \( \text{rank}(\partial^{-1}W_1/W_2) = 2 \).

We now calculate the pullback of \( \partial^{-1}W_2 \). We observed in case 4.2.2 that \( \partial^{-1}W_2 \) is
contained in the pattern subgroup

\[
\begin{pmatrix}
2 & 2 & 2 \\
2 & 2 & 1 \\
2 & 1 & 1
\end{pmatrix}
\]

Let

\[
x = \begin{pmatrix}
3s_{0,0} + t_{0,0} & 3s_{0,1} + t_{0,1} & 3s_{0,2} + t_{0,2} \\
3s_{1,0} + t_{1,0} & 3s_{1,1} + t_{1,1} & 3s_{1,2} \\
3s_{2,0} + t_{2,0} & 3s_{2,1} & 3s_{2,2}
\end{pmatrix}
\in \begin{pmatrix}
2 & 2 & 2 \\
2 & 2 & 1 \\
2 & 1 & 1
\end{pmatrix}.
\]

The variables that are in play are those appearing in the matrix

\[
\begin{pmatrix}
0 & t_{0,1} & t_{0,2} \\
t_{1,0} & t_{1,1} & 3s_{1,2} \\
t_{2,0} & 3s_{2,1} & 3s_{2,2}
\end{pmatrix}
\]

In the notation of Case 4.2.2, we are taking \(d_1 = 1, d_2 = c_3, d_3 = 0, d_4 = c_4, e_2 = 0, e_3 = 1, \) and \(e_4 = c_5.\)

We wish to identify values \(a_1, a_2, a_3, a_4 \in \mathbb{Z}_9\) such that \(\partial_1 x \equiv a_1 m_1 + a_2 m_2 + a_3 m_3 + a_4 m_4 \pmod{I}.\)
We see that $\partial_1 x \in W_2$ if and only if

$$t_{1,1} \equiv a_4 \quad \text{(A1)}$$

$$t_{2,0} \equiv c_4 a_3 + c_5 a_4 \quad \text{(A2)}$$

$$s_{2,2} \equiv c_1 t_{1,1} \quad \text{(A3)}$$

$$s_{1,2} \equiv c_3 a_3 - c_1 s_{2,1} \quad \text{(A4)}$$

$$t_{2,0} \equiv -a_3 \quad \text{(A5)}.$$

We wish to identify values $b_1, b_2, b_3, b_4 \in \mathbb{Z}_9$ such that $\partial_2 x \equiv b_1 m_1 + b_2 m_2 + b_3 m_3 + b_4 m_4 \, (\text{mod } I)$.

We see that $\partial_2 x \in W_2$ if and only if

$$t_{0,2} \equiv b_4 \quad \text{(B1)}$$

$$t_{0,2} \equiv -s_{2,2} \quad \text{(B2)}$$

$$t_{1,1} \equiv c_4 b_3 + c_5 b_4 \quad \text{(B3)}$$

$$t_{0,2} \equiv -c_1 t_{1,1} \quad \text{(B4)}$$

$$s_{1,2} \equiv c_1 t_{0,1} + c_1 t_{0,2} + c_1 c_3 b_3 \quad \text{(B5)}$$

$$s_{2,1} \equiv b_3 - t_{0,1} \quad \text{(B6)}.$$

Using (A5), (A1), and (B4) to rewrite (A2) we obtain $t_{2,0} \equiv -c_4 t_{2,0} - c_1 c_5 t_{0,2}$ which becomes $(c_4 + 1)t_{2,0} \equiv -c_1 c_5 t_{0,2}$. Using (B4), (B6), and (B1) we can rewrite (B3) as $-c_1 t_{0,2} \equiv c_4 t_{0,1} + c_4 s_{2,1} + c_5 t_{0,2}$ which becomes $s_{2,1} \equiv -t_{0,1} - c_4 (c_1 + c_5) t_{0,2}$. We can use this last congruence to write (B6) as $b_3 \equiv -c_4 (c_1 + c_5) t_{0,2}$. Now (B5) becomes $s_{1,2} \equiv c_1 t_{0,1} + c_1 t_{0,2} - c_1 c_4 c_3 t_{0,2}$ which says $s_{1,2} \equiv c_1 t_{0,1} + (c_1 - c_3 c_4 - c_1 c_3 c_4) t_{0,2}$.  

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For convenience let \( q \equiv c_4 + c_1 c_4 c_5 - c_1 + c_3 c_4 + c_1 c_3 c_4 c_5 \). Using (B5) and (B3) we obtain \( s_{1,2} + c_1 s_{2,1} \equiv -qt_{0,2} \). Using this last congruence along with (A5) we write (A4) as \(-c_3 t_{2,0} \equiv -qt_{0,2} \) which says \( c_3 t_{2,0} \equiv qt_{0,2} \).

Hence \( x \in \partial^{-1} W_2 \) if and only if

\[
(c_4 + 1)t_{2,0} \equiv -c_1 c_5 t_{0,2} \quad \text{(A2)}
\]

\[
c_3 t_{2,0} \equiv qt_{0,2} \text{ where } q = c_4 + c_1 c_4 c_5 - c_1 + c_3 c_4 + c_1 c_3 c_4 c_5 \quad \text{(A4)}
\]

\[
s_{2,2} \equiv -t_{0,2} \quad \text{(B2)}
\]

\[
s_{2,1} \equiv -t_{0,1} - c_4(c_1 + c_5)t_{0,2} \quad \text{(B3)}
\]

\[
t_{1,1} \equiv -c_1 t_{0,2} \quad \text{(B4)}
\]

\[
s_{1,2} \equiv c_1 t_{0,1} + (c_1 - c_3 c_4 - c_1 c_3 c_4 c_5)t_{0,2} \quad \text{(B5).}
\]

**Lemma 7.2.1.** Let \( w = \begin{bmatrix} 0 & 0 & 3f_3 \\ 0 & 3f_2 & 0 \\ 3f_1 & 0 & 0 \end{bmatrix} \) for some unknowns \( f_1, f_2, f_3 \in \{0, 1, 2\} \).

The condition \( w \in W_2 \) holds if and only if \( f_1 \equiv 0 \) and \( f_3 \equiv -c_1 f_2 \).

**Proof.** Let \( w = \begin{bmatrix} 0 & 0 & 3f_3 \\ 0 & 3f_2 & 0 \\ 3f_1 & 0 & 0 \end{bmatrix} \). We want to find conditions on \( f_1, f_2, f_3 \) for which \( w \in W_2 \) or equivalently \( w \equiv a_1 m_1 + a_2 m_2 + a_3 m_3 + a_4 m_4 (\text{mod} I) \). Comparing (0,1)-entries, we get \( a_4 \equiv 0 \). Comparing (1,0)-entries, we get \( c_4 a_3 + c_5 a_4 \equiv 0 \). Since \( a_4 \equiv 0 \), we get \( c_4 a_3 \equiv 0 \). Since \( c_4^2 \equiv 1 \), we get \( a_3 \equiv 0 \). Comparing (1,1)-entries, we get \( f_2 \equiv a_1 \). Comparing (0,0)-entries, we get \( c_1 a_1 + a_2 \equiv 0 \). Since \( a_1 \equiv f_2 \) this becomes
$c_1 f_2 + a_2 \equiv 0$, which may be written as $a_2 \equiv -c_1 f_2$. Comparing $(2, 0)$-entries, we get $f_1 \equiv -c_1 a_1 - a_2 + a_3$. Since $a_3 \equiv 0$ this becomes $f_1 \equiv -c_1 a_1 - a_2$. Recalling that $a_1 \equiv f_2$ and $a_2 \equiv -c_1 f_2$, this becomes $f_1 \equiv -c_1 f_2 + c_1 f_2 \equiv 0$ so $f_1 \equiv 0$. Comparing $(0, 2)$-entries, we get $f_3 \equiv -c_1 a_1 + c_3 a_3$. Since $a_3 \equiv 0$ and $a_1 \equiv f_2$ this is equivalent to $f_3 \equiv -c_1 f_2$. Hence we conclude that $w \in W_2$ is and only if $f_1 \equiv 0$ and $f_3 \equiv -c_1 f_2$. \( \square \)

It is convenient to consider the cases $c_3 \neq 0$ and $c_3 = 0$ separately.

**Case 4.2.2.5.1** Suppose $c_3 \neq 0$. Thus $c_3 \in \{1, 2\}$ and $c_3^2 \equiv 1$. (A4) becomes $t_{2,0} \equiv c_3 q t_{0,2}$. Substitution of this last congruence allows us to rewrite (A2) as $(c_4 + 1)c_3 q t_{0,2} \equiv -c_1 c_5 t_{0,2}$. For convenience write $r = (c_4 + 1)c_3 q + c_1 c_5$. Thus (A2) becomes $r t_{0,2} \equiv 0$.

Hence $x \in \partial^{-1} W_2$ if and only if

$$rt_{0,2} \equiv 0 \text{ where } r = (c_4 + 1)c_3 q + c_1 c_5 \quad \text{(A2)}$$

$$t_{2,0} \equiv c_3 q t_{0,2} \text{ where } q \equiv c_1 c_4 c_5 - c_1 + c_3 c_4 + c_1 c_3 c_4 c_5 \quad \text{(A4)}$$

$$s_{2,2} \equiv -t_{0,2} \quad \text{(B2)}$$

$$s_{2,1} \equiv -t_{0,1} - c_4(c_1 + c_5) t_{0,2} \quad \text{(B3)}$$

$$t_{1,1} \equiv -c_1 t_{0,2} \quad \text{(B4)}$$

$$s_{1,2} \equiv c_1 t_{0,1} + (c_1 - c_3 c_4 - c_1 c_3 c_4 c_5) t_{0,2} \quad \text{(B5)}.$$

We will consider the following three cases separately. The first case is when $r \neq 0$. The second case is when $r \equiv 0$ and $q \neq 0$. The third case is when $r \equiv 0$ and $q \equiv 0$.
Case 4.2.2.5.1.1 Suppose \( r \neq 0 \). Now (A2) yields \( t_{0,2} \equiv 0 \).

Hence \( x \in \partial^{-1}W_2 \) if and only if

\[
\begin{align*}
t_{0,2} & \equiv 0 \\
t_{2,0} & \equiv 0 \\
s_{1,2} & \equiv c_1 t_{0,1} \\
s_{2,2} & \equiv 0 \\
s_{2,1} & \equiv -t_{0,1} \\
t_{1,1} & \equiv 0.
\end{align*}
\]

These are the same congruences as those in Case 4.2 and therefore we conclude that \( \partial^{-1}W_2 = \partial^{-1}W_1 \). Thus \( \text{rank}(\partial^{-1}W_2/W_2) = \text{rank}(\partial^{-1}W_1/W_2) \), \( W_2 \) is terminal, and \( W_2 \not\in \hat{L}_2 \).

Case 4.2.2.5.1.2 Suppose \( r \equiv 0 \) and \( q \neq 0 \). Now (A2) is automatically satisfied and may be ignored.

Hence \( x \in \partial^{-1}W_2 \) if and only if

\[
\begin{align*}
t_{2,0} & \equiv c_3 q t_{0,2} \quad (A4) \\
s_{2,2} & \equiv -t_{0,2} \quad (B2) \\
s_{2,1} & \equiv -t_{0,1} - c_4 (c_1 + c_3) t_{0,2} \quad (B3) \\
t_{1,1} & \equiv -c_1 t_{0,2} \quad (B4) \\
s_{1,2} & \equiv c_1 t_{0,1} + (c_1 - c_3 c_4 - c_1 c_3 c_4 c_5) t_{0,2} \quad (B5).
\end{align*}
\]
We regard $t_{0,1}, t_{1,0},$ and $t_{0,2}$ as free variables. Taking $t_{0,1} \equiv 1$, $t_{1,0} = 0$, and $t_{0,2} = 0$, the matrix $x$ becomes

$$v_1 = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 3c_1 \\ 0 & -3 & 0 \end{bmatrix}.$$  

Taking $t_{1,0} \equiv 1$, $t_{0,1} = 0$, and $t_{0,2} = 0$, the matrix $x$ becomes

$$v_2 = \begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$  

Taking $t_{0,2} \equiv 1$, $t_{0,1} = 0$, and $t_{1,0} = 0$, the matrix $x$ becomes

$$v_3 = \begin{bmatrix} 0 & 0 & 1 \\ 0 & -c_1 & 3(c_1 - c_3c_4 - c_1c_3c_4c_5) \\ c_3q & -3c_4(c_1 + c_5) & 0 \end{bmatrix}.$$  

Thus $\partial^{-1}W_2 = \langle \partial^{-1}W_0, v_1, v_2, v_3 \rangle$. Recall that $\partial^{-1}W_1 = \langle \partial^{-1}W_0, v_1, v_2 \rangle$. Thus $\partial^{-1}W_2 = \langle \partial^{-1}W_1, v_3 \rangle$. Note that $v_3 \notin \partial^{-1}W_1$ but that

$$3v_3 = \begin{bmatrix} 0 & 0 & 3 \\ 0 & -3c_1 & 0 \\ 3c_3q & 0 & 0 \end{bmatrix} \in \partial^{-1}W_1.$$  

Thus $|\partial^{-1}W_2/\partial^{-1}W_1| = 3$. Recall $|\partial^{-1}W_1| = 3^9$. Hence $|\partial^{-1}W_2| = 3^{10}$. Recall $|W_2| = 3^7$. Hence $|\partial^{-1}W_2/W_2| = 3^3$. Recall that $\partial^{-1}W_1/W_2 \cong \mathbb{Z}_3 \times \mathbb{Z}_3$. Since $\partial^{-1}W_2/W_2$ has order $3^3$ and contains $\partial^{-1}W_1/W_2$ as a subgroup, we have that
\( \partial^{-1}W_2/W_2 \) is isomorphic to either \( \mathbb{Z}_3 \times \mathbb{Z}_3 \times \mathbb{Z}_3 \) or \( \mathbb{Z}_9 \times \mathbb{Z}_3 \). From Lemma 7.2.1, we see that \( 3v_3 \not\in W_2 \), which tells us that \( v_3 + W_2 \) is an element of order 9 in the group \( \partial^{-1}W_2/W_2 \). Hence \( \partial^{-1}W_2/W_2 \cong \mathbb{Z}_9 \times \mathbb{Z}_3 \) and \( \text{rank}(\partial^{-1}W_2/W_2) = 2 \). Recalling that \( \text{rank}(\partial^{-1}W_1/W_2) = 2 \), we deduce that \( \text{rank}(\partial^{-1}W_1/W_2) = \text{rank}(\partial^{-1}W_2/W_2) \), \( W_2 \) is terminal, and \( W_2 \not\in \hat{L}_2 \).

**Case 4.2.2.5.1.3** Suppose \( r \equiv 0 \) and \( q \equiv 0 \). We shall now argue that these assumptions imply \( c_3 = 1, c_4 \equiv -c_1, \) and \( c_5 = 0 \). Since \( r = (c_4 + 1)c_3q + c_1c_5 \) while \( r \equiv 0 \) and \( q \equiv 0 \), we get \( c_1c_5 \equiv 0 \). Since \( c_1 \in \{1, 2\} \) we have \( c_1^2 \equiv 1 \), and so \( c_5 \equiv 0 \). Recalling that \( c_5 \in \{0, 1, 2\} \), we deduce that \( c_5 = 0 \), as desired. Since \( c_5 = 0 \) we get \( q = c_4 - c_1 + c_3c_4 \). Because \( q \equiv 0 \) we thus have \( c_4 - c_1 + c_3c_4 \equiv 0(\star) \). Since we are assuming \( c_3 \neq 0 \), we have \( c_3 \in \{1, 2\} \). If \( c_3 = 2 \) then \( (\star) \) becomes \( c_1 \equiv 0 \), which is a contradiction since we know \( c_3 \in \{1, 2\} \). Therefore we have \( c_3 = 1 \), as desired. Because \( c_3 = 1 \), \( (\star) \) becomes \( c_4 \equiv -c_1 \), as desired. Thus we have \( c_3 = 1, c_4 \equiv -c_1, \) and \( c_5 = 0 \), as desired. Because \( r \equiv 0 \), (A2) holds automatically and may be ignored.

Because \( q \equiv 0 \), (A4) becomes \( t_{2,0} \equiv 0 \).
Hence $x \in \partial^{-1}W_2$ if and only if

\[
\begin{align*}
t_{2,0} &\equiv 0 \\
 s_{2,2} &\equiv -t_{0,2} \\
 s_{2,1} &\equiv -t_{0,1} + t_{0,2} \\
 t_{1,1} &\equiv -c_1 t_{0,2} \\
 s_{1,2} &\equiv c_1 t_{0,1} - c_1 t_{0,2}.
\end{align*}
\]

We regard $t_{0,1}, t_{1,0},$ and $t_{0,2}$ as free variables. Taking $t_{0,1} \equiv 1$, $t_{1,0} = 0$, and $t_{0,2} = 0$, the matrix $x$ becomes

\[
v_1 = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 3c_1 \\ 0 & -3 & 0 \end{bmatrix}.
\]

Taking $t_{1,0} \equiv 1$, $t_{0,1} = 0$, and $t_{0,2} = 0$, the matrix $x$ becomes

\[
v_2 = \begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}.
\]

Taking $t_{0,2} \equiv 1$, $t_{0,1} = 0$, and $t_{1,0} = 0$, the matrix $x$ becomes

\[
v_3 = \begin{bmatrix} 0 & 0 & 1 \\ 0 & -c_1 & -3c_1 \\ 0 & 3 & 0 \end{bmatrix}.
\]
Thus $\partial^{-1}W_2 = \langle \partial^{-1}W_0, v_1 v_2, v_3 \rangle$. Recall $\partial^{-1}W_1 = \langle \partial^{-1}W_0, v_1, v_2 \rangle$. Thus

$\partial^{-1}W_2 = \langle \partial^{-1}W_1, v_3 \rangle$. Note that $v_3 \notin W_1$ but

$$3v_3 = \begin{bmatrix} 0 & 0 & 3 \\ 0 & -3c_1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \in \partial^{-1}W_1.$$

Thus $|\partial^{-1}W_2 \partial^{-1}W_1| = 3$. Recall $|\partial^{-1}W_1| = 3^9$. Hence $|\partial^{-1}W_2| = 3^{10}$. Recall $|W_2| = 3^7$. Hence $|\partial^{-1}W_2/W_2| = 3^3$. Recall that $\partial^{-1}W_1/W_2 \cong \mathbb{Z}_3 \times \mathbb{Z}_3$. Since $\partial^{-1}W_2/W_2$ has order $3^3$ and contains $\partial^{-1}W_1/W_2$ as a subgroup, indeed $\partial^{-1}W_2/W_2$ is isomorphic to $\mathbb{Z}_3 \times \mathbb{Z}_3 \times \mathbb{Z}_3$ or $\mathbb{Z}_9 \times \mathbb{Z}_3$. Note that

$$-c_1m_1 + m_2 = c_1 \begin{bmatrix} c_1 & 0 & -3c_1 \\ 0 & 3 & 0 \\ -3c_1 & 0 & 0 \end{bmatrix} + \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ -3 & 0 & 0 \end{bmatrix} = \begin{bmatrix} -c_1^2 & 0 & 3 \\ 0 & -3c_1 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

We see that $3v_3 \equiv -c_1m_1 + m_2 \pmod{I}$. Thus $3v_3 \in W_2$. We deduce that $\partial^{-1}W_2/W_2 \cong \mathbb{Z}_3 \times \mathbb{Z}_3 \times \mathbb{Z}_3$ and so $\text{rank}(\partial^{-1}W_2/W_2) = 3$. Recall $\text{rank}(\partial^{-1}W_1/W_2) = 2$. Thus $\text{rank}(\partial^{-1}W_1/W_2) < \text{rank}(\partial^{-1}W_2/W_2)$ and $W_2$ is nonterminal. It appears that the vector space $\partial^{-1}W_2/W_2$ has basis $y_1 + W_2, y_3 + W_2, v_3 + W_2$ while its subspace $\partial^{-1}W_1/W_2$ has basis $y_1 + W_2, y_3 + W_2$.

Let $c_6, c_7 \in \{0, 1, 2\}$. Let $m_5 = c_6y_1 + c_7y_3 + v_3$. Thus

$$m_5 = \begin{bmatrix} 0 & 0 & 1 + 3c_7 \\ 0 & -c_1 & -3c_1 \\ 3c_6 & 3 & -3 \end{bmatrix}.$$
Let $W_3 = <W_2, m_5>$. The number of subgroups of this type is 9. Since $m_5 \not\in W_2$ while $3m_5 \in W_2$ we have $|W_3/W_2| = 3$. Recalling $|W_2| = 3^7$ we get $|W_3| = 3^8$. Since $|W_3/W_2| = 3$ and the antidiagonal of $m_5$ has an entry not divisible by 3, by the Modified Terminal Lemma we conclude that $\partial^{-1}W_3 = \partial^{-1}W_2$. Thus $\text{rank}(\partial^{-1}W_3/W_3) = \text{rank}(\partial^{-1}W_2/W_3)$, $W_3$ is terminal, and $W_3 \not\in \hat{L}_3$.

**Case 4.2.2.5.2** Suppose $c_3 = 0$. Thus $q = c_4 + c_1c_4c_5 - c_1$. Hence $x \in \partial^{-1}W_2$ if and only if

\[(c_4 + 1)t_{2,0} \equiv -c_1c_5t_{0,2} \quad (A2)\]
\[qt_{0,2} \equiv 0 \quad (A4)\]
\[s_{2,2} \equiv -t_{0,2} \quad (B2)\]
\[s_{2,1} \equiv -t_{0,1} - c_4(c_1 + c_5)t_{0,2} \quad (B3)\]
\[t_{1,1} \equiv -c_1t_{0,2} \quad (B4)\]
\[s_{1,2} \equiv c_1t_{0,1} + c_1t_{0,2} \quad (B5).\]

Recall that $c_4 \in \{1, 2\}$. It is convenient to consider the cases $c_4 = 1$ and $c_4 = 2$ separately.

**Case 4.2.2.5.2.1** Suppose $c_4 = 1$. Thus $q = 1 + c_1c_5 - c_1$ and (A2) becomes

$t_{2,0} \equiv c_1c_5t_{0,2}$. 

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Hence \( x \in \partial^{-1}W_2 \) if and only if

\[
\begin{align*}
t_{2,0} & \equiv c_1c_5 t_{0,2} \quad \text{(A2)} \\
qt_{0,2} & \equiv 0 \quad \text{(A4)} \\
s_{2,2} & \equiv -t_{0,2} \quad \text{(B2)} \\
s_{2,1} & \equiv -t_{0,1} - (c_1 + c_5)t_{0,2} \quad \text{(B3)} \\
t_{1,1} & \equiv -c_1 t_{0,2} \quad \text{(B4)} \\
s_{1,2} & \equiv c_1 t_{0,1} + c_1 t_{0,2} \quad \text{(B5)}.
\end{align*}
\]

We consider the cases \( q \not\equiv 0 \) and \( q \equiv 0 \) separately.

**Case 4.2.5.2.1.1** Suppose \( q \not\equiv 0 \). Then (A4) is equivalent to \( t_{0,2} \equiv 0 \). Hence \( x \in \partial^{-1}W_2 \) if and only if

\[
\begin{align*}
t_{2,0} & \equiv 0 \\
t_{0,2} & \equiv 0 \\
s_{2,2} & \equiv 0 \\
s_{2,1} & \equiv -t_{0,1} \\
t_{1,1} & \equiv 0 \\
s_{1,2} & \equiv c_1 t_{0,1}.
\end{align*}
\]

These are the same congruences as those in Case 4.2. Therefore \( \partial^{-1}W_2 = \partial^{-1}W_1 \).

Thus \( \text{rank}(\partial^{-1}W_2/W_2) = \text{rank}(\partial^{-1}W_1/W_2) \), \( W_2 \) is terminal, and \( W_2 \not\in \mathcal{L}_2 \).
Case 4.2.2.5.2.1.2 Suppose \( q \equiv 0 \). Thus \( 1 + c_1 c_5 - c_1 \equiv 0 \). We argue that \( (c_1, c_5) \) is equal to either \((1, 0)\) or \((2, 2)\), from which it will follow that \( c_1 + c_5 \equiv 1 \) so that (B3) becomes \( s_{2,1} \equiv -t_{0,1} - t_{0,2} \). Recall \( c_1 \in \{1, 2\} \). If \( c_1 = 1 \), then \( 1 + c_5 - 1 \equiv 0 \) and so \( c_5 = 0 \), as desired. If \( c_1 = 2 \), then \( 1 - c_5 + 1 \equiv 0 \) and so \( c_5 \equiv 2 \) and \( c_5 = 2 \), as desired. Since \( q \equiv 0 \), congruence (A4) is automatic and may be ignored.

Hence \( x \in \partial^{-1} W_2 \) if and only if

\[
\begin{align*}
t_{2,0} &\equiv c_1 c_5 t_{0,2} \quad \text{(A2)} \\
s_{2,2} &\equiv -t_{0,2} \quad \text{(B2)} \\
s_{2,1} &\equiv -t_{0,1} - t_{0,2} \quad \text{(B3)} \\
t_{1,1} &\equiv -c_1 t_{0,2} \quad \text{(B4)} \\
s_{1,2} &\equiv c_1 t_{0,1} + c_1 t_{0,2} \quad \text{(B5)}.
\end{align*}
\]

We now consider the cases \((c_1, c_5) = (2, 2)\) and \((c_1, c_5) = (1, 0)\) separately.

Case 4.2.2.5.2.1.2.1 Suppose \((c_1, c_5) = (2, 2)\).

Hence \( x \in \partial^{-1} W_2 \) if and only if

\[
\begin{align*}
t_{2,0} &\equiv t_{0,2} \quad \text{(A2)} \\
s_{2,2} &\equiv -t_{0,2} \quad \text{(B2)} \\
s_{2,1} &\equiv -t_{0,1} - t_{0,2} \quad \text{(B3)} \\
t_{1,1} &\equiv t_{0,2} \quad \text{(B4)} \\
s_{1,2} &\equiv -t_{0,1} - t_{0,2} \quad \text{(B5)}.
\end{align*}
\]

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We regard $t_{0,1}, t_{1,0},$ and $t_{0,2}$. Taking $t_{0,1} \equiv 1$, $t_{1,0} = 0$, and $t_{0,2} = 0$, the matrix $x$ becomes

$$v_1 = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & -3 \\ 0 & -3 & 0 \end{bmatrix}.$$ 

Taking $t_{1,0} \equiv 1$, $t_{0,1} = 0$, and $t_{0,2} = 0$, the matrix $x$ becomes

$$v_2 = \begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$ 

Taking $t_{0,2} \equiv 1$, $t_{0,1} = 0$, and $t_{1,0} = 0$, the matrix $x$ becomes

$$v_3 = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & -3 \\ 1 & -3 & -3 \end{bmatrix}.$$ 

By Lemma 7.2.1, we see that $3v_3 \not\in W_2$. Hence $v_3 + W_2$ is an element of order 9. Hence $\partial^{-1}W_2/W_2 \cong \mathbb{Z}_9 \times \mathbb{Z}_3$. Thus $\text{rank}(\partial^{-1}W_2/W_2) = \text{rank}(\partial^{-1}W_1/W_2)$, $W_2$ is terminal, and $W_2 \not\in \hat{L}_2$. 

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Case 4.2.5.2.1.2.2  Suppose \((c_1, c_5) = (1, 0)\). Hence \(x \in \partial^{-1}W_2\) if and only if

\[
\begin{align*}
t_{2,0} &\equiv 0 \quad \text{(A2)} \\
s_{2,2} &\equiv -t_{0,2} \quad \text{(B2)} \\
s_{2,1} &\equiv -t_{0,1} - t_{0,2} \quad \text{(B3)} \\
t_{1,1} &\equiv -t_{0,2} \quad \text{(B4)} \\
s_{1,2} &\equiv t_{0,1} + t_{0,2} \quad \text{(B5)}. 
\end{align*}
\]

We regard \(t_{0,1}, t_{1,0}, \text{ and } t_{0,2}\). Taking \(t_{0,1} \equiv 1, t_{1,0} = 0, \text{ and } t_{0,2} = 0\), the matrix \(x\) becomes

\[
v_1 = \begin{bmatrix}
0 & 1 & 0 \\
0 & 0 & 3 \\
0 & -3 & 0
\end{bmatrix}.
\]

Taking \(t_{1,0} \equiv 1, t_{0,1} = 0, \text{ and } t_{0,2} = 0\), the matrix \(x\) becomes

\[
v_2 = \begin{bmatrix}
0 & 0 & 0 \\
1 & 0 & 0 \\
0 & 0 & 0
\end{bmatrix}.
\]

Taking \(t_{0,2} \equiv 1, t_{0,1} = 0, \text{ and } t_{1,0} = 0\), the matrix \(x\) becomes

\[
v_3 = \begin{bmatrix}
0 & 0 & 1 \\
0 & -1 & 3 \\
1 & -3 & -3
\end{bmatrix}.
\]
Note

\[
m_2 - m_1 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ -3 & 0 & 0 \end{bmatrix} - \begin{bmatrix} 1 & 0 & -3 \\ 0 & 3 & 0 \\ -3 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 3 \\ 0 & -3 & 0 \\ 0 & 0 & 0 \end{bmatrix} = 3v_3,
\]

which says \(3v_3 \in W_1 \subseteq W_2\). Hence element \(v_3 + W_2\) has order 3. We get \(\partial^{-1}W_2/W_2 \cong \mathbb{Z}_3 \times \mathbb{Z}_3 \times \mathbb{Z}_3\) so \(\text{rank}(\partial^{-1}W_2/W_2) = 2 < 3 = \text{rank}(\partial^{-1}W_1/W_2)\). So \(W_2\) is nonterminal.

The basis for \(\partial^{-1}W_2/W_2\) is \(y_1 + W_2, y_3 + W_2, v_3 + W_2\) while basis for \(\partial^{-1}W_1/W_2\) is \(y_1 + W_2, y_3 + W_2\).

Let \(c_6, c_7 \in \{0, 1, 2\}\). Let \(m_5 = c_6y_1 + c_7y_3 + v_3\). Thus

\[
m_5 = \begin{bmatrix} 0 & 0 & 1 + 3c_7 \\ 0 & -1 & 3 \\ 3c_6 & -3 & -3 \end{bmatrix}.
\]

Let \(W_3 =< W_2, m_5 >\). The number of subgroups of this type is 9. Since \(m_5 \not\in W_2\) while \(3m_5 = m_2 - m_1 \in W_2\) we have \(|W_3/W_2| = 3\). Recalling \(|W_2| = 3^7\) we get \(|W_3| = 3^8\). Since \(|W_3/W_2| = 3\) and the antidiagonal of \(m_5\) has an entry not divisible by 3, by the Modified Terminal Lemma we conclude that \(\partial^{-1}W_3 = \partial^{-1}W_2\). Thus \(\text{rank}(\partial^{-1}W_3/W_3) = \text{rank}(\partial^{-1}W_2/W_3)\), \(W_3\) is terminal, and \(W_3 \not\in \hat{L}_3\).

**Case 4.2.2.5.2.2** Suppose \(c_4 = 2\). Thus \(q \equiv -1 - c_1c_5 - c_1\). Furthermore, (A2) and (B3) are equivalent to \(c_1c_5t_{0,2} \equiv 0\) and \(s_{2,1} \equiv -t_{0,1} + (c_1 + c_5)t_{0,2}\) respectively.
Hence \( x \in \partial^{-1}W_2 \) if and only if

\[
c_1 c_5 t_{0,2} \equiv 0 \quad \text{(A2)}
\]
\[
q t_{0,2} \equiv 0 \quad \text{(A4)}
\]
\[
s_{2,2} \equiv -t_{0,2} \quad \text{(B2)}
\]
\[
s_{2,1} \equiv -t_{0,1} + (c_1 + c_5) t_{0,2} \quad \text{(B3)}
\]
\[
t_{1,1} \equiv -c_1 t_{0,2} \quad \text{(B4)}
\]
\[
s_{1,2} \equiv c_1 t_{0,1} + c_1 t_{0,2} \quad \text{(B5)}.
\]

Using \( q \equiv -1 - c_1 c_5 - c_1 \), we now argue that \( q \equiv 0 \) if and only if \((c_1, c_5)\) is equal to either \((1, 1)\) or \((2, 0)\). Recall \( c_1 \in \{1, 2\} \). If \( c_1 = 1 \), then \( 1 + c_1 c_5 + c_1 \equiv 0 \) is equivalent to \( 1 + c_5 + 1 \equiv 0 \) which says \( c_5 \equiv 1 \), as desired. If \( c_1 = 2 \), then \( 1 + c_1 c_5 + c_1 \equiv 0 \) is equivalent to \( 1 - c_5 + 2 \equiv 0 \) which says \( c_5 \equiv 0 \), as desired. We now consider the cases \((c_1, c_5) \neq (2, 0)\) and \((c_1, c_5) = (2, 0)\) separately.

**Case 4.2.2.5.2.2.1** Suppose \((c_1, c_5) \neq (2, 0)\). We now show that (A2) and (A4) together are equivalent to \( t_{0,2} \equiv 0 \). For this we examine the cases \( c_5 \neq 0 \) and \( c_4 = 0 \) separately. First suppose \( c_5 \neq 0 \). Since \( c_5 \in \{0, 1, 2\} \) and \( c_1 \in \{1, 2\} \) we obtain \( c_1 c_5 \neq 0 \) and so (A2) is equivalent to \( t_{0,2} \equiv 0 \), as desired. Now suppose \( c_5 = 0 \). By hypothesis it follows that \( c_1 \neq 2 \). Since \( c_1 \neq 2 \) and \( c_5 = 0 \), the paragraph preceding this case tells us that \( q \neq 0 \). Thus (A4) is equivalent to \( t_{0,2} \equiv 0 \), as desired.
Hence \( x \in \partial^{-1}W_2 \) if and only if

\[
\begin{align*}
t_{0,2} &\equiv 0 \\
s_{2,2} &\equiv 0 \\
s_{2,1} &\equiv -t_{0,1} \\
t_{1,1} &\equiv 0 \\
s_{1,2} &\equiv c_1 t_{0,1}.
\end{align*}
\]

We regard \( t_{0,1}, t_{1,0}, \) and \( t_{2,0} \) as free variables. Taking \( t_{0,1} \equiv 1, t_{1,0} = 0, \) and \( t_{2,0} = 0, \) the matrix \( x \) becomes

\[
v_1 = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 3 \\ 0 & -3 & 0 \end{bmatrix}.
\]

Taking \( t_{1,0} \equiv 1, t_{0,1} = 0, \) and \( t_{2,0} = 0, \) the matrix \( x \) becomes

\[
v_2 = \begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}.
\]

Taking \( t_{2,0} \equiv 1, t_{0,1} = 0, \) and \( t_{1,0} = 0, \) the matrix \( x \) becomes

\[
v_3 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix}.
\]

From Lemma 7.2.1 we see that \( 3v_3 \not\in W_2. \) Hence element \( v_3 + W_2 \) is an element of order 9. Hence \( \partial^{-1}W_2/W_2 \cong \mathbb{Z}_9 \times \mathbb{Z}_3 \) and \( \text{rank}(\partial^{-1}W_2/W_2) = 2. \) Recalling
that \( \text{rank}(\partial^{-1}W_1/W_2) = 2 \), we deduce that \( \text{rank} (\partial^{-1}W_1/W_2) = \text{rank}(\partial^{-1}W_2/W_2) \),

\( W_2 \) is terminal, and \( W_2 \not\in \hat{L}_2 \).

**Case 4.2.2.5.2.2.2** Suppose \((c_1, c_5) = (2, 0)\). Thus \( q \equiv 0 \) and so \((A4)\) is automatic.

Since \( c_5 = 0 \), congruence \((A2)\) is automatic. Hence \( x \in \partial^{-1}W_2 \) if and only if

\[
\begin{align*}
s_{2,2} &\equiv -t_{0,2} \quad \text{(B2)} \\
s_{2,1} &\equiv -t_{0,1} - t_{0,2} \quad \text{(B3)} \\
t_{1,1} &\equiv t_{0,2} \quad \text{(B4)} \\
s_{1,2} &\equiv -t_{0,1} - t_{0,2} \quad \text{(B5)}. 
\end{align*}
\]

We regard \( t_{0,1}, t_{1,0}, t_{0,2}, \) and \( t_{2,0} \) as free variables. Taking \( t_{0,1} \equiv 1, t_{1,0} = 0, t_{0,2} = 0, \) and \( t_{2,0} = 0 \), the matrix \( x \) becomes

\[
v_1 = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & -3 \\ 0 & -3 & 0 \end{bmatrix}
\]

Taking \( t_{1,0} \equiv 1, t_{0,1} = 0, t_{0,2} = 0, \) and \( t_{2,0} = 0 \), the matrix \( x \) becomes

\[
v_2 = \begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}
\]

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Taking \( t_{0,2} \equiv 1, t_{0,1} = 0, t_{1,0} = 0 \), and \( t_{2,0} = 0 \), the matrix \( x \) becomes

\[
v_3 = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & -3 \\ 0 & -3 & -3 \end{bmatrix}.
\]

Taking \( t_{2,0} \equiv 1, t_{0,1} = 0, t_{1,0} = 0 \), and \( t_{0,2} = 0 \), the matrix \( x \) becomes

\[
v_4 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix}.
\]

Note

\[
m_1 + m_2 = \begin{bmatrix} 2 & 3 & 0 \\ 0 & 3 & 0 \\ 3 & 0 & 0 \end{bmatrix} + \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ -3 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 3 & 0 & 3 \\ 0 & 3 & 0 \\ 0 & 0 & 0 \end{bmatrix} \cong \begin{bmatrix} 0 & 0 & 3 \\ 0 & 3 & 0 \\ 0 & 0 & 0 \end{bmatrix} = 3v_3.
\]

Hence \( 3v_3 \in W_2 \) and so \( v_3 + W_2 \) is an element of order 3. But since \( 3v_4 = y_1 \) by Lemma 7.2.1 we get \( 3v_4 \not\in W_2 \) so \( v_4 + W_2 \) is an element of order 9. So \( \partial^{-1}W_2/W_2 \cong \mathbb{Z}_9 \times \mathbb{Z}_3 \times \mathbb{Z}_3 \). So \( \text{rank}(\partial^{-1}W_1/W_2) = 2 < 3 = \text{rank}(\partial^{-1}W_2/W_2) \) so \( W_2 \) is nonterminal.

Note \( \Omega_1(\partial^{-1}W_2/W_2) \cong \mathbb{Z}_3 \times \mathbb{Z}_3 \times \mathbb{Z}_3 \). Note \( \Omega_1(\partial^{-1}W_2/W_2) \) has basis \( y_1 + W_1, y_3 + W_2, v_3 + W_2 \) while its subspace \( \partial^{-1}W_1/W_2 \) has basis \( y_1 + W_1, y_3 + W_2 \).

Let \( c_6, c_7 \in \{0, 1, 2\} \). Let \( m_5 = c_6y_1 + c_7y_3 + v_3 \). Thus

\[
m_5 = \begin{bmatrix} 0 & 0 & 1 + 3c_7 \\ 0 & 1 & -3 \\ 3c_6 & -3 & -3 \end{bmatrix}.
\]
Let $W_3 = \langle W_2, m_5 \rangle$. The number of subgroups of this type is 9. Since $m_5 \not\in W_2$ while $3m_5 \in W_2$ we have $|W_3/W_2| = 3$. Recalling $|W_2| = 3^7$ we get $|W_3| = 3^8$. Since $|W_3/W_2| = 3$ and the antidiagonal of $m_5$ has an entry not divisible by 3, by the Modified Terminal Lemma we conclude that $\partial^{-1}W_3 = \partial^{-1}W_2$. Thus $\text{rank}(\partial^{-1}W_3/W_3) = \text{rank}(\partial^{-1}W_2/W_3)$, $W_3$ is terminal, and $W_3 \not\in \mathcal{L}_3$.

Case 4.2.2.6

We fix arbitrary values $c_3, c_4, c_5, c_6$ where $\{(c_3, c_4), (c_5, c_6)\}$ is an ordered pair of linearly independent vectors in $\mathbb{Z}_3 \times \mathbb{Z}_3$. There are 48 ways to choose the values these ordered pairs. Let $m_3 = y_1 + c_3v_1 + c_4v_2$ and $m_4 = y_3 + c_5v_1 + c_6v_2$. Thus

$$m_3 = \begin{bmatrix} 0 & c_3 & 0 \\ c_4 & 0 & 3c_1c_3 \\ 3 & -3c_3 & 0 \end{bmatrix} \quad \text{and} \quad m_4 = \begin{bmatrix} 0 & c_5 & 3 \\ c_6 & 0 & 3c_1c_5 \\ 0 & -3c_5 & 0 \end{bmatrix}.$$ 

Let $W_2 = \langle W_1, m_3, m_4 \rangle \in \mathcal{L}_2$. The number of subgroups $W_2$ of this type is 48. Since $m_3, m_4 \not\in W_1$ and $3m_1, 3m_4 \in W_1$ we have $|W_2/W_1| = 3^2$. Since $|W_1| = 3^5$ it follows that $|W_2| = 3^7$. Note that $\partial^{-1}W_1/W_2 \cong \mathbb{Z}_3 \times \mathbb{Z}_3$ and $\text{rank}(\partial^{-1}W_1/W_2) = 2$.

We now calculate the pullback of $\partial^{-1}W_2$. We observed in case 4.2.2 that $\partial^{-1}W_2$ is contained in the pattern subgroup

$$\begin{bmatrix} 2 & 2 & 2 \\ 2 & 2 & 1 \\ 2 & 1 & 1 \end{bmatrix}.$$
Let
\[
x = \begin{bmatrix}
3s_{0,0} + t_{0,0} & 3s_{0,1} + t_{0,1} & 3s_{0,2} + t_{0,2} \\
3s_{1,0} + t_{1,0} & 3s_{1,1} + t_{1,1} & 3s_{1,2} \\
3s_{2,0} + t_{2,0} & 3s_{2,1} & 3s_{2,2}
\end{bmatrix}
\in \begin{bmatrix}
2 & 2 & 2 \\
2 & 2 & 1 \\
2 & 1 & 1
\end{bmatrix}.
\]

The variables that are in play are those appearing in the matrix
\[
\begin{bmatrix}
0 & t_{0,1} & t_{0,2} \\
t_{1,0} & t_{1,1} & 3s_{1,2} \\
t_{2,0} & 3s_{2,1} & 3s_{2,2}
\end{bmatrix}.
\]

In the notation of Case 4.2.2, we are taking \( d_1 = 1, \) \( d_2 = 0, \) \( d_3 = c_3, d_4 = c_4, \)
\( e_2 = 1, \) \( e_3 = c_5, \) and \( e_4 = c_6. \) We wish to identify values \( a_1, a_2, a_3, a_4 \in \mathbb{Z}_9 \) such that
\[
\partial_1 x \equiv a_1 m_1 + a_2 m_2 + a_3 m_3 + a_4 m_4 \pmod{I}.
\]

We see that \( \partial_1 x \in W_2 \) if and only if

1. \( t_{1,1} \equiv c_3 a_3 + c_5 a_4 \quad (A1) \)
2. \( t_{2,0} \equiv c_4 a_3 + c_6 a_4 \quad (A2) \)
3. \( s_{2,2} \equiv c_1 t_{1,1} \quad (A3) \)
4. \( s_{1,2} \equiv a_4 - c_1 s_{2,1} \quad (A4) \)
5. \( t_{2,0} \equiv -a_3 \quad (A5). \)

We wish to identify values \( b_1, b_2, b_3, b_4 \in \mathbb{Z}_9 \) such that \( \partial_2 x \equiv b_1 m_1 + b_2 m_2 + b_3 m_3 + b_4 m_4 \pmod{I}. \)
We see that $\partial_2 x \in W_2$ if and only if

$$t_{0,2} \equiv c_3 b_3 + c_5 b_4 \quad (B1)$$

$$t_{0,2} \equiv -s_{2,2} \quad (B2)$$

$$t_{1,1} \equiv c_4 b_3 + c_6 b_4 \quad (B3)$$

$$t_{0,2} \equiv -c_1 t_{1,1} \quad (B4)$$

$$s_{1,2} \equiv c_1 t_{0,1} + c_1 t_{0,2} + c_1 b_4 \quad (B5)$$

$$s_{2,1} \equiv b_3 - t_{0,1} \quad (B6).$$

Rewriting (A4) and (A5) we obtain $a_4 \equiv s_{1,2} + c_1 s_{2,1}$ and $a_3 \equiv -t_{2,0}$. Substituting $a_3, a_4,$ and (B4) into (A1) we obtain $c_3 t_{2,0} - c_1 t_{0,2} \equiv c_5 (s_{1,2} + c_1 s_{2,1})$. Substituting $a_3$ and $a_4$ into (A2) we obtain $(c_4 + 1)t_{2,0} \equiv c_6 (s_{1,2} + c_1 s_{2,1})$. Rewriting (B5) and (B6) we obtain $b_4 \equiv c_1 s_{1,2} - t_{0,1} - t_{0,2}$ and $b_3 \equiv s_{2,1} + t_{0,1}$. Substituting $b_3$ and $b_4$ into (B1) we obtain $c_3 s_{2,1} + c_1 c_5 s_{1,2} \equiv (c_5 - c_3) t_{0,1} + (c_5 + 1)t_{0,2}$. Substituting $b_3, b_4$ and (B4) into (B3) we obtain $c_4 s_{2,1} + c_1 c_6 s_{1,2} \equiv (c_6 - c_4) t_{0,1} + (c_6 - c_1)t_{0,2}.$

Hence $x \in \partial^{-1}W_2$ if and only if

$$c_3 t_{2,0} - c_1 t_{0,2} \equiv c_5 (s_{1,2} + c_1 s_{2,1}) \quad (A1)$$

$$(c_4 + 1)t_{2,0} \equiv c_6 (s_{1,2} + c_1 s_{2,1}) \quad (A2)$$

$$c_3 s_{2,1} + c_1 c_5 s_{1,2} \equiv (c_5 - c_3) t_{0,1} + (c_5 + 1)t_{0,2} \quad (B1)$$

$$s_{2,2} \equiv -t_{0,2} \quad (B2)$$

$$c_4 s_{2,1} + c_1 c_6 s_{1,2} \equiv (c_6 - c_4) t_{0,1} + (c_6 - c_1)t_{0,2} \quad (B3)$$

$$t_{0,2} \equiv -c_1 t_{0,2} \quad (B4).$$
Let \{ (c_3, c_4), (c_5, c_6) \} be a pair of linearly independent vectors in \( \mathbb{Z}_3 \times \mathbb{Z}_3 \).

Hence

\[
\begin{bmatrix}
  c_3 & c_4 \\
  c_5 & c_6 
\end{bmatrix} \in GL(2, 3).
\]

Thus the determinant is not equivalent to zero, \( c_3 c_6 - c_4 c_5 \neq 0 \). There are three cases where this holds. The first case is \( c_3 c_6 \neq 0 \) and \( c_4 c_5 = 0 \) which has 20 possibilities.

The second case is \( c_3 c_6 \equiv 0 \) and \( c_4 c_5 \neq 0 \) which has 20 possibilities. The third case is \( 0 \neq c_3 c_6 \neq c_4 c_5 \neq 0 \) which has 8 possibilities.

Lemma 7.2.2. Let \( w = \begin{bmatrix}
  0 & 0 & 3f_3 \\
  0 & 3f_2 & 0 \\
  3f_1 & 0 & 0 
\end{bmatrix} \) for unknowns \( f_1, f_2, f_3 \in \{0, 1, 2\} \). The condition \( w \in W_2 \) holds if and only if \( f_1 \equiv 0 \) and \( f_3 \equiv -c_1 f_2 \).

Proof. Let \( w = \begin{bmatrix}
  0 & 0 & 3f_3 \\
  0 & 3f_2 & 0 \\
  3f_1 & 0 & 0 
\end{bmatrix} \) for unknowns \( f_1, f_2, f_3 \in \{0, 1, 2\} \). We want to find conditions on \( f_1, f_2, f_3 \) for which \( w \in W_2 \) or equivalently \( w \equiv a_1 m_1 + a_2 m_2 + a_3 m_3 + a_4 m_4 (\text{mod } I) \). Recall that \( a_1 m_1 + a_2 m_2 + a_3 m_3 + a_4 m_4 (\text{mod } I) \)

\[
= \begin{bmatrix}
  c_1 a_1 + a_2 & c_3 a_3 + c_5 a_4 & 3(-c_1 a_1 + a_4) \\
  c_4 a_3 + c_6 a_4 & 3a_1 & 3(c_1 c_3 a_3 + c_1 c_5 a_4) \\
  3(-c_1 a_1 - a_2 + a_3) & 3(-c_3 a_3 - c_5 a_4) & 0 
\end{bmatrix}.
\]

Comparing (0,1)-entries we get, \( c_3 a_3 + c_5 a_4 \equiv 0 \). Comparing (1,0)-entries, we get \( c_4 a_3 + c_6 a_4 \equiv 0 \). The preceding pair of congruences is equivalent to the matrix
equation \[
\begin{bmatrix}
c_3 & c_5 \\
c_4 & c_6
\end{bmatrix}
\begin{bmatrix}
a_3 \\
a_4
\end{bmatrix} = \begin{bmatrix}
0 \\
0
\end{bmatrix}
\] working in the field with 3 elements. This says \[
\begin{bmatrix}
a_3 \\
a_4
\end{bmatrix}
\] is in nullspace of the matrix. Since the columns of the matrix are linearly independent, the nullspace is trivial. Thus we obtain \(a_3 \equiv 0\) and \(a_4 \equiv 0\). Comparing \((1, 1)\)-entries, we get \(f_2 \equiv a_1\). Comparing \((0, 0)\)-entries, we get \(c_1a_1 + a_2 \equiv 0\). Since \(a_1 \equiv f_2\) this says \(a_2 \equiv -c_1f_2\). Comparing \((2, 0)\)-entries, we get \(f_1 \equiv -c_1a_1 - a_2 + a_3\). Substituting \(a_1, a_2, a_3\) we get \(f_1 \equiv -c_1f_2 - (-c_1f_2)\). Therefore \(f_1 \equiv 0\). Comparing \((0, 2)\)-entries, we get \(f_3 \equiv -c_1a_1 + a_4\). Substituting \(a_1, a_4\) we get \(f_3 \equiv -c_1f_2\). Thus we conclude that \(w \in W_2\) if and only if \(f_1 \equiv -\) and \(f_3 \equiv -c_1f_2\). \(\square\)

**Case 4.2.2.6.1** Suppose \(c_3c_6 \neq 0\) and \(c_4c_5 \equiv 0\). It is convenient to consider the cases \(c_5 \equiv 0\) and \(c_5 \neq 0\) separately.

**Case 4.2.2.6.1.1** Suppose \(c_3 \neq 0, c_6 \neq 0, \text{ and } c_5 = 0\).

Hence \(x \in \partial^{-1}W_2\) if and only if

\[
t_{2,0} \equiv c_1c_3t_{0,2} \quad \text{(A1)}
\]

\[(c_4 + 1)t_{2,0} \equiv c_6(s_{1,2} + c_1s_{2,1}) \quad \text{(A2)}
\]

\[c_3s_{2,1} \equiv -c_3t_{0,1} + t_{0,2} \quad \text{(B1)}
\]

\[s_{2,2} \equiv -t_{0,2} \quad \text{(B2)}
\]

\[c_4s_{2,1} + c_1c_6s_{1,2} \equiv (c_6 - c_4)t_{0,1} + (c_6 - c_1)t_{0,2} \quad \text{(B3)}
\]

\[t_{0,2} \equiv -c_1t_{0,2} \quad \text{(B4)}.
\]
The congruence (B1) becomes $s_{2,1} \equiv -t_{0,1} + c_3 t_{0,2}$. (B3) becomes $c_1 c_6 s_{1,2} \equiv (c_6 - c_4) t_{0,1} + (c_6 - c_1) t_{0,2} - c_4 (-t_{0,1} + c_3 t_{0,2})$ which says $c_1 c_6 s_{1,2} \equiv c_6 t_{0,1} + (c_6 - c_1 - c_3 c_4) t_{0,2}$. Therefore $s_{1,2} \equiv c_1 t_{0,1} + (c_1 - c_6 - c_1 c_3 c_4 c_6) t_{0,2}$. Note $s_{1,2} + c_1 s_{2,1} \equiv q t_{0,2}$ where $q \equiv c_1 - c_6 - c_1 c_3 c_4 c_6 + c_1 c_3$. Then (A2) becomes $(c_4 + 1) c_1 c_3 t_{0,2} \equiv c_6 q t_{0,2}$ which says $r t_{0,2} \equiv 0$ where $r = (c_4 + 1) c_1 c_3 - c_6 q$. It is convenient to consider the cases $r \not\equiv 0$ and $r \equiv 0$ separately.

**Case 4.2.2.6.1.1.1** Suppose $r \not\equiv 0$. Then (A2) forces $t_{0,2} \equiv 0$. Then (A1) forces $t_{2,0} \equiv 0$. Hence $t_{0,2} \equiv t_{2,0} \equiv s_{2,2} \equiv t_{1,1} \equiv 0$, $s_{1,2} \equiv c_1 t_{0,1}$ and $s_{2,1} \equiv -t_{0,1}$.

These are the same congruences as those in Case 4.2 and therefore we conclude that $\partial^{-1} W_2 = \partial^{-1} W_1$. Thus $\text{rank}(\partial^{-1} W_2/W_2) = \text{rank}(\partial^{-1} W_1/W_2)$, $W_2$ is terminal, and $W_2 \not\in \hat{L}_2$.

**Case 4.2.2.6.1.1.2** Suppose $r \equiv 0$. Then (A2) holds automatically and $t_{0,2}$ is a new free variable. We regard $t_{0,1}$, $t_{1,0}$, and $t_{0,2}$ as free variables. Taking $t_{0,1} \equiv 1$, $t_{1,0} = 0$, and $t_{0,2} = 0$, the matrix $x$ becomes

$$v_1 = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 3 c_1 \\ 0 & -3 & 0 \end{bmatrix}.$$
Taking $t_{1,0} \equiv 1$, $t_{0,1} = 0$, and $t_{0,2} = 0$, the matrix $x$ becomes

\[
v_2 = \begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}.
\]

Taking $t_{0,2} \equiv 1$, $t_{0,1} = 0$, and $t_{1,0} = 0$, the matrix $x$ becomes

\[
v_3 = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 3(c_1 - c_6 - c_1c_3c_4c_6) \\ c_1c_3 & 3c_3 & -3 \end{bmatrix}.
\]

Since $c_1c_3 \neq 0$, then from Lemma 7.2.2 we see that neither $v_3$ nor $3v_3$ is contained in $W_2$ and that $\partial^{-1}W_2 = \langle \partial^{-1}W_0, v_1, v_2, v_3 \rangle$. We know that $v_3 \in \partial^{-1}W_2$ and since $3v_3 \notin W_2$ then $v_3 + W_2$ is an element of the group $\partial^{-1}W_2/W_2$ whose order is larger than 3. Recall that $|\partial^{-1}W_2/W_2| = 3^4$ and that $\partial^{-1}W_1/W_2$ has rank 2. Since $\partial^{-1}W_2$ has an element of order larger than 3 then $\text{rank}(\partial^{-1}W_2/W_2)$ is not greater than 2 since $\partial^{-1}W_2/W_2 \cong \mathbb{Z}_9 \times \mathbb{Z}_3$ and therefore $W_2$ is terminal and $W_2 \notin \mathcal{L}_2$. 

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Case 4.2.2.6.1.2 Now consider the case \( c_5 \neq 0 \). Thus \( c_4 = 0 \), \( c_3 \neq 0 \), and \( c_6 \neq 0 \).

Hence \( x \in \partial^{-1}W_2 \) if and only if

\[
\begin{align*}
    &c_3 t_{2,0} - c_1 t_{0,2} \equiv c_5 (s_{1,2} + c_1 s_{2,1}) \quad \text{(A1)} \\
    &t_{2,0} \equiv c_6 (s_{1,2} + c_1 s_{2,1}) \quad \text{(A2)} \\
    &c_3 s_{2,1} + c_1 c_5 s_{1,2} \equiv (c_5 - c_3) t_{0,1} + (c_5 + 1) t_{0,2} \quad \text{(B1)} \\
    &s_{2,2} \equiv -t_{0,2} \quad \text{(B2)} \\
    &c_1 c_6 s_{1,2} \equiv c_6 t_{0,1} + (c_6 - c_1) t_{0,2} \quad \text{(B3)} \\
    &t_{0,2} \equiv -c_1 t_{0,2} \quad \text{(B4)}.
\end{align*}
\]

Multiplying (B3) by \( c_1 c_6 \) we obtain \( s_{1,2} \equiv c_1 t_{0,1} + (c_1 - c_6) t_{0,2} \). Substituting (B3) into (B1) we obtain \( s_{2,1} \equiv -t_{0,1} + c_3 (1 + c_1 c_5 c_6) t_{0,2} \). Note that \( s_{1,2} + c_1 s_{2,1} \equiv [c_1 - c_6 + c_1 c_3 (1 + c_1 c_5 c_6)] t_{0,2} \). Thus \( s_{1,2} + c_1 s_{2,1} \equiv q t_{0,2} \) where \( q = c_1 - c_6 + c_1 c_3 + c_3 c_5 c_6 \).

Then (A1) becomes \( t_{2,0} \equiv c_3 (c_1 + c_5 q) t_{0,2} \) and (A2) becomes \( t_{2,0} \equiv c_6 q t_{0,2} \). Combining (A1) and (A2) we get \( (c_1 c_3 + c_3 c_5 q - c_6 q) t_{0,2} \equiv 0 \), which we call the new (A1). For convenience let \( r = c_1 c_3 + c_3 c_5 q - c_6 q \). Thus (A1) becomes \( r t_{0,2} \equiv 0 \).
Hence \( x \in \partial^{-1}W_2 \) if and only if

\[
rt_{0,2} \equiv 0 \quad \text{(A1)}
\]

\[
t_{2,0} \equiv c_6qt_{0,2} \quad \text{(A2)}
\]

\[
s_{2,1} \equiv -t_{0,1} + c_3(1 + c_1c_5c_6)t_{0,2} \quad \text{(B1)}
\]

\[
s_{2,2} \equiv -t_{0,2} \quad \text{(B2)}
\]

\[
s_{1,2} \equiv c_1t_{0,1} + (c_1 - c_6)t_{0,2} \quad \text{(B3)}
\]

\[
t_{0,2} \equiv c_1t_{0,2} \quad \text{(B4)}.
\]

We will consider the cases \( r \neq 0 \) and \( r \equiv 0 \) separately.

**Case 4.2.2.6.1.2.1** Suppose \( r \neq 0 \). Thus (A1) says \( t_{0,2} \equiv 0 \). Now using (A2) we get \( t_{2,0} \equiv 0 \). Hence \( t_{0,2} \equiv t_{2,0} \equiv s_{2,2} \equiv t_{1,1} \equiv 0 \), \( s_{1,2} \equiv c_1t_{0,1} \) and \( s_{2,1} \equiv -t_{0,1} \). These are the same congruences as those in Case 4.2 and therefore we conclude that \( \partial^{-1}W_2 = \partial^{-1}W_1 \). Thus \( \text{rank}(\partial^{-1}W_2/W_2) = \text{rank}(\partial^{-1}W_1/W_2) \), \( W_2 \) is terminal, and \( W_2 \notin \mathcal{L}_2 \).

**Case 4.3.3.6.1.2.2** Suppose \( r \equiv 0 \). Thus (A1) holds automatically so we may ignore it. Thus \( t_{0,2} \) is a new free variable. We may now argue that \( q \neq 0 \). For if \( q \equiv 0 \) then using \( r = c_1c_3 + c_3c_5q - c_6q \) we obtain \( r \equiv c_1c_3 \neq 0 \) which is contrary to hypothesis that \( r \equiv 0 \). Hence \( q \neq 0 \). Recall that \( c_6 \neq 0 \) and that (A2) says \( t_{2,0} \equiv c_6qt_{0,2} \). We regard \( t_{0,1}, t_{1,0}, \) and \( t_{2,0} \) as free variables. Taking \( t_{0,1} \equiv 1, t_{1,0} = 0, \) and \( t_{2,0} = 0, \) the
matrix $x$ becomes

$$v_1 = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 3c_1 \\ 0 & -3 & 0 \end{bmatrix}.$$  

Taking $t_{1,0} \equiv 1$, $t_{0,1} = 0$, and $t_{2,0} = 0$, the matrix $x$ becomes

$$v_2 = \begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$  

Taking $t_{2,0} \equiv 1$, $t_{0,1} = 0$, and $t_{1,0} = 0$, the matrix $x$ becomes

$$v_3 = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 3(c_1 - c_6) \\ c_6q & 3c_3(1 + c_1c_5c_6) & -3 \end{bmatrix}.$$  

From Lemma 7.2.2 we see that neither $v_3$ nor $3v_3$ is contained in $W_2$ and that $\partial^{-1}W_2 = < \partial^{-1}W_0, v_1, v_2, v_3 >$. We know that $v_3 \in \partial^{-1}W_2$ and since $3v_3 \notin W_2$ then $v_3 + W_2$ is an element of the group $\partial^{-1}W_2/W_2$ whose order is larger than 3. Recall that $|\partial^{-1}W_2/W_2| = 3^4$ and that $\partial^{-1}W_1/W_2$ has rank 2. Since $\partial^{-1}W_2$ has an element of order larger than 3 then rank($\partial^{-1}W_2/W_2$) is not greater than 2 since $\partial^{-1}W_2/W_2 \cong \mathbb{Z}_9 \times \mathbb{Z}_3$ and therefore $W_2$ is terminal and $W_2 \notin \hat{\mathcal{L}}_2$.

**Case 4.2.6.2** Suppose $c_3c_6 \equiv 0$ and $c_4c_5 \neq 0$. It is convenient to consider the cases $c_3 \equiv 0$ and $c_3 \neq 0$ separately.
Case 4.2.2.6.2.1 First we consider the case $c_3 = 0$. Since $c_4c_5 \neq 0$ then $c_4 \neq 0$ and $c_5 \neq 0$. Hence $x \in \partial^{-1}W_2$ if and only if

$$-c_1t_{0,2} \equiv c_5(s_{1,2} + c_1s_{2,1}) \quad \text{(A1)}$$

$$(c_4 + 1)t_{2,0} \equiv c_6(s_{1,2} + c_1s_{2,1}) \quad \text{(A2)}$$

$$s_{2,2} \equiv c_1t_{1,1} \quad \text{(A3)}$$

$$c_1c_5s_{1,2} \equiv c_5t_{0,1} + (c_5 + 1)t_{0,2} \quad \text{(B1)}$$

$$s_{2,2} \equiv -t_{0,2} \quad \text{(B2)}$$

$$c_4s_{2,1} + c_1c_6s_{1,2} \equiv (c_6 - c_4)t_{0,1} + (c_6 - c_1)t_{0,2} \quad \text{(B3)}$$

$$t_{0,2} \equiv -c_1t_{0,2} \quad \text{(B4)}.$$

Since $c_1, c_4, c_5 \in \{1, 2\}$, the congruence (B1) becomes $s_{1,2} \equiv c_1t_{0,1} + (c_1c_5 + c_1)t_{0,2}$. Substituting (B1) into (B3) we obtain $s_{2,1} \equiv -t_{0,1} - c_4(c_1 + c_5c_6)t_{0,2}$. Note that $s_{1,2} + c_1s_{2,1} \equiv (c_1c_5 + c_1 - c_4 - c_1c_4c_5c_6)t_{0,2}$. For convenience we write $q = c_1c_5 + c_1 - c_4 - c_1c_4c_5c_6$. Then (A1) becomes $-c_1t_{0,2} \equiv c_5qt_{0,2}$ which can we written as $(c_5q + c_1)t_{0,2} \equiv 0$. (A2) becomes $(c_4 + 1)t_{2,0} \equiv c_6qt_{0,2}$. We can write $r = c_5q + c_1$. Note that $r = c_1c_5 - c_1c_4c_5 - c_1c_4c_6$. Then (A1) becomes $rt_{0,2} \equiv 0$. 

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Hence \( x \in \partial^{-1}W_2 \) if and only if

\[
rt_{0,2} \equiv 0 \quad \text{(A1)}
\]
\[
(c_4 + 1)t_{2,0} \equiv c_6qt_{0,2} \quad \text{(A2)}
\]
\[
s_{1,2} \equiv c_1t_{0,1} + (c_1c_5 + c_1)t_{0,2} \quad \text{(B1)}
\]
\[
s_{2,2} \equiv -t_{0,2} \quad \text{(B2)}
\]
\[
s_{2,1} \equiv -t_{0,1} - c_4(c_1 + c_5c_6)t_{0,2} \quad \text{(B3)}
\]
\[
t_{1,1} \equiv -c_1t_{0,2} \quad \text{(B4)}.
\]

It is convenient to consider the following six cases separately. The first case is \( r \not\equiv 0 \), and \( c_4 = 1 \). The second case is \( r \not\equiv 0 \) and \( c_4 = 2 \). The third case is \( r \equiv 0 \), \( c_4 = 1 \), and \( c_6 = 0 \). The fourth case is \( r \equiv 0 \), \( c_4 = 2 \), and \( c_6 = 0 \). The fifth case is \( r \equiv 0 \), \( c_4 = 1 \), and \( c_6 \in \{1, 2\} \). The sixth case is \( r \equiv 0 \), \( c_4 = 2 \), and \( c_6 \in \{1, 2\} \).

**Case 4.2.6.2.1.1** First we suppose \( r \not\equiv 0 \) and \( c_4 = 1 \). Since \( r \not\equiv 0 \), congruence (A1) implies that \( t_{0,2} \equiv 0 \). Now since \( c_1 \equiv 1 \) (A2) becomes \(-t_{2,0} \equiv 0\), which says \( t_{2,0} \equiv 0 \). Hence \( t_{0,2} \equiv t_{2,0} \equiv s_{2,2} \equiv t_{1,1} \equiv 0 \), \( s_{1,2} \equiv c_1t_{0,1} \) and \( s_{2,1} \equiv -t_{0,1} \). These are the same congruences as those in Case 4.2 and therefore we conclude that \( \partial^{-1}W_2 = \partial^{-1}W_1 \).

Thus \( \text{rank}(\partial^{-1}W_2/W_2) = \text{rank}(\partial^{-1}W_1/W_2) \), \( W_2 \) is terminal, and \( W_2 \notin \hat{\mathcal{L}}_2 \).

**Case 4.2.6.2.1.2** Now suppose \( r \not\equiv 0 \) and \( c_4 = 2 \). Since \( r \not\equiv 0 \), (A1) becomes \( t_{0,2} \equiv 0 \). Since \( c_4 = 2 \) and \( t_{0,2} \equiv 0 \), (A2) holds automatically. So \( t_{2,0} \) will be a new free variable. It follows that \( s_{2,2} \equiv t_{1,1} \equiv t_{0,2} \equiv 0 \), \( s_{1,2} \equiv c_1t_{0,1} \), and \( s_{2,1} \equiv -t_{0,1} \).
Hence $x \in \partial^{-1}W_2$ if and only if

$$
t_{0,2} \equiv 0
$$

$$
s_{2,2} \equiv 0
$$

$$
t_{1,1} \equiv 0
$$

$$
s_{1,2} \equiv c_1 t_{0,1}
$$

$$
s_{2,1} \equiv -t_{0,2}.
$$

We regard $t_{0,1}, t_{1,0},$ and $t_{2,0}$ as free variables. Taking $t_{0,1} \equiv 1$, $t_{1,0} = 0$, and $t_{2,0} = 0$, the matrix $x$ becomes

$$
v_1 = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 3c_1 \\ 0 & -3 & 0 \end{pmatrix}.
$$

Taking $t_{1,0} \equiv 1$, $t_{0,1} = 0$, and $t_{2,0} = 0$, the matrix $x$ becomes

$$
v_2 = \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.
$$

Taking $t_{2,0} \equiv 1$, $t_{0,1} = 0$, and $t_{1,0} = 0$, the matrix $x$ becomes

$$
v_3 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}.
$$

From Lemma 7.2.2 we see that neither $v_3$ nor $3v_3$ is contained in $W_2$ and that $\partial^{-1}W_2 = \langle \partial^{-1}W_0, v_1, v_2, v_3 \rangle$. We know that $v_3 \in \partial^{-1}W_2$ and since $3v_3 \notin W_2$
then \( v_3 + W_2 \) is an element of the group \( \partial^{-1}W_2/W_2 \) whose order is larger than 3. Recall that \( |\partial^{-1}W_2/W_2| = 3^4 \) and that \( \partial^{-1}W_1/W_2 \) has rank 2. Since \( \partial^{-1}W_2 \) has an element of order larger than 3 then \( \text{rank}(\partial^{-1}W_2/W_2) \) is not greater than 2 since \( \partial^{-1}W_2/W_2 \cong \mathbb{Z}_9 \times \mathbb{Z}_3 \) and therefore \( W_2 \) is terminal and \( W_2 \notin \hat{L}_2 \).

**Case 4.2.2.6.2.1.3** Now suppose \( r \equiv 0, c_4 = 1, \) and \( c_6 = 0 \). By earlier work, this forces \( c_1 = 2 \) and \( c_5 = 2 \). Since \( r \equiv 0, (A1) \) is holds automatically and \( t_{0,2} \) is a new free variable. Since \( c_4 = 1 \) and \( c_6 = 0, (A2) \) becomes \(-t_{2,0} \equiv 0 \) which says \( t_{2,0} \equiv 0 \). (B1) becomes \( s_{1,2} \equiv -t_{0,1} \). Hence \( x \in \partial^{-1}W_2 \) if and only if

\[
\begin{align*}
t_{2,0} & \equiv 0 \\
t_{1,1} & \equiv t_{0,2} \\
s_{1,2} & \equiv -t_{0,1} \\
s_{2,2} & \equiv -t_{0,2} \\
s_{2,1} & \equiv -t_{0,1} + t_{0,2}.
\end{align*}
\]

We regard \( t_{0,1}, t_{1,0}, \) and \( t_{0,2} \) as free variables. Taking \( t_{0,1} \equiv 1, t_{1,0} = 0, \) and \( t_{0,2} = 0, \) the matrix \( x \) becomes

\[
v_1 = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 3c_1 \\ 0 & -3 & 0 \end{bmatrix}.
\]

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Taking \( t_{1,0} \equiv 1, t_{0,1} = 0, \) and \( t_{0,2} = 0, \) the matrix \( x \) becomes

\[
v_2 = \begin{bmatrix}
0 & 0 & 0 \\
1 & 0 & 0 \\
0 & 0 & 0
\end{bmatrix}.
\]

Taking \( t_{0,2} \equiv 1, t_{0,1} = 0, \) and \( t_{1,0} = 0, \) the matrix \( x \) becomes

\[
v_3 = \begin{bmatrix}
0 & 0 & 1 \\
0 & 1 & 0 \\
0 & 3 & -3
\end{bmatrix}.
\]

We see that \( \partial^{-1}W_2 = \langle \partial^{-1}W_1, v_3 \rangle. \) Since \( v_3 \notin \partial^{-1}W_1 \) and \( 3v_3 \in \partial^{-1}W_1 \) we know \( |\partial^{-1}W_2 / \partial^{-1}W_1| = 3. \) Since \( |\partial^{-1}W_1| = 3^9 \) then \( |\partial^{-1}W_2| = 3^{10}. \) But because \( |W_2| = 3^7 \) then \( |\partial^{-1}W_2 / W_2| = 3^3. \) Recall that \( |\partial^{-1}W_2 / W_2| = 3^3, \) we obtain \( \partial^{-1}W_2 / W_2 \cong \mathbb{Z}_3 \times \mathbb{Z}_3 \times \mathbb{Z}_3. \) Since \( \text{rank} \left( \partial^{-1}W_2 / W_2 \right) = 3 \) and \( \text{rank} \left( \partial^{-1}W_1 / W_2 \right) = 2, \) then \( \text{rank} \left( \partial^{-1}W_1 / W_2 \right) < \text{rank} \left( \partial^{-1}W_2 / W_2 \right). \) Hence \( W_2 \) is nonterminal and \( W_2 \in \hat{\mathcal{L}}_2. \)

Also, \( y_1, y_3, v_3 \in \partial^{-1}W_2 \) but not contained in \( W_2. \) Each of \( 3y_1, 3y_3, 3v_3 \) is contained in \( W_2. \) So \( y_1 + W_2, y_3 + W_2, v_3 + W_2 \) are elements of order 3 in the group \( \partial^{-1}W_2 / W_2. \)

A basis for \( \partial^{-1}W_2 / W_2 \) is \( y_1 + W_2, y_3 + W_2, v_3 + W_2 \) while its subspace \( \partial^{-1}W_1 / W_2 \) has basis \( y_1 + W_2, y_3 + W_2. \)

We fix arbitrary values \( c_7, c_8 \in \{0, 1, 2\}. \) There are \( 3^2 = 9 \) ways to choose these values. Let \( m_5 = c_7 y_1 + c_8 y_3 + v_3. \) Thus

\[
m_5 = \begin{bmatrix}
0 & 0 & 1 + 3c_8 \\
0 & 1 & 0 \\
3c_7 & 3 & -3
\end{bmatrix}.
\]
Let $W_3 = <W_2, m_5>$. There are 9 subgroups of this type. Since $m_5 \in W_2$ and $3m_5 \in W_2$ then $|W_3/W_2| = 3$. Since $|W_2| = 3^7$ then $|W_3| = 3^8$. Then $\partial^{-1}W_2/W_3 \cong \mathbb{Z}_3 \times \mathbb{Z}_3$ and $\text{rank}(\partial^{-1}W_2/W_3) = 2$.

Since $|W_3/W_2| = 3$ and the antidiagonal of $m_5$ has an entry not divisible by 3, by the Modified Terminal Lemma we conclude that $\partial^{-1}W_3 = \partial^{-1}W_2$. Thus $\text{rank}(\partial^{-1}W_3/W_3) = \text{rank}(\partial^{-1}W_2/W_3)$, $W_3$ is terminal, and $W_3 \notin \hat{\mathcal{L}}_3$.

**Case 4.2.2.6.2.1.4** Now suppose $r \equiv 0$, $c_4 = 2$, and $c_6 = 0$. By earlier work this forces $c_1 = 1$ and $c_5 = 2$. Since $r \equiv 0$, (A1) holds automatically and $t_{0,2}$ is a new free variable. Since $c_4 = 2$ and $c_6 = 0$, (A2) holds automatically and $t_{2,0}$ is also a free variable.

Hence $x \in \partial^{-1}W_2$ if and only if

$$t_{1,1} \equiv -t_{0,2}$$

$$s_{1,2} \equiv t_{0,1}$$

$$s_{2,2} \equiv -t_{0,2}$$

$$s_{2,1} \equiv -t_{0,1} + t_{0,2}$$

We regard $t_{0,1}$, $t_{1,0}$, $t_{0,2}$, and $t_{2,0}$ as free variables. Taking $t_{0,1} \equiv 1$, $t_{1,0} = 0$, $t_{0,2} = 0$, and $t_{2,0} = 0$, the matrix $x$ becomes

$$v_1 = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 3c_1 \\ 0 & -3 & 0 \end{bmatrix}.$$
Taking $t_{1,0} \equiv 1$, $t_{0,1} = 0$, $t_{0,2} = 0$, and $t_{2,0} = 0$, the matrix $x$ becomes

$$v_2 = \begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$ 

Taking $t_{0,2} \equiv 1$, $t_{0,1} = 0$, $t_{1,0} = 0$, and $t_{2,0} = 0$, the matrix $x$ becomes

$$v_3 = \begin{bmatrix} 0 & 0 & 1 \\ 0 & -1 & 0 \\ 0 & 3 & -3 \end{bmatrix}.$$ 

Taking $t_{2,0} \equiv 1$, $t_{0,1} = 0$, $t_{1,0} = 0$, and $t_{0,2} = 0$, the matrix $x$ becomes

$$v_4 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix}.$$ 

We see that $\partial^{-1}W_1 = < \partial^{-1}W_0, v_1, v_2, v_3, v_4 >$. From Lemma 7.2.2 we see that neither $v_4$ nor $3v_4$ is contained in $W_2$ and that $\partial^{-1}W_2 = < \partial^{-1}W_0, v_1, v_2, v_3, v_4 >$. We know that $v_4 \in \partial^{-1}W_2$ and since $3v_4 \notin W_2$ then $v_4 + W_2$ is an element of the group $\partial^{-1}W_2/W_2$ whose order is larger than 3. Since $v_3 \notin \partial^{-1}W_1$, and $3v_3 \in \partial^{-1}W_1$, we know $|\partial^{-1}W_2/\partial^{-1}W_1| = 3^2$. Since $|\partial^{-1}W_1| = 3^9$ then $|\partial^{-1}W_2| = 3^{11}$. Because $|W_2| = 3^7$ then $|\partial^{-1}W_2/W_2| = 3^4$ and we obtain $\partial^{-1}W_2/W_2 \cong \mathbb{Z}_9 \times \mathbb{Z}_3 \times \mathbb{Z}_3$. Since rank($\partial^{-1}W_2/W_2$) = 3 and rank($\partial^{-1}W_1/W_2$) = 2, then rank($\partial^{-1}W_1/W_2$) < rank($\partial^{-1}W_2/W_2$). Hence $W_2$ is nonterminal and $W_2 \in \hat{L}_2$. Also, $y_1, y_3, v_3 \in \partial^{-1}W_2$ but not contained in $W_2$. Each of $3y_1, 3y_3, 3v_3$ is contained in $W_2$. So $y_1 + W_2, y_3 +
$W_2, v_3 + W_2$ are elements of order 3 in the group $\partial^{-1}W_2/W_2$. $\Omega_1(\partial^{-1}W_2/W_2) \cong \mathbb{Z}_3 \times \mathbb{Z}_3 \times \mathbb{Z}_3$ has basis $y_1 + W_2, y_3 + W_2, v_3 + W_2$ while its subspace $\partial^{-1}W_1/W_2$ has basis $y_1 + W_2, y_3 + W_2, v_3 + W_2$.

We fix arbitrary values $c_7, c_8 \in \{0, 1, 2\}$. There are $3^2 = 9$ ways to choose these values. Let $m_5 = c_7y_1 + c_8y_3 + v_3$. Thus

$$m_5 = \begin{bmatrix} 0 & 0 & 1 + 3c_8 \\ 0 & -1 & 0 \\ 3c_7 & 3 & -3 \end{bmatrix}.$$ 

Let $W_3 = \langle W_2, m_5 \rangle$. There are 9 subgroups of this type. Since $m_5 \in W_3$ and $3m_5 \in W_3$ then $|W_3/W_2| = 3$. Since $|W_2| = 3^7$ then $|W_3| = 3^8$. Then $\partial^{-1}W_2/W_3 \cong \mathbb{Z}_3 \times \mathbb{Z}_3$ and $\text{rank}(\partial^{-1}W_2/W_3) = 2$.

Since $|W_3/W_2| = 3$ and the antidiagonal of $m_5$ has an entry not divisible by 3, by the Modified Terminal Lemma we conclude that $\partial^{-1}W_3 = \partial^{-1}W_2$. Thus $\text{rank}(\partial^{-1}W_3/W_3) = \text{rank}(\partial^{-1}W_2/W_3)$, $W_3$ is terminal, and $W_3 \notin \hat{L}_3$.

**Case 4.2.2.6.2.1.5** Now suppose that $r \equiv 0$, $c_4 = 1$, and $c_6 \in \{1, 2\}$. Since $r \equiv 0$, (A1) automatically holds and $t_{0,2}$ is a new free variable. Since $c_4 = 1$, (A2) becomes $t_{2,0} \equiv -c_6gt_{0,2}$. Since $c_5q + c_1 \equiv r \equiv 0$ and $c_1 \not\equiv 0$, we deduce that $q \not\equiv 0$. By our assumption that $c_6 \not\equiv 0$ we obtain $c_6q \not\equiv 0$. 

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Hence \( x \in \partial^{-1}W_2 \) if and only if

\[
\begin{align*}
t_{2,0} &\equiv -c_6 t_{0,2} \\
t_{1,1} &\equiv -c_1 t_{0,2} \\
s_{1,2} &\equiv c_1 t_{0,1} + (c_1 c_5 + c_1) t_{0,2} \\
s_{2,2} &\equiv -t_{0,2} \\
s_{2,1} &\equiv -t_{0,1} - (c_1 + c_5 c_6) t_{0,2}.
\end{align*}
\]

We regard \( t_{0,1}, t_{1,0}, \) and \( t_{0,2} \) as free variables. Taking \( t_{0,1} \equiv 1, t_{1,0} = 0, \) and \( t_{0,2} = 0, \) the matrix \( x \) becomes

\[
v_1 = \begin{bmatrix}
0 & 1 & 0 \\
0 & 0 & 3 c_1 \\
0 & -3 & 0
\end{bmatrix}.
\]

Taking \( t_{1,0} \equiv 1, t_{0,1} = 0, \) and \( t_{0,2} = 0, \) the matrix \( x \) becomes

\[
v_2 = \begin{bmatrix}
0 & 0 & 0 \\
1 & 0 & 0 \\
0 & 0 & 0
\end{bmatrix}.
\]

Taking \( t_{0,2} \equiv 1, t_{0,1} = 0, \) and \( t_{1,0} = 0, \) the matrix \( x \) becomes

\[
v_3 = \begin{bmatrix}
0 & 0 & 1 \\
0 & -c_1 & 3(c_1 c_5 + c_1) \\
-c_6 q & -3(c_1 + c_5 c_6) & -3
\end{bmatrix}.
\]

From Lemma 7.2.2 we see that neither \( v_3 \) nor \( 3v_3 \) is contained in \( W_2 \) and that \( \partial^{-1}W_2 = \langle \partial^{-1}W_0, v_1, v_2, v_3 \rangle. \) We know that \( v_3 \in \partial^{-1}W_2 \) and since \( 3v_3 \notin W_2 \)
then $v_3 + W_2$ is an element of the group $\partial^{-1}W_2/W_2$ whose order is larger than 3. Recall that $|\partial^{-1}W_2/W_2| = 3^4$ and that $\partial^{-1}W_1/W_2$ has rank 2. Since $\partial^{-1}W_2$ has an element of order larger than 3 then $\text{rank}(\partial^{-1}W_2/W_2)$ is not greater than 2 since $\partial^{-1}W_2/W_2 \cong \mathbb{Z}_9 \times \mathbb{Z}_3$ and therefore $W_2$ is terminal and $W_2 \notin \hat{L}_2$.

**Case 4.2.2.6.2.1.6** Now we suppose $r \equiv 0$, $c_4 = 2$, and $c_6 \in \{1, 2\}$. Since $r \equiv 0$, (A1) automatically holds. By argument used in Case 4.2.2.6.2.1.5 we get $q \neq 0$. By our assumption that $c_6 \neq 0$ we get $c_6q \neq 0$. Since $c_4 = 2$, (A2) becomes $-c_3qt_{0,2} \equiv 0$. Since $c_6q \neq 0$ this says $t_{0,2} \equiv 0$ and $t_{2,0}$ is a new free variable. It then follows that $t_{1,1} \equiv s_{2,2} \equiv t_{0,2} \equiv 0$, $s_{1,2} \equiv c_1t_{0,1}$ and $s_{2,1} \equiv -t_{0,1}$.

Hence $x \in \partial^{-1}W_2$ if and only if

$$
\begin{align*}
& t_{0,2} \equiv 0 \\
& s_{2,2} \equiv 0 \\
& t_{1,1} \equiv 0 \\
& s_{1,2} \equiv c_1t_{0,1} \\
& s_{2,1} \equiv -t_{0,1}.
\end{align*}
$$

We regard $t_{0,1}$, $t_{1,0}$, and $t_{2,0}$ as free variables. Taking $t_{0,1} \equiv 1$, $t_{1,0} = 0$, and $t_{2,0} = 0$, the matrix $x$ becomes

$$
v_1 = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 3c_1 \\ 0 & -3 & 0 \end{bmatrix}.
$$
Taking \( t_{1,0} \equiv 1, t_{0,1} = 0, \) and \( t_{2,0} = 0, \) the matrix \( x \) becomes
\[
v_2 = \begin{bmatrix}
0 & 0 & 0 \\
1 & 0 & 0 \\
0 & 0 & 0
\end{bmatrix}.
\]

Taking \( t_{2,0} \equiv 1, t_{0,1} = 0, \) and \( t_{1,0} = 0, \) the matrix \( x \) becomes
\[
v_3 = \begin{bmatrix}
0 & 0 & 0 \\
0 & 0 & 0 \\
1 & 0 & 0
\end{bmatrix}.
\]

From Lemma 7.2.2 we see that neither \( v_3 \) nor \( 3v_3 \) is contained in \( W_2 \) and that \( \partial^{-1}W_2 = < \partial^{-1}W_0, v_1, v_2, v_3 >. \) We know that \( v_3 \in \partial^{-1}W_2 \) and since \( 3v_3 \notin W_2 \) then \( v_3 + W_2 \) is an element of the group \( \partial^{-1}W_2/W_2 \) whose order is larger than 3. Recall that \( |\partial^{-1}W_2/W_2| = 3^4 \) and that \( \partial^{-1}W_1/W_2 \) has rank 2. Since \( \partial^{-1}W_2 \) has an element of order larger than 3 then rank(\( \partial^{-1}W_2/W_2 \)) is not greater than 2 since \( \partial^{-1}W_2/W_2 \cong \mathbb{Z}_9 \times \mathbb{Z}_3 \) and therefore \( W_2 \) is terminal and \( W_2 \notin \hat{\mathcal{L}}_2. \)

**Case 4.2.2.6.2.2** Now suppose that \( c_3 \neq 0. \) Recall that \( c_4 \neq 0 \) and \( c_5 \neq 0. \) Thus \( c_6 = 0. \)
Hence \( x \in \partial^{-1}W_2 \) if and only if

\[
c_3 t_{2,0} - c_1 t_{0,2} \equiv c_5 (s_{1,2} + c_1 s_{2,1}) \quad (A1)
\]

\[
(c_4 + 1)t_{2,0} \equiv 0 \quad (A2)
\]

\[
c_3 s_{2,1} + c_1 c_5 s_{1,2} \equiv (c_5 - c_3)t_{0,1} + (c_5 + 1)t_{0,2} \quad (B1)
\]

\[
s_{2,2} \equiv -t_{0,2} \quad (B2)
\]

\[
c_4 s_{2,1} \equiv -c_4 t_{0,1} - c_1 t_{0,2} \quad (B3)
\]

\[
t_{1,1} \equiv -c_1 t_{0,2} \quad (B4).
\]

The congruence \((B3)\) can be rewritten as \(s_{2,1} \equiv -t_{0,1} - c_1 c_4 t_{0,2}\). Substituting \((B3)\) into \((B1)\) we obtain \(s_{1,2} \equiv c_1 t_{0,1} + (c_1 + c_1 c_5 + c_3 c_4 c_5) t_{0,2}\). Note that \(s_{1,2} + c_1 s_{2,1} \equiv (c_1 + c_1 c_5 + c_3 c_4 c_5 - c_4) t_{0,2}\). For convenience we will write \(r = c_1 + c_1 c_5 + c_3 c_4 c_5 - c_4\). \((A1)\) becomes \(c_3 t_{2,0} \equiv (c_1 + c_5 r)t_{0,2}\) which says \(t_{2,0} \equiv c_3 (c_1 + c_5 r)t_{0,2}\). We will write \(q = c_3 (c_1 + c_5 r)\) and note that \(q \equiv c_1 c_3 c_5 - c_1 c_3 + c_4 - c_3 c_4 c_5\).

Hence \( x \in \partial^{-1}W_2 \) if and only if

\[
t_{2,0} \equiv q t_{0,2} \quad \text{where} \quad q = c_1 c_3 c_5 - c_1 c_3 + c_4 - c_3 c_4 c_5 \quad (A1)
\]

\[
(c_4 + 1)t_{2,0} \equiv 0 \quad (A2)
\]

\[
s_{1,2} \equiv t_{0,1} + (c_1 + c_1 c_5 + c_3 c_4 c_5)t_{0,2} \quad (B1)
\]

\[
s_{2,2} \equiv -t_{0,2} \quad (B2)
\]

\[
s_{2,1} \equiv -t_{0,1} - c_1 c_4 t_{0,2} \quad (B3)
\]

\[
t_{1,1} \equiv -c_1 t_{0,2} \quad (B4).
\]
We shall soon see that $W_2$ is nonterminal precisely when $q \equiv 0$. Recall $c_1, c_3, c_4, c_5 \in \{1, 2\}$. We shall now determine those values of $c_1, c_3, c_4, c_5$ for which $q \equiv 0$.

**Lemma 7.2.3.** $q \equiv 0$ if and only if we have $c_3 = 1$ and we have either $c_1 = c_4$ or $c_5 = 1$.

**Proof.** We examine the cases $c_3 = 1$ and $c_3 = 2$ separately. First suppose $c_3 = 1$. Thus $q \equiv c_1c_5 - c_1 + c_4 - c_4c_5$ and so $q \equiv (c_1 - c_4)(c_5 - 1)$. It is now clear that $q \equiv 0$ if and only if either $c_1 = c_4$ or $c_5 = 1$. Now suppose $c_3 = 2$. This $q \equiv -c_1c_5 + c_1 + c_4c_4c_5$ and so $q \equiv c_1(1 - c_5) + c_4(c_5 + 1)$. We argue that $q \not\equiv 0$. Since $c_5 \in \{1, 2\}$ we examine the cases $c_5 = 1$ and $c_5 = 2$ separately. If $c_5 = 1$ then $q \equiv -c_4$, and since $c_4 \in \{1, 2\}$ this yields $q \not\equiv 0$, as desired. If $c_5 = 2$ then $q \equiv -c_1$, and since $c_1 \in \{1, 2\}$ this yields $q \not\equiv 0$, as desired.

We now consider the cases $q \equiv 0$ and $q \not\equiv 0$ separately. By Lemma 7.2.3, the condition $q \equiv 0$ holds if we assume that $c_3 = 1$ and also that either $c_1 = c_4$ or $c_5 = 1$. This is Case 4.2.2.6.2.2.1 below. By Lemma 7.2.3, the condition $q \not\equiv 0$ holds if we assume that either $c_3 = 2$ or that both $c_1 \neq c_4$ and $c_5 = 2$. This is Case 4.2.2.6.2.2.2 below.

**Case 4.2.2.6.2.2.1** Suppose that $c_3 = 1$ and also that either $c_1 = c_4$ or $c_5 = 1$. Thus $q \equiv 0$. Now (A1) becomes $t_{2,0} \equiv 0$. Thus (A2) is automatically satisfied.
Hence $x \in \partial^{-1}W_2$ if and only if

\begin{align*}
t_{2,0} &\equiv 0 \quad \text{(A1)} \\
s_{2,2} &\equiv c_1 t_{1,1} \quad \text{(A3)} \\
s_{1,2} &\equiv t_{0,1} + (c_1 + c_1 c_5 + c_4 c_5) t_{0,2} \quad \text{(B1)} \\
s_{2,2} &\equiv -t_{0,2} \quad \text{(B2)} \\
s_{2,1} &\equiv -t_{0,1} - c_1 c_4 t_{0,2} \quad \text{(B3)} \\
t_{1,1} &\equiv -c_1 t_{0,2} \quad \text{(B4)}.
\end{align*}

We regard $t_{0,1}$, $t_{1,0}$, and $t_{0,2}$ as free variables. Taking $t_{0,1} \equiv 1$, $t_{1,0} = 0$, and $t_{0,2} = 0$, the matrix $x$ becomes

$$v_1 = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 3c_1 \\ 0 & -3 & 0 \end{bmatrix}. $$

Taking $t_{1,0} \equiv 1$, $t_{0,1} = 0$, and $t_{0,2} = 0$, the matrix $x$ becomes

$$v_2 = \begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}. $$

Taking $t_{0,2} \equiv 1$, $t_{0,1} = 0$, and $t_{1,0} = 0$, the matrix $x$ becomes

$$v_3 = \begin{bmatrix} 0 & 0 & 1 \\ 0 & -c_1 & 3(c_1 + c_1 c_5 + c_4 c_5) \\ 0 & -3c_1 c_4 & -3 \end{bmatrix}. $$

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We see that $\partial^{-1}W_2 = < \partial^{-1}W_1, v_3 >$. Since $v_3 \notin \partial^{-1}W_1$ and $3v_3 \in \partial^{-1}W_1$ we know $|\partial^{-1}W_2/\partial^{-1}W_1| = 3$. Since $|\partial^{-1}W_1| = 3^9$ then $|\partial^{-1}W_2| = 3^{10}$. Because $|W_2| = 3^7$ then $|\partial^{-1}W_2/W_2| = 3^3$. Recall that $|\partial^{-1}W_2/W_2| = 3^3$, we obtain $\partial^{-1}W_2/W_2 \cong \mathbb{Z}_3 \times \mathbb{Z}_3 \times \mathbb{Z}_3$. Since rank($\partial^{-1}W_2/W_2$) = 3 and rank($\partial^{-1}W_1/W_2$) = 2, then rank($\partial^{-1}W_1/W_2$) < rank($\partial^{-1}W_2/W_2$). Hence $W_2$ is nonterminal and $W_2 \in \hat{L}_2$. Also, $y_1, y_3, v_3 \in \partial^{-1}W_2$ but not contained in $W_2$. Each of $3y_1, 3y_3, 3v_3$ is contained in $W_2$. So $y_1 + W_2, y_3 + W_2, v_3 + W_2$ are elements of order 3 in the group $\partial^{-1}W_2/W_2$. A basis for $\partial^{-1}W_2/W_2$ is $y_1 + W_2, y_3 + W_2, v_3 + W_2$ while its subspace $\partial^{-1}W_1/W_2$ has basis $y_1 + W_2, y_3 + W_2$.

We fix arbitrary values $c_7, c_8 \in \{0, 1, 2\}$. There are $3^2 = 9$ ways to choose these values. Let $m_5 = c_7y_1 + c_8y_3 + v_3$. Thus

$$m_5 = \begin{bmatrix}
0 & 0 & 1 + 3c_8 \\
0 & -c_1 & 0 \\
3c_7 & -3c_1c_4 & -3
\end{bmatrix}.$$

Let $W_3 = < W_2, m_5 >$. There are 9 subgroups of this type. Since $m_5 \in W_3$ and $3m_2 \in W_3$ then $|W_3/W_2| = 3$. Since $|W_2| = 3^7$ then $|W_3| = 3^8$. Then $\partial^{-1}W_2/W_3 \cong \mathbb{Z}_3 \times \mathbb{Z}_3$ and rank($\partial^{-1}W_2/W_3$) = 2.

Since $|W_3/W_2| = 3$ and the antidiagonal of $m_5$ has an entry not divisible by 3, by the Modified Terminal Lemma we conclude that $\partial^{-1}W_3 = \partial^{-1}W_2$. Thus rank($\partial^{-1}W_3/W_3$) = rank($\partial^{-1}W_2/W_3$), $W_3$ is terminal, and $W_3 \notin \hat{L}_3$.

**Case 4.2.6.2.2.2** Now suppose that either $c_3 = 2$ or that both $c_1 \neq c_4$ and $c_5 = 2$. Recall that $c_4 \in \{1, 2\}$. We need to consider the cases $c_4 = 1$ and $c_4 = 2$ separately.
Case 4.2.6.2.2.2.1 Suppose $c_4 = 1$. The congruence (A2) is thus equivalent to $t_{2,0} \equiv 0$. Then (A1) becomes $qt_{0,2} \equiv 0$, and since $q \not\equiv 0$ this is equivalent to $t_{0,2} \equiv 0$. Then (B1) and (B3) become $s_{1,2} \equiv c_1 t_{0,1}$ and $s_{1,2} \equiv -t_{0,1}$ respectively. Finally, (B2) and (B4) become $s_{2,2} \equiv 0$ and $t_{1,1} \equiv 0$. Thus $\partial^{-1}W_2 = \partial^{-1}W_1$ and $W_2$ is terminal.

Case 4.2.6.2.2.2 Suppose $c_4 = 2$. Then (A2) automatically holds so we may ignore it. Hence $x \in \partial^{-1}W_2$ if and only if

\begin{align*}
t_{2,0} &\equiv qt_{0,2} \quad \text{(A1)} \\
s_{1,2} &\equiv t_{0,1} + (c_1 + c_1 c_5 - c_3 c_5)t_{0,2} \quad \text{(B1)} \\
s_{2,2} &\equiv -t_{0,2} \quad \text{(B2)} \\
s_{2,1} &\equiv -t_{0,1} + c_1 t_{0,2} \quad \text{(B3)} \\
t_{1,1} &\equiv -c_1 t_{0,2} \quad \text{(B4)}.
\end{align*}

We regard $t_{0,1}$, $t_{1,0}$, and $t_{0,2}$ as free variables. Taking $t_{0,1} \equiv 1$, $t_{1,0} = 0$, and $t_{0,2} = 0$, the matrix $x$ becomes

$$v_1 = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 3c_1 \\ 0 & -3 & 0 \end{bmatrix}. $$

Taking $t_{1,0} \equiv 1$, $t_{0,1} = 0$, and $t_{0,2} = 0$, the matrix $x$ becomes

$$v_2 = \begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}. $$
Taking \( t_{0,2} \equiv 1, t_{0,1} = 0, \) and \( t_{1,0} = 0, \) the matrix \( x \) becomes
\[
v_3 = \begin{bmatrix}
0 & 0 & 1 \\
0 & -c_1 & 3(c_1 + c_1 c_5 - c_3 c_5) \\
q & 3c_1 & -3
\end{bmatrix}.
\]

Since \( q \neq 0, \) from Lemma 7.2.2 we see that neither \( v_3 \) nor \( 3v_3 \) is contained in \( W_2 \) and that \( \partial^{-1}W_2 = \langle \partial^{-1}W_0, v_1, v_2, v_3 \rangle. \) We know that \( v_3 \in \partial^{-1}W_2 \) and since \( 3v_3 \not\in W_2 \) then \( v_3 + W_2 \) is an element of the group \( \partial^{-1}W_2/W_2 \) whose order is larger than 3. Recall that \( |\partial^{-1}W_2/W_2| = 3^4 \) and that \( \partial^{-1}W_1/W_2 \) has rank 2. Since \( \partial^{-1}W_2 \) has an element of order larger than 3 then \( \text{rank}(\partial^{-1}W_2/W_2) \) is not greater than 2 since \( \partial^{-1}W_2/W_2 \cong \mathbb{Z}_9 \times \mathbb{Z}_3 \) and therefore \( W_2 \) is terminal and \( W_2 \not\in \hat{L}_2. \)

**Case 4.2.2.6.3** Suppose \( 0 \neq c_3 c_6 \neq c_4 c_5 \neq 0. \) Thus in particular each of \( c_3, c_4, c_5, c_6 \) is equal to either 1 or 2. Let \( D = c_3 c_6 - c_4 c_5 \) (for determinant) and note \( D \neq 0. \) Hence \( D^2 \equiv 1. \) Also let \( q_1 = -c_4 - c_1 c_3 \) and \( q_2 = c_6 + c_1 c_5. \) We multiply (B1) by \( c_4 \) and multiply (B3) by \( c_3 \) and we subtract the first from the second to obtain
\[c_1 D s_{1,2} \equiv D t_{0,2} + (D + q_1) t_{0,2}.\]
We multiply the last congruence by \( c_1 D \) and we obtain
\[s_{1,w} \equiv c_1 t_{0,2} + c_1 (1 + D q_1) t_{0,2} \]
which we call the new (B3). We multiply (B1) by \( c_6 \) and we multiply the original (B3) by \( c_5 \) and we subtract the second from the first to obtain
\[D s_{2,1} \equiv -D t_{0,1} + q_2 t_{0,2}.\]
We multiply this last congruence by \( D \) and get
\[s_{2,1} \equiv -t_{0,1} + D q_2 t_{0,2} \]
which we call the new (B1). Write \( q = 1 + D(q_1 + q_2). \) Thus
\[s_{1,2} + c_2 s_{2,1} \equiv c_1 q t_{0,2}. \]
The congruence (A1) becomes
\[t_{2,0} \equiv c_1 c_3 t_{0,2} + c_3 c_5 (s_{1,2} + c_1 s_{2,1}) \]
which becomes
\[t_{2,0} \equiv c_1 c_3 t_{0,2} + c_1 c_3 c_5 q t_{0,2} \]
which becomes
\[t_{2,0} \equiv c_1 c_3 (1 + c_5 q) t_{0,2}. \]
congruence (A2) becomes $(c_4 + 1)t_{2,0} \equiv c_1 c_6 q t_{0,2}$. Hence $x \in \partial^{-1}W_2$ if and only if

$$t_{2,0} \equiv c_1 c_3 (1 + c_5 q) t_{0,2} \quad \text{(A1)}$$

$$(c_4 + 1)t_{2,0} \equiv c_1 c_6 q t_{0,2} \quad \text{(A2)}$$

$$s_{2,1} \equiv -t_{0,1} + D q_2 t_{0,2} \quad \text{(B1)}$$

$$s_{2,2} \equiv -t_{0,2} \quad \text{(B2)}$$

$$s_{1,2} \equiv c_1 t_{0,1} + c_1 (1 + D q_1) t_{0,2} \quad \text{(B3)}$$

$$t_{0,2} \equiv -c_1 t_{0,2} \quad \text{(B4)}.$$

It is convenient to consider the following four cases separately. The first case is when $c_4 = 2, q \equiv 0$. The second case is when $c_4 = 2, q \neq 0$. The third case is when $c_4 = 1, q \equiv 0$. The fourth case is when $c_4 = 1, q \neq 0$.

**Case 4.2.2.6.3.1** Suppose $c_4 = 2, q \equiv 0$. Then (A2) holds automatically and we may ignore it. The congruence (A1) becomes $t_{2,0} \equiv c_1 c_3 t_{0,2}$. So $t_{0,2}$ is a new free variable. We regard $t_{0,1}, t_{1,0},$ and $t_{0,2}$ as free variables. Taking $t_{0,1} \equiv 1, t_{1,0} = 0,$ and $t_{0,2} = 0,$ the matrix $x$ becomes

$$v_1 = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 3c_1 \\ 0 & -3 & 0 \end{bmatrix}.$$
Taking \( t_{1,0} \equiv 1, t_{0,1} = 0, \) and \( t_{0,2} = 0, \) the matrix \( x \) becomes

\[
v_2 = \begin{bmatrix}
0 & 0 & 0 \\
1 & 0 & 0 \\
0 & 0 & 0
\end{bmatrix}.
\]

Taking \( t_{0,2} \equiv 1, t_{0,1} = 0, \) and \( t_{1,0} = 0, \) the matrix \( x \) becomes

\[
v_3 = \begin{bmatrix}
0 & 0 & 1 \\
0 & 0 & 3c_1(1 + Dq_1) \\
c_1c_3 & 3Dq_2 & -3
\end{bmatrix}.
\]

Since \( c_1c_3 \neq 0, \) from Lemma 7.2.2 we see that neither \( v_3 \) nor \( 3v_3 \) is contained in \( W_2 \) and that \( \partial^{-1}W_2 = \langle \partial^{-1}W_0, v_1, v_2, v_3 \rangle. \) We know that \( v_3 \in \partial^{-1}W_2 \) and since \( 3v_3 \not\in W_2 \) then \( v_3 + W_2 \) is an element of the group \( \partial^{-1}W_2/W_2 \) whose order is larger than 3. Recall that \( |\partial^{-1}W_2/W_2| = 3^4 \) and that \( \partial^{-1}W_1/W_2 \) has rank 2. Since \( \partial^{-1}W_2 \) has an element of order larger than 3 then \( \text{rank}(\partial^{-1}W_2/W_2) \) is not greater than 2 since \( \partial^{-1}W_2/W_2 \cong \mathbb{Z}_9 \times \mathbb{Z}_3 \) and therefore \( W_2 \) is terminal and \( W_2 \not\in \hat{\mathcal{L}}_2. \)

**Case 4.2.2.6.3.2** Suppose \( c_4 = 2, q \neq 0. \) Then (A2) becomes \( c_1c_6qt_{0,2} \equiv 0, \) with \( c_1c_6q \neq 0 \) forcing \( t_{0,2} \equiv 0. \) Since \( t_{0,2} \equiv 0, \) (A1) forces \( t_{2,0} \equiv 0. \) Hence \( t_{0,2} \equiv t_{2,0} \equiv s_{2,2} \equiv t_{1,1} \equiv 0, s_{1,2} \equiv c_1t_{0,1} \) and \( s_{2,1} \equiv -t_{0,1}. \) These are the same congruences as those in Case 4.2 and therefore we conclude that \( \partial^{-1}W_2 = \partial^{-1}W_1. \)

Thus \( \text{rank}(\partial^{-1}W_2/W_2) = \text{rank}(\partial^{-1}W_1/W_2), \) \( W_2 \) is terminal, and \( W_2 \not\in \hat{\mathcal{L}}_2. \)
**Case 4.2.2.6.3.3** Suppose $c_4 = 1, q \equiv 0$. (A2) becomes $-t_{2,0} \equiv 0$ and so $t_{2,0} \equiv 0$. Since $t_{2,0} \equiv 0$ and $q \equiv 0$, (A1) becomes $c_1 c_3 t_{0,2} \equiv 0$. Since $c_1 c_3 \neq 0$ this yields $t_{0,2} \equiv 0$. Hence $t_{0,2} \equiv t_{2,0} \equiv s_{2,2} \equiv t_{1,1} \equiv 0$, $s_{1,2} \equiv c_1 t_{0,1}$ and $s_{2,1} \equiv -t_{0,1}$. These are the same congruences as those in Case 4.2 and therefore we conclude that $\partial^{-1}W_2 = \partial^{-1}W_1$. Thus $\text{rank}(\partial^{-1}W_2/W_2) = \text{rank}(\partial^{-1}W_1/W_2)$, $W_2$ is terminal, and $W_2 \notin \hat{L}_2$.

**Case 4.2.2.6.3.4** Suppose $c_4 = 1, q \neq 0$. (A2) becomes $t_{2,0} \equiv -c_1 c_6 q t_{0,2}$. Combining this with (A1) we obtain $c_1 (c_3 + c_5 c_2 q + c_6 q) t_{0,2} \equiv 0$ which we will denote by $\star$. Now we consider the subcases $c_3 + c_5 c_2 q + c_6 q \neq 0$ and $c_3 + c_5 c_2 q + c_6 q \equiv 0$.

**Case 4.2.2.6.3.4.1** Suppose $c_3 + c_5 c_2 q + c_6 q \neq 0$. Recall $c_1 \neq 0$. Then $t_{0,2} \equiv 0$, by $\star$, which then forces $t_{2,0} \equiv 0$ by (A2). Hence $t_{0,2} \equiv t_{2,0} \equiv s_{2,2} \equiv t_{1,1} \equiv 0$, $s_{1,2} \equiv c_1 t_{0,1}$ and $s_{2,1} \equiv -t_{0,1}$. These are the same congruences as those in Case 4.2 and therefore we conclude that $\partial^{-1}W_2 = \partial^{-1}W_1$. Thus $\text{rank}(\partial^{-1}W_2/W_2) = \text{rank}(\partial^{-1}W_1/W_2)$, $W_2$ is terminal, and $W_2 \notin \hat{L}_2$.

**Case 4.2.2.6.3.4.2** Suppose $c_3 c_5 q + c_6 q \equiv 0$. Then $\star$ holds automatically. Thus $t_{0,2}$ is a new free variable. We regard $t_{0,1}, t_{1,0},$ and $t_{0,2}$ as free variables. Taking $t_{0,1} \equiv 1$, $t_{1,0} = 0$, and $t_{0,2} = 0$, the matrix $x$ becomes

$$v_1 = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 3c_1 \\ 0 & -3 & 0 \end{bmatrix}.$$
Taking \( t_{1,0} \equiv 1, t_{0,1} = 0, \) and \( t_{0,2} = 0, \) the matrix \( x \) becomes

\[
v_2 = \begin{bmatrix}
0 & 0 & 0 \\
1 & 0 & 0 \\
0 & 0 & 0
\end{bmatrix}.
\]

Taking \( t_{0,2} \equiv 1, t_{0,1} = 0, \) and \( t_{1,0} = 0, \) the matrix \( x \) becomes

\[
v_3 = \begin{bmatrix}
0 & 0 & 1 \\
0 & 0 & 3c_1(1 + Dq_1) \\
-c_1c_6q & 3Dq_2 & -3
\end{bmatrix}.
\]

Since \(-c_1c_6q \neq 0,\) from Lemma 7.2.2 we see that neither \( v_3 \) nor \( 3v_3 \) is contained in \( W_2 \) and that \( \partial^{-1}W_2 = \langle \partial^{-1}W_0, v_1, v_2, v_3 \rangle. \) We know that \( v_3 \in \partial^{-1}W_2 \) and since \( 3v_3 \notin W_2 \) then \( v_3 + W_2 \) is an element of the group \( \partial^{-1}W_2/W_2 \) whose order is larger than 3. Recall that \( |\partial^{-1}W_2/W_2| = 3^4 \) and that \( \partial^{-1}W_1/W_2 \) has rank 2. Since \( \partial^{-1}W_2 \) has an element of order larger than 3 then \( \text{rank}(\partial^{-1}W_2/W_2) \) is not greater than 2 since \( \partial^{-1}W_2/W_2 \cong \mathbb{Z}_9 \times \mathbb{Z}_3 \) and therefore \( W_2 \) is terminal and \( W_2 \notin \hat{L}_2. \)

In Case 4 we found 4 subgroups \( W_1 \in L_1, \) all of which satisfy \( |W_1| = 3^5 \) and all of which are nonterminal. We found that 2 of these members of \( L_1 \) are each contained in 9 members of \( L_2 \) and 2 of these members of \( L_1 \) are each contained in 117 members of \( L_2. \) Thus we found 252 subgroups \( W_2 \in L_2, \) 90 of which satisfy \( |W_2| = 3^6 \) and 162 of which satisfy \( |W_2| = 3^7. \) Each of the 90 members of \( L_2 \) that satisfy \( |W_2| = 3^6 \) is terminal. Exactly 134 members of \( L_2 \) that satisfy \( |W_2| = 3^7 \) are terminal. Exactly 28 members of \( L_2 \) that satisfy \( |W_2| = 3^7 \) are nonterminal. Each of these 28 nonterminal members of \( L_2 \) is contained in 9 members of \( L_3. \) Thus we found
252 subgroups $W_3 \in \mathcal{L}_3$, all satisfying $|W_3| = 3^8$. Every member of $\mathcal{L}_3$ is terminal.

So in Case 4 we found a total of $4 + 252 + 252 = 508$ subgroups.
We fix arbitrary values $c_1, c_2, c_3 \in \{0, 1, 2\}$ such that $(c_1, c_2) \neq (0, 0)$ and $c_3 \neq 0$. Let $m_1 = y_1 + c_1y_2 + c_2y_4$ and $m_2 = y_3 + c_3y_4$. Thus

$$m_1 = \begin{bmatrix} c_2 & 0 & -3c_2 \\ 0 & 3c_1 & 0 \\ 3(1 - c_2) & 0 & 0 \end{bmatrix} \quad \text{and} \quad m_2 = \begin{bmatrix} c_3 & 0 & 3(1 - c_3) \\ 0 & 0 & 0 \\ -3c_3 & 0 & 0 \end{bmatrix}.$$ 

Let $W_1 = \langle W_0, m_1, m_2 \rangle \in \mathcal{L}_1$. The number of subgroups of this type is 16.

Note that $|W_1| = 3^5$ and $\partial^{-1}W_0/W_1 \cong \mathbb{Z}_3 \times \mathbb{Z}_3$ and rank $(\partial^{-1}W_0/W_1) = 2$.

We now calculate the pullback $\partial^{-1}W_1$. The subgroup $W_1$ is contained in the pattern subgroup

$$\begin{bmatrix} 2 & 1 & 1 \\ 1 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}.$$ 

Thus the pullback $\partial^{-1}W_1$ is contained in the pattern subgroup

$$\begin{bmatrix} 2 & 2 & 2 \\ 2 & 1 & 1 \\ 1 & 1 & 0 \end{bmatrix} \cap \begin{bmatrix} 2 & 2 & 1 \\ 2 & 1 & 1 \\ 2 & 1 & 0 \end{bmatrix} = \begin{bmatrix} 2 & 2 & 1 \\ 2 & 1 & 1 \\ 1 & 1 & 0 \end{bmatrix}.$$
Let
\[
x = \begin{bmatrix}
3s_{0,0} + t_{0,0} & 3s_{0,1} + t_{0,1} & 3s_{0,2} \\
3s_{1,0} + t_{1,0} & 3s_{1,1} & 3s_{1,2} \\
3s_{2,0} & 3s_{2,1} & 0
\end{bmatrix}
\in \begin{bmatrix}
2 & 2 & 1 \\
2 & 1 & 1 \\
1 & 1 & 0
\end{bmatrix}.
\]

Thus
\[
\partial_1 x = \begin{bmatrix}
3s_{1,0} + t_{1,0} & 3s_{1,1} & 3s_{1,2} \\
3s_{2,0} & 3s_{2,1} & 0 \\
-3t_{1,0} & 0 & 0
\end{bmatrix}
\quad \text{and} \quad
\partial_2 x = \begin{bmatrix}
3s_{0,1} + t_{0,1} & 3s_{0,2} & -3t_{0,1} \\
3s_{1,1} & 3s_{1,2} & 0 \\
3s_{2,1} & 0 & 0
\end{bmatrix}.
\]

The variables that are in play are those appearing in the matrix
\[
\begin{bmatrix}
0 & t_{0,1} & 0 \\
t_{1,0} & 0 & 3s_{1,2} \\
0 & 3s_{2,1} & 0
\end{bmatrix}.
\]

We want to identify a value \(a_1, a_2 \in \mathbb{Z}_9\) such that \(\partial_1 x \equiv a_1 m_1 + a_2 m_2 (\mod I)\). A formal expression for \(a_1 m_1 + a_2 m_2\) is
\[
\begin{bmatrix}
c_2 & 0 & -3c_2 \\
0 & 3c_1 & 0 \\
3(1 - c_2) & 0 & 0
\end{bmatrix}
+ a_2
\begin{bmatrix}
c_3 & 0 & 3(1 - c_3) \\
0 & 0 & 0 \\
-3c_3 & 0 & 0
\end{bmatrix}
\begin{bmatrix}
a_1c_2 + a_2c_3 & 0 & 3(a_2(1 - c_3) - a_1c_2) \\
0 & 3a_1c_1 & 0 \\
3(a_1(1 - c_2) - a_2c_3) & 0 & 0
\end{bmatrix}.
\]
Comparing (0, 1) - entries, we get $0 \equiv 3s_{1,1}$ which gives no information.

Comparing (1, 0) - entries, we get $0 \equiv s_{2,0}$ which gives no information.

Comparing (0, 0) - entries, we get $a_1c_2 + a_2c_3 \equiv t_{1,0}$. Since $c_3 \in \{1, 2\}$, then $c_3^2 \equiv 1$.

Multiplying by $c_3$ we obtain $a_2 \equiv c_3t_{1,0} - a_1c_2c_3$.

Comparing (2, 0) - entries, we get $a_1(1 - c_2) - a_2c_3 \equiv -t_{1,0}$. Substituting $a_2$ we obtain $a_1 \equiv 0$. Therefore, $a_2 \equiv c_3t_{1,0}$.

Comparing (1, 1)-entries, we get $s_{2,1} \equiv a_1c_1$. Since $a_1 \equiv 0$ we obtain $s_{2,1} \equiv 0$.

Comparing (0, 2) - entries, we get $a_2(1 - c_3) - a_1c_2 \equiv s_{1,2}$. Substituting $a_1$ and $a_2$ we obtain $s_{1,2} \equiv (c_3 - 1)t_{1,0}$.

We see that $\partial_1 x \in W_2$ if and only if

\[ s_{2,1} \equiv 0 \quad (A1) \]
\[ s_{1,2} \equiv (c_3 - 1)t_{1,0} \quad (A2). \]

We want to identify a value $b_1, b_2 \in \mathbb{Z}_9$ such that $\partial_2 x \equiv b_1m_1 + b_2m_2(\text{mod } I)$. A formal expression for $b_1m_1 + b_2m_2$ is

\[
\begin{pmatrix}
    c_2 & 0 & -3c_2 \\
    0 & 3c_1 & 0 \\
    3(1 - c_2) & 0 & 0
\end{pmatrix}
\begin{pmatrix}
    b_1 \\
    + b_2 \\
    \end{pmatrix}
\begin{pmatrix}
    c_3 & 0 & 3(1 - c_3) \\
    0 & 0 & 0 \\
    -3c_3 & 0 & 0
\end{pmatrix}
\]

\[
= \begin{pmatrix}
    b_1c_2 + b_2c_3 & 0 & 3(b_2(1 - c_3) - b_1c_2) \\
    0 & 3b_1c_1 & 0 \\
    3(b_1(1 - c_2) - b_2c_3) & 0 & 0
\end{pmatrix}
\]

Comparing (0, 1) - entries, we get $0 \equiv 3s_{0,2}$ which gives no information.
Comparing $(1,0)$ - entries, we get $0 \equiv s_{1,1}$ which gives no information.

Comparing $(0,0)$ - entries, we get $b_1 c_2 + b_2 c_3 \equiv t_{0,1}$. Since $c_3 \in \{1, 2\}$, then $c_3^2 \equiv 1$. Multiplying by $c_3$ we obtain $b_2 \equiv c_3 t_{0,1} - b_1 c_2 c_3$.

Comparing $(2,0)$ - entries, we get $s_{2,1} \equiv b_1 (1 - c_2) - b_2 c_3$. Substituting $b_2$ we obtain $b_1 \equiv s_{2,1} + t_{0,1}$. Therefore $b_2 \equiv c_3 (1 - c_2) t_{0,1} - c_2 c_3 s_{2,1}$.

Comparing $(0,2)$-entries, we get $-t_{0,1} \equiv -b_1 c_2 + b_2 - b_2 c_3$. Substituting $b_1, b_2$ we obtain $0 \equiv (1 - c_2) t_{0,1} - c_2 s_{2,1}$.

Comparing $(1,1)$-entries, we get $s_{1,2} \equiv b_1 c_1$. Substituting $b_1$ we obtain $s_{1,2} \equiv c_1 s_{2,1} + c_1 t_{0,1}$.

We see that $\partial_1 x \in W_2$ if and only if

\[ 0 \equiv (1 - c_2) t_{0,1} - c_2 s_{2,1} \quad \text{(B1)} \]
\[ s_{1,2} \equiv c_1 s_{2,1} + c_1 t_{0,1} \quad \text{(B2)}. \]

By (A1) the congruences (B1) and (B2) become $0 \equiv (1 - c_2) t_{0,1}$ and $s_{1,2} \equiv c_1 t_{0,1}$ respectively. Combining (A2) and (B2) we obtain $c_1 t_{0,1} \equiv (c_3 - 1) t_{1,0}$ which we will denote as our new (B2).

Hence $x \in \partial^{-1} W_2$ if and only if

\[ s_{2,1} \equiv 0 \quad \text{(A1)} \]
\[ s_{1,2} \equiv (c_3 - 1) t_{1,0} \quad \text{(A2)} \]
\[ 0 \equiv (1 - c_2) t_{0,1} \quad \text{(B1)} \]
\[ c_1 t_{0,1} \equiv (c_3 - 1) t_{1,0} \quad \text{(B2)}. \]
Lemma 8.0.4. Let \( w = \begin{bmatrix} 0 & 0 & 3r_3 \\ 0 & 3r_2 & 0 \\ 3r_1 & 0 & 0 \end{bmatrix} \) for \( r_1, r_2, r_3 \in \{0, 1, 2\} \). The condition
\( w \in W_1 \) holds if and only if \( r_3 \equiv -c_2c_3r_1 \) and \( r_2 \equiv c_1r_1 \). In particular, if \( (r_1, r_2, r_3) = (1, 0, 0) \) then \( w \not\in W_1 \).

Proof. Let \( w = \begin{bmatrix} 0 & 0 & 3r_3 \\ 0 & 3r_2 & 0 \\ 3r_1 & 0 & 0 \end{bmatrix} \) for \( r_1, r_2, r_3 \in \{0, 1, 2\} \). Comparing (0,0)-entries, we get \( c_2a_1 + c_3a_2 \equiv 0 \). This tells us that \( a_2 \equiv -c_2c_3a_1 \). Comparing (2,0)-entries, we get \( (1 - c_2)a_1 - c_3a_2 \equiv r_1 \). Comparing (1,1)-entries, we get \( c_1a_1 \equiv r_2 \). Comparing (0,2)-entries, we get \( (1 - c_3)a_2 - c_2a_1 \equiv r_3 \). Substituting \( a_2 \) into (2,0) we obtain \( r_1 \equiv a_1 \). Substituting \( a_1, a_2 \) into \( r_3 \) we obtain \( r_3 \equiv -c_2c_3r_1 \). Therefore \( r_2 \) becomes \( r_2 \equiv c_1r_1 \). Hence we see that \( w \in W_1 \) if and only if \( r_3 \equiv -c_2c_3r_1 \) and \( r_2 \equiv c_1r_1 \).

However, we look at the case when \( (r_1, r_2, r_3) = (1, 0, 0) \). Then this would force \( 0 \equiv r_3 \equiv -c_2c_3r_1 \equiv -c_2c_3 \), which would force \( c_2 \equiv 0 \) since \( c_3 \neq 0 \). Then we obtain \( 0 \equiv r_2 \equiv c_1r_1 \equiv c_1 \). Then \( c_1 \equiv 0 \) but this contradicts the assumption that \( (c_1, c_2) \neq (0, 0) \). Therefore we conclude that \( w \in W_1 \) if and only if \( r_3 \equiv -c_2c_3r_1 \) and \( r_2 \equiv c_1r_1 \) and \( (r_1, r_2, r_3) \neq (1, 0, 0) \). \( \square \)

Recall \( c_3 \in \{1, 2\} \). It is convenient to consider the cases \( c_3 = 2 \) and \( c_3 = 1 \) separately.
8.1 Case 5.1

Suppose \( c_3 = 2 \). Hence \( x \in \partial^{-1}W_1 \) if and only if

\[
\begin{align*}
  s_{2,1} &\equiv 0 \quad (A1) \\
  s_{1,2} &\equiv t_{1,0} \quad (A2) \\
  0 &\equiv (1 - c_2)t_{0,1} \quad (B1) \\
  c_1t_{0,1} &\equiv t_{1,0} \quad (B2).
\end{align*}
\]

It is convenient to consider the cases \( c_2 \neq 2 \) and \( c_2 = 2 \) separately.

8.1.1 Case 5.1.1

Suppose \( c_2 \in \{0, 2\} \). Then (B1) is equivalent to \( t_{0,1} \equiv 0 \). Thus (B2) is equivalent to \( t_{1,0} \equiv 0 \). Thus (A2) is equivalent to \( s_{1,2} \equiv 0 \). Hence \( \partial^{-1}W_1 = \partial^{-1}W_0 \). Thus\r
\r
\[\text{rank}(\partial^{-1}W_1/W_1) = \text{rank}(\partial^{-1}W_0/W_1)\]
and \( W_1 \) is terminal and \( W_1 \notin \hat{\mathcal{L}}_1 \).

8.1.2 Case 5.1.2

Suppose \( c_2 = 1 \). Then (B1) is automatic so we may ignore it. Using (B2) to replace \( t_{1,0} \) in (A2), (A2) becomes \( s_{1,2} \equiv c_1t_{0,1} \).

Hence \( x \in \partial^{-1}W_1 \) if and only if

\[
\begin{align*}
  s_{2,1} &\equiv 0 \quad (A1) \\
  s_{1,2} &\equiv c_1t_{0,1} \quad (A2) \\
  t_{1,0} &\equiv c_1t_{0,1} \quad (B2).
\end{align*}
\]
We regard $t_{0,1}$ as a free variable. Taking $t_{0,1} \equiv 1$, the matrix $x$ becomes

$$v_1 = \begin{bmatrix} 0 & 1 & 0 \\ c_1 & 0 & 3c_1 \\ 0 & 0 & 0 \end{bmatrix}.$$ 

We see that $\partial^{-1}W_1 = < \partial^{-1}W_0, v_1 >$. Since $v_1 \notin \partial^{-1}W_0$ and $3v_1 \in \partial^{-1}W_0$, we know $|\partial^{-1}W_1/\partial^{-1}W_0| = 3$. Since $|\partial^{-1}W_0| = 3^7$ then $|\partial^{-1}W_1| = 3^8$. Because $|W_1| = 3^5$ then $|\partial^{-1}W_1/W_1| = 3^3$. Also, $v_1, y_1, y_3 \in \partial^{-1}W_1$ but are not contained in $W_1$. Each of $3v_1, 3y_1, 3y_3$ is contained in $W_1$. So $v_1 + W_1, y_1 + W_1, y_3 + W_1$ are elements of order 3 in the group $\partial^{-1}W_1/W_1$. These three elements form a generating set for the group $\partial^{-1}W_1/W_1$. Recall that $|\partial^{-1}W_1/W_1| = 3^3$ we obtain $\partial^{-1}W_1/W_1 \cong \mathbb{Z}_3 \times \mathbb{Z}_3 \times \mathbb{Z}_3$.

Since rank($\partial^{-1}W_1/W_1$) = 3 and rank($\partial^{-1}W_0/W_1$) = 2, then rank($\partial^{-1}W_0/W_1$) < rank($\partial^{-1}W_1/W_1$). Hence $W_1$ is nonterminal and $W_1 \in \mathcal{L}_1$. A basis for $\partial^{-1}W_1/W_1$ is $v_1 + W_1, y_1 + W_1, y_3 + W_1$. A basis for $\partial^{-1}W_0/W_1$ is $y_1 + W_1, y_3 + W_1$. 

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We fix arbitrary values $c_4, c_5 \in \{0, 1, 2\}$. There are $3^2 = 9$ ways to choose these values. Let $m_3 = c_4y_1 + c_5y_3 + v_1$. Thus

$$m_3 = \begin{bmatrix} 0 & 1 & 3c_5 \\ c_1 & 0 & 3c_1 \\ 3c_4 & 0 & 0 \end{bmatrix}.$$  

Let $W_2 = \langle W_1, m_3 \rangle$. There are 9 subgroups $W_2$ of this type. Since $m_3 \notin W_1$ and $3m_3 \in W_1$, then $|W_2/W_1| = 3$. Also, since $|W_1| = 3^5$ then $|W_2| = 3^6$. Recall $|\partial^{-1}W_1| = 3^8$, then $\partial^{-1}W_1/W_2 \cong \mathbb{Z}_3 \times \mathbb{Z}_3$ and $\operatorname{rank}(\partial^{-1}W_1/W_2) = 2$.

The subgroup $W_2$ is contained in the pattern subgroup

$$\begin{bmatrix} 2 & 2 & 1 \\ 2 & 1 & 1 \\ 1 & 0 & 0 \end{bmatrix}.$$  

The pullback $\partial^{-1}W_2$ is contained in the pattern subgroup

$$\begin{bmatrix} 2 & 2 & 2 \\ 2 & 1 & 0 \end{bmatrix} \cap \begin{bmatrix} 2 & 2 & 2 \\ 2 & 1 & 0 \end{bmatrix} = \begin{bmatrix} 2 & 2 & 2 \\ 2 & 1 & 0 \end{bmatrix}.$$  

Let

$$x = \begin{bmatrix} 3s_{0,0} + t_{0,0} & 3s_{0,1} + t_{0,1} & 3s_{0,2} + t_{0,2} \\ 3s_{1,0} + t_{1,0} & 3s_{1,1} + t_{1,1} & 3s_{1,2} \\ 3s_{2,0} + t_{2,0} & 3s_{2,1} & 0 \end{bmatrix} \in \begin{bmatrix} 2 & 2 & 2 \\ 2 & 2 & 1 \\ 2 & 1 & 0 \end{bmatrix}.$$  

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Thus

\[
\partial_1 x = \begin{bmatrix}
3s_{1,0} + t_{1,0} & 3s_{1,1} + t_{1,1} & 3s_{1,2} \\
3s_{2,0} + t_{2,0} & 3s_{2,1} & 0 \\
-3(t_{1,0} + t_{2,0}) & -3t_{1,1} & 0 \\
\end{bmatrix}
\]

and

\[
\partial_2 x = \begin{bmatrix}
3s_{0,1} + t_{0,1} & 3s_{0,2} + t_{0,2} & -3(t_{0,1} + t_{0,2}) \\
3s_{1,1} + t_{1,1} & 3s_{1,2} & -3t_{1,1} \\
3s_{2,1} & 0 & 0 \\
\end{bmatrix}
\]

The variables that are in play are those appearing in the matrix

\[
\begin{bmatrix}
0 & t_{0,1} & t_{0,2} \\
t_{1,0} & t_{1,1} & 3s_{1,2} \\
t_{2,0} & 3s_{2,1} & 0 \\
\end{bmatrix}
\]

We want to identify a value \(a_1, a_2, a_3 \in \mathbb{Z}_9\) such that \(\partial_1 x \equiv a_1 m_1 + a_2 m_2 + a_3 m_3 \pmod{I}\). A formal expression for \(a_1 m_1 + a_2 m_2 + a_3 m_3\) is

\[
a_1 \begin{bmatrix} 1 & 0 & -3 \\ 0 & 3c_1 & 0 \end{bmatrix} + a_2 \begin{bmatrix} 2 & 0 & -3 \\ 0 & 0 & 0 \end{bmatrix} + a_3 \begin{bmatrix} 0 & 1 & 3c_5 \\ c_1 & 0 & 3c_1 \\ 3c_4 & 0 & 0 \end{bmatrix}
\]

\[
= \begin{bmatrix}
a_1 - a_2 & a_3 & 3(-a_1 - a_2 + a_3 c_5) \\
a_3 c_1 & 3a_1 c_1 & 3a_3 c_1 \\
3(a_2 + a_3 c_4) & 0 & 0 \\
\end{bmatrix}
\]

Comparing \((2,1)\)-entries, we get \(t_{1,1} \equiv 0\).

Comparing \((0,1)\)-entries, we get \(a_3 \equiv t_{1,1}\). Therefore \(a_3 \equiv 0\).
Comparing (1, 0)-entries, we get $t_{2,0} \equiv a_3 c_1$. Therefore we get $t_{2,0} \equiv 0$.

Comparing (1, 2)-entries, we get $a_3 c_1 \equiv 0$ which gives no new information.

Comparing (2, 0)-entries, we get $a_2 + a_3 c_4 \equiv -t_{1,0} - t_{2,0}$. Substituting $a_3, t_{2,0}$ we obtain $a_2 \equiv -t_{1,0}$.

Comparing (0, 0)-entries, we get $a_1 - a_2 \equiv t_{1,0}$. Substituting $a_2$ we obtain $a_1 \equiv 0$.

Comparing (1, 1)-entries, we get $a_1 c_1 \equiv s_{1,2}$. Substituting $a_1$ we obtain $s_{1,2} \equiv 0$.

Comparing (0, 2)-entries, we get $-a_1 - a_2 + a_3 c_5 \equiv s_{1,2}$. Substituting $a_1, a_2, a_3$ we obtain $t_{1,0} \equiv s_{1,2}$.

We see that $\partial_1 x \in W_2$ if and only if

$$t_{1,1} \equiv 0 \quad (A1)$$

$$t_{2,0} \equiv 0 \quad (A2)$$

$$s_{1,2} \equiv t_{1,0} \quad (A3).$$

We want to identify a value $b_1, b_2, b_3 \in \mathbb{Z}_9$ such that $\partial_1 x \equiv b_1 m_1 + b_2 m_2 + b_3 m_3 \pmod{I}$. A formal expression for $b_1 m_1 + b_2 m_2 = b_3 m_3$ is

$$b_1 \begin{bmatrix} 1 & 0 & -3 \\ 0 & 3c_1 & 0 \\ 0 & 0 & 0 \end{bmatrix} + b_2 \begin{bmatrix} 2 & 0 & -3 \\ 0 & 0 & 0 \\ 3 & 0 & 0 \end{bmatrix} + b_3 \begin{bmatrix} 0 & 1 & 3c_5 \\ c_1 & 0 & 3c_1 \\ 3c_4 & 0 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} b_1 - b_2 & b_3 & 3(-b_1 - b_2 + b_3 c_5) \\ b_3 c_1 & 3b_1 c_1 & 3b_3 c_1 \\ 3(b_2 + b_3 c_4) & 0 & 0 \end{bmatrix}.$$
Comparing \((0, 1)\)-entries, we get \(b_3 \equiv t_{1,1}\).

Comparing \((1, 0)\)-entries, we get \(b_3 c_1 \equiv t_{1,1}\).

Comparing \((1, 2)\)-entries, we get \(b_3 c_1 \equiv -t_{1,1}\). Substituting \(b_3 c_1\) we obtain \(t_{1,1} \equiv 0\).

Comparing \((2, 0)\)-entries, we get \(b_2 \equiv s_{2,1} - b_3 c_4\). Substituting \(b_3\) we obtain \(b_2 \equiv s_{2,1} - c_4 t_{0,2}\).

Comparing \((0, 0)\)-entries, we get \(b_1 \equiv t_{0,1} + b_2\). Substituting \(b_2\) we obtain \(b_1 \equiv t_{0,1} + s_{2,1} - c_4 t_{0,2}\).

Comparing \((0, 2)\)-entries, we get \(-b_1 - b_2 + b_3 c_5 \equiv -t_{0,1} - t_{0,2}\). Substituting \(b_1, b_2, b_3\) we obtain \(s_{2,1} \equiv (c_4 - c_5 - 1)t_{0,2}\).

Comparing \((1, 1)\)-entries, we obtain \(b_1 c_1 \equiv s_{1,2}\). Substituting \(b_1, s_{1,2}\) we obtain \(s_{1,2} \equiv c_1 t_{0,1} + (-c_1 c_5 - c_1)t_{0,2}\).

We see that \(\partial_1 x \in W_2\) if and only if

\[
t_{1,1} \equiv 0 \quad \text{(B1)}
\]

\[
s_{2,1} \equiv (c_4 - c_5 - 1)t_{0,2} \quad \text{(B2)}
\]

\[
s_{1,2} \equiv c_1 t_{0,1} + (-c_1 c_5 - c_1)t_{0,2} \quad \text{(B3)}.
\]

(B1) is redundant with (A1) we so may ignore it. Combining (A3) with (B3) we obtain \(t_{1,0} \equiv c_1 t_{0,1} + (-c_1 c_5 - c_1)t_{0,2}\) which we will denote as our new (A3).
Hence \( x \in \partial^{-1}W_2 \) if and only if

\[
t_{1,1} \equiv 0 \quad \text{(A1)}
\]
\[
t_{2,0} \equiv 0 \quad \text{(A2)}
\]
\[
t_{1,0} \equiv c_1 t_{0,1} + (-c_1 c_5 - c_1) t_{0,2} \quad \text{(A3)}
\]
\[
s_{2,1} \equiv (c_4 - c_5 - 1) t_{0,2} \quad \text{(B2)}
\]
\[
s_{1,2} \equiv c_1 t_{0,1} + (-c_1 c_5 - c_1) t_{0,2} \quad \text{(B3)}.
\]

We regard \( t_{0,2} \) as our new free variable. Taking \( t_{0,2} \equiv 1 \) and \( t_{0,1} = 0 \), the matrix \( x \) becomes

\[
v_2 = \begin{bmatrix}
0 & 0 & 1 \\
-c_1 c_5 - c_1 & 0 & 3(-c_1 c_5 - c_1) \\
0 & 3(c_4 - c_5 - 1) & 0
\end{bmatrix}.
\]

We see that neither \( v_2 \) nor \( 3v_2 \) is contained in \( W_2 \) and that \( \partial^{-1}W_2 =< \partial^{-1}W_0, v_1, v_2 > \). We know that \( v_2 \in \partial^{-1}W_2 \) and since \( 3v_2 \not\in W_2 \) then \( v_2 + W_2 \) is an element of the group \( \partial^{-1}W_2/W_2 \) whose order is larger than 3. Recall that \( |\partial^{-1}W_2/W_2| = 3^4 \) and that \( \partial^{-1}W_1/W_2 \) has rank 3. Since \( \partial^{-1}W_2 \) has an element of order larger than 3 then rank(\( \partial^{-1}W_2/W_2 \)) is not greater than 3 since \( \partial^{-1}W_2/W_2 \cong \mathbb{Z}_9 \times \mathbb{Z}_3 \times \mathbb{Z}_3 \) and therefore \( W_2 \) is terminal and \( W_2 \not\in \hat{L}_2 \).
Suppose $c_3 = 1$. Hence $x \in \partial^{-1}W_1$ if and only if

\[
\begin{align*}
    s_{2,1} &\equiv 0 \quad (A1) \\
    s_{1,2} &\equiv 0 \quad (A2) \\
    0 &\equiv (1 - c_2)t_{0,1} \quad (B1) \\
    c_1t_{0,1} &\equiv 0 \quad (B2).
\end{align*}
\]

Since $(c_1, c_2) \neq (0, 0)$ there are exactly 8 possibilities for $(c_1, c_2)$.

8.2.1 Case 5.2.1

Suppose $(c_1, c_2) \neq (0, 1)$. Thus either $c_1 \neq 0$ or $c_2 \neq 1$. If $c_1 \neq 0$ then (B2) is equivalent to $t_{0,1} \equiv 0$ and so (B1) is automatic. If $c_2 \neq 1$ then (B1) is equivalent to $t_{0,1} \equiv 0$ and so (B2) is automatic.

Hence $x \in \partial^{-1}W_1$ if and only if

\[
\begin{align*}
    s_{2,1} &\equiv 0 \\
    s_{1,2} &\equiv 0 \\
    t_{0,1} &\equiv 0.
\end{align*}
\]

We regard $t_{1,0}$ as a free variable. Taking $t_{1,0} \equiv 1$, the matrix $x$ becomes

\[
v_1 = \begin{bmatrix}
    0 & 0 & 0 \\
    1 & 0 & 0 \\
    0 & 0 & 0
\end{bmatrix}.
\]
We see that \( \partial^{-1}W_1 =< \partial^{-1}W_0, v_1 > \). Since \( v_1 \notin \partial^{-1}W_0 \) and \( 3v_1 \in \partial^{-1}W_0 \), we know \( |\partial^{-1}W_1/\partial^{-1}W_0| = 3 \). Since \( |\partial^{-1}W_0| = 3^7 \) then \( |\partial^{-1}W_1| = 3^8 \). Because \( |W_1| = 3^5 \) then \( |\partial^{-1}W_1/W_1| = 3^3 \).

It is convenient to consider the cases \( c_1 = 0 \) and \( c_1 \neq 0 \) separately.

Case 5.2.1.1

Suppose \( c_1 = 0 \). Since \( (c_1, c_2) \) is neither \((0, 0)\) or \((0, 1)\) we deduce that \( c_2 = 2 \). Thus

\[
m_1 = \begin{bmatrix} 2 & 0 & 3 \\ 0 & 0 & 0 \\ -3 & 0 & 0 \end{bmatrix}, \quad m_2 = \begin{bmatrix} 1 & 0 \\ 0 & 0 \\ -3 & 0 \end{bmatrix}, \quad \text{and} \quad m_1 + m_2 = \begin{bmatrix} 3 & 0 & 3 \\ 0 & 0 & 0 \\ 3 & 0 & 0 \end{bmatrix} = y_1 + y_3.
\]

Since \( W_1 =< W_0, m_1, m_2 > \) we see that \( W_1 =< W_0, m_2, m_1 + m_2 > \). Since \( y_1 + y_3 \in W_1 \), we see that \( y_1 + W_1, y_3 + W_1 \) is not a basis for the 2-dimensional vector space \( \partial^{-1}W_0/W_1 \). However, we see that \( y_2 + W_1, y_1 - y_3 + W_1 \) is indeed a basis for \( \partial^{-1}W_0/W_1 \).

Let \( c_4, c_5 \in \{0, 1, 2\} \) and \( m_3 = c_4y_2 + c_5(y_1 - y_3) + v_1 \). Thus

\[
m_3 = \begin{bmatrix} 0 & 0 & -3c_5 \\ 1 & 3c_4 & 0 \\ 3c_5 & 0 & 0 \end{bmatrix}.
\]

Let \( W_2 =< W_1, m_3 > \). There are 9 subgroups \( W_2 \) of this type. Since \( m_3 \notin W_1 \) and \( 3m_3 \in W_1 \), then \( |W_2/W_1| = 3 \). Also, since \( |W_1| = 3^5 \) then \( |W_2| = 3^6 \). Recall \( |\partial^{-1}W_1| = 3^8 \), then \( \partial^{-1}W_1/W_2 \cong \mathbb{Z}_3 \times \mathbb{Z}_3 \) and \( \text{rank}(\partial^{-1}W_1/W_2) = 2 \).
The subgroup $W_2$ is contained in the pattern subgroup

$$\begin{bmatrix}
2 & 1 & 1 \\
2 & 1 & 0 \\
1 & 0 & 0
\end{bmatrix}.$$ 

The pullback $\partial^{-1}W_2$ is contained in the pattern subgroup

$$\begin{bmatrix}
2 & 2 & 2 \\
2 & 1 & 1 \\
2 & 1 & 0
\end{bmatrix} \cap \begin{bmatrix}
2 & 2 & 1 \\
2 & 2 & 1 \\
2 & 1 & 0
\end{bmatrix} = \begin{bmatrix}
2 & 2 & 1 \\
2 & 1 & 1 \\
2 & 1 & 0
\end{bmatrix}.$$ 

Let

$$x = \begin{bmatrix}
3s_{0,0} + t_{0,0} \\
3s_{0,1} + t_{0,1} \\
3s_{0,2} \\
3s_{1,0} + t_{1,0} \\
3s_{1,1} \\
3s_{1,2} \\
3s_{2,0} + t_{2,0} \\
3s_{2,1} \\
0
\end{bmatrix} \in \begin{bmatrix}
2 & 2 & 1 \\
2 & 1 & 1 \\
2 & 1 & 0
\end{bmatrix}.$$ 

Thus

$$\partial_1 x = \begin{bmatrix}
3s_{1,0} + t_{1,0} \\
3s_{1,1} \\
3s_{1,2} \\
3s_{2,0} + t_{2,0} \\
3s_{2,1} \\
0 \\
-3(t_{1,0} + t_{2,0}) \\
0 \\
0
\end{bmatrix} \quad \text{and} \quad \partial_2 x = \begin{bmatrix}
3s_{0,1} + t_{0,1} \\
3s_{0,2} \\
-3t_{0,1} \\
3s_{1,1} \\
3s_{1,2} \\
0 \\
3s_{2,1} \\
0 \\
0
\end{bmatrix}.$$ 

The variables that are in play are those appearing in the matrix

$$\begin{bmatrix}
0 & t_{0,1} & 0 \\
t_{1,0} & 0 & 3s_{1,2} \\
t_{2,0} & 3s_{2,1} & 0
\end{bmatrix}.$$
We want to identify a value \( a_1, a_2, a_3 \in \mathbb{Z}_9 \) such that \( \partial_1 x \equiv a_1 m_1 + a_2 m_2 + a_3 m_3 \pmod{I} \). A formal expression for \( a_1 m_1 + a_2 m_2 + a_3 m_3 \) is

\[
\begin{bmatrix}
2 & 0 & 3 \\
0 & 0 & 0 \\
-3 & 0 & 0
\end{bmatrix} a_1 + \begin{bmatrix}
1 & 0 & 0 \\
0 & 0 & 0 \\
-3 & 0 & 0
\end{bmatrix} a_2 + \begin{bmatrix}
0 & 0 & -3c_5 \\
1 & 3c_4 & 0 \\
3c_5 & 0 & 0
\end{bmatrix} a_3
\]

\[
= \begin{bmatrix}
2a_1 + a_2 & 0 & 3(a_1 - a_3c_5) \\
a_3 & 3a_3c_4 & 0 \\
3(-a_1 - a_2 + a_3c_5) & 0 & 0
\end{bmatrix}
\]

Comparing \((0,1)\)-entries, we get \( 0 \equiv 3s_{1,1} \), which gives us no information.

Comparing \((1,0)\)-entries, we get \( a_3 \equiv t_{2,0} \).

Comparing \((1,1)\)-entries, we get \( a_3 c_4 \equiv s_{2,1} \). Recall \( a_3 \equiv t_{2,0} \) we obtain \( s_{2,1} \equiv c_4 t_{2,0} \).

Comparing \((0,2)\)-entries, we get \( a_1 \equiv s_{1,2} + a_3 c_5 \). Substituting \( a_3 \) we obtain \( a_1 \equiv s_{1,2} + c_5 t_{2,0} \).

Comparing \((0,0)\)-entries, we get \( a_2 \equiv t_{1,0} + a_1 \). Substituting \( a_1 \) we obtain \( a_2 \equiv t_{1,0} + s_{1,2} + c_5 t_{2,0} \).

Comparing \((2,0)\)-entries, we get \( -a_1 - a_2 + a_3 c_5 \equiv -t_{1,0} - t_{2,0} \). Substituting \( a_1, a_2, a_3 \) we obtain \( s_{1,2} \equiv (c_5 - 1)t_{2,0} \).

We see that \( \partial_1 x \in W_2 \) if and only if

\[ s_{2,1} \equiv c_4 t_{2,0} \quad \text{(A1)} \]

\[ s_{1,2} \equiv (c_5 - 1)t_{2,0} \quad \text{(A2)}. \]
We want to identify a value \( b_1, b_2, b_3 \in \mathbb{Z}_9 \) such that \( \partial_1 x \equiv b_1 m_1 + b_2 m_2 + b_3 m_3 \pmod{I} \). A formal expression for \( b_1 m_1 + b_2 m_2 = b_3 m_3 \) is

\[
\begin{bmatrix}
2 & 0 & 3 \\
0 & 0 & 0 \\
-3 & 0 & 0
\end{bmatrix}
+ b_2
\begin{bmatrix}
1 & 0 & 0 \\
0 & 0 & 0 \\
-3 & 0 & 0
\end{bmatrix}
+ b_3
\begin{bmatrix}
0 & 0 & -3c_5 \\
1 & 3c_4 & 0 \\
3c_5 & 0 & 0
\end{bmatrix}
\]

Comparing (0,1)-entries, we get \( 0 \equiv 3s_{0,2} \), which gives us no information.

Comparing (1,0)-entries, we get \( b_3 \equiv 0 \).

Comparing (1,1)-entries, we get \( b_3 c_4 \equiv s_{1,2} \). Substituting \( b_3 \equiv 0 \) we obtain \( s_{1,2} \equiv 0 \).

Comparing (0,2)-entries, we get \( b_1 - b_3 c_5 \equiv -t_{0,1} \). Substituting \( b_3 \) we obtain \( b_1 \equiv -t_{0,1} \).

Comparing (0,0)-entries, we get \( b_2 \equiv t_{0,1} + b_1 \). Substituting \( b_1 \) we obtain \( b_2 \equiv 0 \).

Comparing (2,0)-entries, we get \( -b_1 - b_2 + b_3 c_5 \equiv s_{2,1} \). Substituting \( b_1, b_2, b_3 \) we obtain \( s_{2,1} \equiv t_{0,1} \).

We see that \( \partial_1 x \in W_2 \) if and only if

\[
\begin{align*}
s_{1,2} &\equiv 0 \quad \text{(B1)} \\
s_{2,1} &\equiv t_{0,1} \quad \text{(B2)}
\end{align*}
\]
Substituting (B1) into (A2) we obtain \( 0 \equiv (c_5 - 1)t_{2,0} \) which we will denote as our new (A2). Combining (A1) and (B2) we obtain \( t_{0,1} \equiv c_4 t_{2,0} \) which we will denote as our new (B2).

Hence \( x \in \partial^{-1}W_2 \) if and only if

\[
\begin{align*}
s_{2,1} &\equiv c_4 t_{2,0} \quad \text{(A1)} \\
0 &\equiv (c_5 - 1)t_{2,0} \quad \text{(A2)} \\
s_{1,2} &\equiv 0 \quad \text{(B1)} \\
t_{0,1} &\equiv c_4 t_{2,0} \quad \text{(B2)}.
\end{align*}
\]

If \( c_5 - 1 \neq 0 \) then \( t_{2,0} \equiv 0 \). Thus \( s_{2,1} \equiv t_{0,1} \equiv 0 \). Hence \( \partial^{-1}W_2 = \partial^{-1}W_1 \).

Thus \( \text{rank}(\partial^{-1}W_2/W_2) = \text{rank} (\partial^{-1}W_1/W_2) \) and \( W_2 \) is terminal and \( W_2 \notin \hat{L}_2 \).

If \( c_5 - 1 \equiv 0 \), then \( c_5 = 1 \) and \( t_{2,0} \) is a free variable. Taking \( t_{2,0} \equiv 1 \) and \( t_{1,0} = 0 \), the matrix \( x \) becomes

\[
v_2 = \begin{bmatrix} 0 & c_4 & 0 \\ 0 & 0 & 0 \\ 1 & 3c_4 & 0 \end{bmatrix}.
\]

We see that neither \( v_2 \) nor \( 3v_2 \) is contained in \( W_2 \) and that \( \partial^{-1}W_2 = \langle \partial^{-1}W_0, v_1, v_2 \rangle \). We know that \( v_2 \in \partial^{-1}W_2 \) and since \( 3v_2 \notin W_2 \) then \( v_2 + W_2 \) is an element of the group \( \partial^{-1}W_2/W_2 \) whose order is larger than 3. Recall that \( |\partial^{-1}W_2/W_2| = 3^4 \) and that \( \partial^{-1}W_1/W_2 \) has rank 3. Since \( \partial^{-1}W_2 \) has an element of order larger than 3 then \( \text{rank}(\partial^{-1}W_2/W_2) \) is not greater than 3 since \( \partial^{-1}W_2/W_2 \cong \mathbb{Z}_9 \times \mathbb{Z}_3 \times \mathbb{Z}_3 \) and therefore \( W_2 \) is terminal and \( W_2 \notin \hat{L}_2 \).
Case 5.2.1.2

Suppose \( c_1 \neq 0 \). We now show that \( y_1 + W_1, y_3 + W_1 \) is a basis for the 2-dimensional vector space \( \partial^{-1}W_0/W_1 \). To show this we examine the three cases \( c_2 = 0 \), \( c_2 = 1 \), and \( c_2 = 2 \).

First suppose \( c_2 = 0 \). Then

\[
m_1 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 3c_1 & 0 \\ 3 & 0 & 0 \end{bmatrix}
\]

and we see that \( y_1 + W_1, y_3 + W_1 \) is a basis for \( \partial^{-1}W_0/W_1 \) as desired.

Next suppose \( c_2 = 1 \). Then

\[
m_1 = \begin{bmatrix} 1 & 0 & -3 \\ 0 & 3c_1 & 0 \\ 0 & 0 & 0 \end{bmatrix}
\]

and

\[
m_1 - m_2 = \begin{bmatrix} 0 & 0 & -3 \\ 0 & 3c_1 & 0 \\ 3 & 0 & 0 \end{bmatrix}
\]

Since \( W_1 = \langle W_0, m_1, m_2 \rangle \) we have \( W_1 = \langle W_0, m_2, m_1 - m_2 \rangle \). Thus \( y_1 + W_1, y_3 + W_1 \) is a basis for \( \partial^{-1}W_0/W_1 \) as desired.

Finally suppose \( c_2 = 2 \). Then

\[
m_1 = \begin{bmatrix} 2 & 0 & 3 \\ 0 & 3c_1 & 0 \\ -3 & 0 & 0 \end{bmatrix}
\]

and

\[
m_1 + m_2 = \begin{bmatrix} 3 & 0 & 3 \\ 0 & 3c_1 & 0 \\ 3 & 0 & 0 \end{bmatrix}
\]

Since \( W_1 = \langle W_0, m_1, m_2 \rangle \) we have \( W_1 = \langle W_0, m_2, m_1 + m_3 \rangle \). Thus \( y_1 + W_1, y_3 + W_1 \) is a basis for \( \partial^{-1}W_0/W_1 \) as desired. Now \( \partial^{-1}W_0/W_1 \) has basis \( y_1 + W_1, y_3 + W_1 \) while \( \partial^{-1}W_1/W_1 \) has basis \( y_1 + W_1, y_3 + W_1, v_1 + W_1 \).

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We fix arbitrary values $c_4, c_5 \in \{0, 1, 2\}$. There are $3^2 = 9$ ways to choose these values. Let $m_3 = c_4y_1 + c_5y_3 + v_1$. Thus

$$m_3 = \begin{bmatrix}
0 & 0 & 3c_5 \\
1 & 0 & 0 \\
3c_4 & 0 & 0
\end{bmatrix}.$$ 

Let $W_2 = \langle W_1, m_3 \rangle$. There are 9 subgroups $W_2$ of this type. Since $m_3 \notin W_1$ and $3m_3 \in W_1$, then $|W_2/W_1| = 3$. Also, since $|W_1| = 3^5$ then $|W_2| = 3^6$. Recall $|\partial^{-1}W_1| = 3^8$, then $\partial^{-1}W_1/W_2 \cong \mathbb{Z}_3 \times \mathbb{Z}_3$ and $\text{rank}(\partial^{-1}W_1/W_2) = 2$.

The subgroup $W_2$ is contained in the pattern subgroup

$$\begin{bmatrix}
2 & 1 & 1 \\
2 & 1 & 0 \\
1 & 0 & 0
\end{bmatrix}.$$ 

The pullback $\partial^{-1}W_2$ is contained in the pattern subgroup

$$\begin{bmatrix}
2 & 2 & 2 \\
2 & 1 & 1 \\
2 & 1 & 0
\end{bmatrix} \cap \begin{bmatrix}
2 & 2 & 1 \\
2 & 1 & 0
\end{bmatrix} = \begin{bmatrix}
2 & 2 & 1 \\
2 & 1 & 1
\end{bmatrix}.$$ 

Let

$$x = \begin{bmatrix}
3s_{0,0} + t_{0,0} & 3s_{0,1} + t_{0,1} & 3s_{0,2} \\
3s_{1,0} + t_{1,0} & 3s_{1,1} & 3s_{1,2} \\
3s_{2,0} + t_{2,0} & 3s_{2,1} & 0
\end{bmatrix} \in \begin{bmatrix}
2 & 2 & 1 \\
2 & 1 & 1
\end{bmatrix}.$$ 

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Thus
\[ \partial_1 x = \begin{bmatrix}
3s_{1,0} + t_{1,0} & 3s_{1,1} & 3s_{1,2} \\
3s_{2,0} + t_{2,0} & 3s_{2,1} & 0 \\
-3(t_{1,0} + t_{2,0}) & 0 & 0
\end{bmatrix} \quad \text{and} \quad \partial_2 x = \begin{bmatrix}
3s_{0,1} + t_{0,1} & 3s_{0,2} & -3t_{0,1} \\
3s_{1,1} & 3s_{1,2} & 0 \\
3s_{2,1} & 0 & 0
\end{bmatrix}. \]

The variables that are in play are those appearing in the matrix
\[ \begin{bmatrix}
0 & t_{0,1} & 0 \\
t_{1,0} & 0 & 3s_{1,2} \\
t_{2,0} & 3s_{2,1} & 0
\end{bmatrix}. \]

We want to identify a value \( a_1, a_2, a_3 \in \mathbb{Z}_9 \) such that \( \partial_1 x \equiv a_1 m_1 + a_2 m_2 + a_3 m_3 \pmod{I} \). A formal expression for \( a_1 m_1 + a_2 m_2 + a_3 m_3 \) is
\[ a_1 \begin{bmatrix}
c_2 & 0 & -3c_2 \\
0 & 3c_1 & 0 \\
3(1-c_2) & 0 & 0
\end{bmatrix} + a_2 \begin{bmatrix}
1 & 0 & 0 \\
0 & 0 & 0 \\
-3 & 0 & 0
\end{bmatrix} + a_3 \begin{bmatrix}
0 & 0 & 3c_5 \\
0 & 0 & 0 \\
3c_4 & 0 & 0
\end{bmatrix} = \begin{bmatrix}
a_1 c_2 + a_2 & 0 & 3(-a_1 c_2 + a_3 c_5) \\
a_3 & 3a_1 c_1 & 0 \\
3(a_1 c_2 a_1 - a_2 + a_3 c_4) & 0 & 0
\end{bmatrix}. \]

Comparing \((0,1)\)-entries, we get \( 0 \equiv 3s_{1,1} \), which gives us no information.

Comparing \((1,0)\)-entries, we get \( a_3 \equiv t_{2,0} \).

Comparing \((1,1)\)-entries, we get \( a_1 c_1 \equiv s_{2,1} \). Since \( c_1 \neq 0 \) we obtain \( a_1 \equiv c_1 s_{2,1} \).

Comparing \((0,0)\)-entries, we get \( a_2 \equiv t_{1,0} - a_1 c_2 \). Substituting \( a_1 \) we obtain \( a_2 \equiv t_{1,0} - c_1 c_2 s_{2,1} \).
Comparing (2, 0)-entries, we get $a_1 - c_2 a_1 - a_2 + a_3 c_4 \equiv -t_{1,0} - t_{2,0}$. Substituting $a_1, a_2, a_3$ we obtain $s_{2,1} \equiv (-c_1 c_4 - c_1) t_{2,0}$.

Comparing (0, 2)-entries, we get $-a_1 c_2 + a_3 c_5 \equiv s_{1,2}$. Substituting $a_1, a_3, s_{2,1}$ we obtain $s_{1,2} \equiv (c_2 c_4 + c_2 + c_5) t_{2,0}$.

We see that $\partial_1 x \in W_2$ if and only if

$$s_{2,1} \equiv (-c_1 c_4 - c_1) t_{2,0} \quad (A1)$$

$$s_{1,2} \equiv (c_2 c_4 + c_2 + c_5) t_{2,0} \quad (A2).$$

We want to identify a value $b_1, b_2, b_3 \in \mathbb{Z}_9$ such that $\partial_1 x \equiv b_1 m_1 + b_2 m_2 + b_3 m_3 \pmod{I}$. A formal expression for $b_1 m_1 + b_2 m_2 = b_3 m_3$ is

$$b_1 \begin{bmatrix} c_2 & 0 & -3c_2 \\ 0 & 3c_4 & 0 \\ 3(1 - c_2) & 0 & 0 \end{bmatrix} + b_2 \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ -3 & 0 & 0 \end{bmatrix} + b_3 \begin{bmatrix} 0 & 0 & 3c_5 \\ 1 & 0 & 0 \\ 3c_4 & 0 & 0 \end{bmatrix} = \begin{bmatrix} b_1 c_2 + b_2 & 0 & 3(-b_1 c_2 + b_3 c_5) \\ b_3 & 3b_1 c_1 & 0 \\ 3(b_1 - c_2 b_1 - b_2 + b_3 c_4) & 0 & 0 \end{bmatrix}.$$}

Comparing (0, 1)-entries, we get $0 \equiv 3s_{0,2}$, which gives us no information.

Comparing (1, 0)-entries, we get $b_3 \equiv 0$.

Comparing (1, 1)-entries, we get $b_1 c_1 \equiv s_{1,2}$. Since $c_1 \neq 0$ we obtain $b_1 \equiv c_1 s_{1,2}$.

Comparing (0, 0)-entries, we get $b_2 \equiv t_{0,1} - b_1 c_2$. Substituting $b_1$ we obtain $b_2 \equiv t_{0,1} - c_1 c_2 s_{1,2}$.
Comparing $(0, 2)$-entries, we get $-b_1c_2 + b_3c_5 \equiv -t_{0,1}$. Substituting $b_1, b_3$ we obtain $t_{0,1} \equiv c_1c_2s_{1,2}$.

Comparing $(2, 0)$-entries, we get $b_1 - c_2b_1 - b_2 + b_3c_4 \equiv s_{2,1}$. Substituting $b_1, b_2, b_3, t_{0,1}$ we obtain $s_{2,1} \equiv (c_1 - c_1c_2)s_{1,2}$.

We see that $\partial_1 x \in W_2$ if and only if

$$t_{0,1} \equiv c_1c_2s_{1,2} \quad (B1)$$

$$s_{2,1} \equiv (c_1 - c_1c_2)s_{1,2} \quad (B2).$$

Combining (A1) and (B2) we obtain $(-c_1c_4 - c_1)t_{2,0} \equiv (c_1 - c_1c_2)s_{1,2}$. Substituting (A2) into this new congruence we obtain $0 \equiv (c_1c_2c_4 + c_1c_2 + c_1c_5 - c_1c_2^2c_4 - c_1c_2^2 - c_1c_2c_5 + c_1c_4 + c_1)t_{2,0}$. For convenience let $q = c_1c_2c_4 + c_1c_2 + c_1c_5 - c_1c_2^2c_4 - c_1c_2^2 - c_1c_2c_5 + c_1c_4 + c_1$. Therefore we obtain $0 \equiv qt_{2,0}$ which we will denote as our new (B2).

Hence $x \in \partial^{-1}W_2$ if and only if

$$s_{2,1} \equiv (-c_1c_4 - c_1)t_{2,0} \quad (A1)$$

$$s_{1,2} \equiv (c_2c_4 + c_2 + c_5)t_{2,0} \quad (A2)$$

$$t_{0,1} \equiv c_1c_2s_{1,2} \quad (B1)$$

$$0 \equiv qt_{2,0} \quad (B2).$$

It is convenient to consider the cases $q \not\equiv 0$ and $q \equiv 0$ separately.

If $q \not\equiv 0$ then $t_{2,0} \equiv 0$. It follows that $s_{2,1} \equiv s_{1,2} \equiv t_{0,1} \equiv 0$. So $\partial^{-1}W_2 = \partial^{-1}W_1$. Thus $\text{rank}(\partial^{-1}W_2/W_2) = \text{rank}(\partial^{-1}W_1/W_2)$ and $W_2$ is terminal and $W_2 \not\in \hat{\mathcal{L}}_2$. 199
If \( q \equiv 0 \) then \( t_{2,0} \) is a new free variable. Taking \( t_{2,0} \equiv 1 \) and \( t_{1,0} = 0 \), the matrix \( x \) becomes

\[
v_2 = \begin{bmatrix}
0 & c_1c_2(c_2c_4 + c_2 + c_5) & 0 \\
0 & 0 & 3(c_2c_4 + c_2 + c_5) \\
1 & 3(-c_1c_4 - c_1) & 0
\end{bmatrix}.
\]

We see that neither \( v_2 \) nor \( 3v_2 \) is contained in \( W_2 \) and that \( \partial^{-1}W_2 = \langle \partial^{-1}W_0, v_1, v_2 \rangle \). We know that \( v_2 \in \partial^{-1}W_2 \) and since \( 3v_2 \notin W_2 \) then \( v_2 + W_2 \) is an element of the group \( \partial^{-1}W_2/W_2 \) whose order is larger than 3. Recall that \( |\partial^{-1}W_2/W_2| = 3^4 \) and that \( \partial^{-1}W_1/W_2 \) has rank 3. Since \( \partial^{-1}W_2 \) has an element of order larger than 3 then \( \text{rank}(\partial^{-1}W_2/W_2) \) is not greater than 3 since \( \partial^{-1}W_2/W_2 \cong \mathbb{Z}_9 \times \mathbb{Z}_3 \times \mathbb{Z}_3 \) and therefore \( W_2 \) is terminal and \( W_2 \notin \hat{L}_2 \).

8.2.2 Case 5.2.2

Suppose \((c_1, c_2) = (0, 1)\). Then (B1) and (B2) hold automatically and \( t_{0,1} \) is a new free variable. Hence \( x \in \partial^{-1}W_2 \) if and only if

\[
s_{2,1} \equiv 0 \quad (A1)
\]

\[
s_{1,2} \equiv 0 \quad (A2).
\]

We regard \( t_{1,0} \) and \( t_{0,1} \) as free variables. Taking \( t_{1,0} \equiv 1 \) and \( t_{0,1} = 0 \), the matrix \( x \) becomes

\[
v_1 = \begin{bmatrix}
0 & 0 & 0 \\
1 & 0 & 0 \\
0 & 0 & 0
\end{bmatrix}
\]
Taking \( t_{0,1} \equiv 1 \) and \( t_{1,0} = 0 \), the matrix \( x \) becomes

\[
v_2 = \begin{bmatrix}
0 & 1 & 0 \\
0 & 0 & 0 \\
0 & 0 & 0
\end{bmatrix}.
\]

We see that \( \partial^{-1}W_1 = \langle \partial^{-1}W_0, v_1, v_2 \rangle \). Since \( v_1 \notin \partial^{-1}W_0 \), \( v_2 \notin \partial^{-1}W_0, v_1 \rangle \), and \( 3v_1 \in \partial^{-1}W_0 \), and \( 3v_2 \in \partial^{-1}W_0, v_1 \rangle \), we know

\[ |\partial^{-1}W_1/\partial^{-1}W_0| = 3^2. \]

Since \( |\partial^{-1}W_0| = 3^7 \) then \( |\partial^{-1}W_1| = 3^9 \). Because \( |W_1| = 3^5 \) then \( |\partial^{-1}W_1/W_1| = 3^4 \). Also, \( v_1, v_2, y_2, y_1 + y_3 \in \partial^{-1}W_1 \) but are not contained in \( W_1 \). Each of \( 3v_1, 3v_2, 3y_2, 3(y_1 + y_3) \) is contained in \( W_1 \). So \( v_1 + W_1, v_2 + W_1, y_2 + W_1, y_1 + y_3 + W_1 \) are elements of order 3 in the group \( \partial^{-1}W_1/W_1 \). These four elements form a generating set for the group \( \partial^{-1}W_1/W_1 \). Recall that \( |\partial^{-1}W_1/W_1| = 3^4 \), we obtain \( \partial^{-1}W_1/W_1 \cong \mathbb{Z}_3 \times \mathbb{Z}_3 \times \mathbb{Z}_3 \times \mathbb{Z}_3 \). Since \( \text{rank}(\partial^{-1}W_1/W_1) = 4 \) and \( \text{rank}(\partial^{-1}W_0/W_1) = 2 \), then \( \text{rank}(\partial^{-1}W_0/W_1) < \text{rank}(\partial^{-1}W_1/W_1) \). Hence \( W_1 \) is nonterminal and \( W_1 \in \hat{L}_1 \).

A basis for \( \partial^{-1}W_1/W_1 \) is \( v_1 + W_1, v_2 + W_1, y_2 + W_1, y_1 + y_3 + W_1 \). A basis for \( \partial^{-1}W_0/W_1 \) is \( y_2 + W_1, y_1 + y_3 + W_1 \).

The subgroups \( W_2 \) belonging to \( L(W_1) \) correspond to the nontrivial proper subspace \( W_2/W_1 \) of \( \partial^{-1}W_1/W_1 \) for which the intersection \( W_2/W_1 \cap \partial^{-1}W_0/W_1 \) is trivial. Since \( \partial^{-1}W_1/W_1 \) has dimension 4 while its subspace \( \partial^{-1}W_0/W_1 \) has dimension 2, every such subspace \( W_2/W_1 \) has dimension either 1 or 2.

To help us define the subgroups \( W_2 \) belonging to \( L_2(W_1) \), it will be convenient to identify each element of the vector space \( \partial^{-1}W_1/W_1 \) with its coordinate vector with respect to the ordered basis \( y_2 + W_1, y_1 + y_3 + W_1, v_1 + W_1, v_2 + W_1 \). In this way, we
identify $\partial^{-1}W_1/W_1$ with the vector space $\mathbb{Z}_3 \times \mathbb{Z}_3 \times \mathbb{Z}_3 \times \mathbb{Z}_3$ consisting of row vectors.

Under this identification, the elements $y_2 + W_1, y_1 + y_3 + W_1, v_1 + W_1, v_2 + W_1$ are associated with the so-called standard basis vectors $[1, 0, 0, 0], [0, 1, 0, 0], [0, 0, 1, 0], [0, 0, 0, 1]$ in $\mathbb{Z}_3 \times \mathbb{Z}_3 \times \mathbb{Z}_3 \times \mathbb{Z}_3$.

The subgroups $W_2$ belonging to $\mathcal{L}(W_1)$ are in one-to-one correspondence with the nontrivial proper subgroup $W_2/W_1$ of $\partial^{-1}W_1/W_1$ for which the intersection $W_2/W_1 \cap \partial^{-1}W_0/W_1$ is trivial. Note that $\partial^{-1}W_0W_1$ is the 2-dimensional subspace generated by the pair of elements $y_2 + W_1$ and $y_1 + y_3 + W_1$. Under our identification, each such subspace $W_2/W_1$ is associated with a subspace $S$ of $\mathbb{Z}_3 \times \mathbb{Z}_3 \times \mathbb{Z}_3 \times \mathbb{Z}_3$ that contains no nonzero vector that is a linear combination of $[1, 0, 0, 0]$ and $[0, 1, 0, 0]$.

Let $m$ denote the unique matrix in reduced row-echelon form that is row equivalent to the matrix whose rows are the members of an arbitrarily-chosen basis of such a subspace $S$.

In Case 5.2.2.1 we consider the 1-dimensional subspaces $W_2/W_1$. There are four possible forms for the matrix $m$. The first form is

$$m = [0, 0, 0, 1]$$

(1 possibility), which is considered in Case 5.2.2.1.1. The second form is

$$m = [0, 0, 1, c_4] \quad \text{for } c_4 \in \{0, 1, 2\}$$

(3 possibilities), which is considered in Case 5.2.2.1.2. The third form is

$$m = [0, 1, c_4, c_5] \quad \text{for } c_4, c_5 \in \{0, 1, 2\}, \ (c_4, c_5) \neq (0, 0)$$

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(8 possibilities), which is considered in Case 5.2.2.1.3. The fourth form is

\[ m = [1, c_4, c_5, c_6] \quad \text{for} \quad c_4, c_5, c_6 \in \{0, 1, 2\} \quad (c_5, c_6) \neq (0, 0) \]

(24 possibilities), which is considered in Case 5.2.2.1.4.

In Case 5.2.2.2 we consider the 2-dimensional subspaces \( W_2/W_1 \). There are six possible forms for the matrix \( m \). The first form is

\[
\begin{bmatrix}
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1
\end{bmatrix}
\]

(1 possibility), which is considered in Case 5.2.2.2.1. The second form is

\[
\begin{bmatrix}
0 & 1 & c_4 & 0 \\
0 & 0 & 0 & 1
\end{bmatrix}
\quad \text{for} \quad c_4 \in \{1, 2\}
\]

(2 possibilities), which is considered in Case 5.2.2.2. The third form is

\[
\begin{bmatrix}
0 & 1 & 0 & c_4 \\
0 & 0 & 1 & c_5
\end{bmatrix}
\quad \text{for} \quad c_4 \in \{1, 2\} \quad \text{and} \quad c_5 \in \{0, 1, 2\}
\]

(6 possibilities), which is considered in Case 5.2.2.2.3. The fourth form is

\[
\begin{bmatrix}
1 & c_4 & c_5 & 0 \\
0 & 0 & 0 & 1
\end{bmatrix}
\quad \text{for} \quad c_4 \in \{0, 1, 2\} \quad \text{and} \quad c_5 \in \{1, 2\}
\]

(6 possibilities), which is considered in Case 5.2.2.2.4 The fifth form is

\[
\begin{bmatrix}
1 & c_4 & 0 & c_5 \\
0 & 0 & 1 & c_6
\end{bmatrix}
\quad \text{for} \quad c_4, c_6 \in \{0, 1, 2\} \quad \text{and} \quad c_5 \in \{1, 2\}
\]
(18 possibilities), which is considered in Case 5.2.2.5. The sixth form is

\[
m = \begin{bmatrix}
1 & 0 & c_4 & c_5 \\
0 & 1 & c_6 & c_7
\end{bmatrix}
\]

where \{(c_4, c_5), (c_6, c_7)\} is an ordered pair of linearly independent vectors in \(\mathbb{Z}_3 \times \mathbb{Z}_3\) (48 possibilities) which is considered in Case 5.2.2.6.

Case 5.2.2.1

We consider the 1-dimensional subspaces \(W_2/W_1\). Let \(d_1, d_2, d_3, d_4\) be unspecified variables. Let \(m_3 = d_1 y_2 + d_2 (y_1 + y_3) + d_3 v_1 + d_4 v_2\). A formal expression for \(m_3\) is

\[
m_3 = d_1 \begin{bmatrix}
0 & 0 & 0 \\
0 & 3 & 0 \\
0 & 0 & 0
\end{bmatrix} + d_2 \begin{bmatrix}
0 & 0 & 3 \\
0 & 0 & 0 \\
3 & 0 & 0
\end{bmatrix} + d_3 \begin{bmatrix}
0 & 0 & 0 \\
1 & 0 & 0 \\
0 & 0 & 0
\end{bmatrix} + d_4 \begin{bmatrix}
0 & 1 & 0 \\
0 & 0 & 0 \\
0 & 0 & 0
\end{bmatrix}
\]

\[
= \begin{bmatrix}
0 & d_4 & 3d_2 \\
d_3 & 3d_1 & 0 \\
3d_2 & 0 & 0
\end{bmatrix}
\]

Let \(W_2 = < W_1, m_3 > \in \mathcal{L}_2\).

We now calculate the pullback \(\partial^{-1} W_2\). The subgroup \(W_2\) is contained in the pattern subgroup

\[
\begin{bmatrix}
2 & 2 & 1 \\
2 & 1 & 0 \\
1 & 0 & 0
\end{bmatrix}
\]
Thus the pullback $\partial^{-1}W_2$ is contained in the pattern subgroup

$$
\begin{bmatrix}
2 & 2 & 2 \\
2 & 2 & 1 \\
2 & 1 & 0
\end{bmatrix} \cap \begin{bmatrix}
2 & 2 & 2 \\
2 & 2 & 1 \\
2 & 1 & 0
\end{bmatrix} = \begin{bmatrix}
2 & 2 & 2 \\
2 & 2 & 1 \\
2 & 1 & 0
\end{bmatrix}.
$$

Let

$$
x = \begin{bmatrix}
3s_{0,0} + t_{0,0} & 3s_{0,1} + t_{0,1} & 3s_{0,2} + t_{0,2} \\
3s_{1,0} + t_{1,0} & 3s_{1,1} + t_{1,1} & 3s_{1,2} \\
3s_{2,0} + t_{2,0} & 3s_{2,1} & 0
\end{bmatrix} \in \begin{bmatrix}
2 & 2 & 2 \\
2 & 2 & 1 \\
2 & 1 & 0
\end{bmatrix}.
$$

Thus

$$
\partial_1 x = \begin{bmatrix}
3s_{1,0} + t_{1,0} & 3s_{1,1} + t_{1,1} & 3s_{1,2} \\
3s_{2,0} + t_{2,0} & 3s_{2,1} & 0 \\
-3(t_{1,0} + t_{2,0}) & -3t_{1,1} & 0
\end{bmatrix}
$$

and

$$
\partial_2 x = \begin{bmatrix}
3s_{0,1} + t_{0,1} & 3s_{0,2} + t_{0,2} & -3(t_{0,1} + t_{0,2}) \\
3s_{1,1} + t_{1,1} & 3s_{1,2} & -3t_{1,1} \\
3s_{2,1} & 0 & 0
\end{bmatrix}.
$$

The variables that are in play are those appearing in the matrix

$$
\begin{bmatrix}
0 & t_{0,1} & t_{0,2} \\
t_{0,1} & t_{1,1} & 3s_{1,2} \\
t_{2,0} & 3s_{2,1} & 0
\end{bmatrix}.
$$
We wish to identify values $a_1, a_2, a_3 \in \mathbb{Z}_9$ such that $\partial_1 x \equiv a_1 m_1 + a_2 m_2 + a_3 m_3$ (mod $I$). A formal expression for $a_1 m_1 + a_2 m_2 + a_3 m_3$ is

\[
\begin{bmatrix}
1 & 0 & -3 \\
0 & 0 & 0 \\
0 & 0 & 0
\end{bmatrix} a_1 + \begin{bmatrix}
1 & 0 & 0 \\
0 & 0 & 0 \\
-3 & 0 & 0
\end{bmatrix} a_2 + \begin{bmatrix}
0 & d_4 & 3d_2 \\
d_3 & 3d_1 & 0 \\
3d_2 & 0 & 0
\end{bmatrix} a_3
= \begin{bmatrix}
a_1 + a_2 & a_3d_4 & 3(-a_1 + a_3d_2) \\
a_3d_3 & 3a_3d_1 & 0 \\
3(-a_2 + a_3d_2) & 0 & 0
\end{bmatrix}.
\]

Comparing (2,1)-entries, we get $t_{1,1} \equiv 0$.

Comparing (0,1)-entries, we get $a_3d_4 \equiv t_{1,1}$. Substituting $t_{1,1}$ we obtain $a_3d_4 \equiv 0$.

Comparing (1,0)-entries, we get $a_3d_3 \equiv t_{2,0}$.

Comparing (0,2)-entries, we get $a_1 \equiv a_3d_2 - s_{1,2}$.

Comparing (0,0)-entries, we get $a_2 \equiv t_{1,0} - a_1$. Substituting $a_1$ we obtain $a_2 \equiv t_{1,0} - a_3d_2 + s_{1,2}$.

Comparing (1,1)-entries, we get $a_3d_1 \equiv s_{2,1}$.

Comparing (2,0)-entries, we get $-a_2 + a_3d_2 \equiv -t_{0,1} - t_{2,0}$. Substituting $a_2$ we obtain $s_{1,2} \equiv t_{2,0} - a_3d_2$.  

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We see that \( \partial_1 x \in W_2 \) if and only if

\[
\begin{align*}
t_{1,1} &\equiv 0 \quad (A1) \\
a_3d_4 &\equiv 0 \quad (A2) \\
a_3d_3 &\equiv t_{2,0} \quad (A3) \\
a_3d_1 &\equiv s_{2,1} \quad (A4) \\
s_{1,2} &\equiv t_{2,0} - a_3d_2 \quad (A5).
\end{align*}
\]

We wish to identify values \( b_1, b_2, b_3 \in \mathbb{Z}_9 \) such that \( \partial_2 x \equiv b_1m_1 + b_2m_2 + b_3m_3 \pmod{I} \). A formal expression for \( b_1m_1 + b_2m_2 + b_3m_3 \) is

\[
\begin{bmatrix}
1 & 0 & -3 \\
0 & 0 & 0 \\
0 & 0 & 0
\end{bmatrix}
\begin{bmatrix}
b_1 \\
b_2 \\
b_3
\end{bmatrix}
+ \begin{bmatrix}
1 & 0 & 0 \\
0 & 0 & 0 \\
-3 & 0 & 0
\end{bmatrix}
\begin{bmatrix}
b_3d_3 \\
b_3d_4 \\
3d_4
\end{bmatrix}
+ \begin{bmatrix}
0 & d_4 & 3d_2 \\
d_3 & 3d_4 & 0 \\
3d_2 & 0 & 0
\end{bmatrix}
\begin{bmatrix}
b_3d_1 \\
b_3d_3 \\
3(\frac{-b_1}{3} + b_3d_2)
\end{bmatrix}.
\]

Comparing \((1,2)\)-entries, we get \( t_{1,1} \equiv 0 \).

Comparing \((1,0)\)-entries, we get \( b_3d_3 \equiv t_{1,1} \). Therefore \( b_3d_3 \equiv 0 \).

Comparing \((0,1)\)-entries, we get \( b_3d_4 \equiv t_{0,2} \).

Comparing \((0,2)\)-entries, we get \( b_1 \equiv b_3d_2 + t_{0,1} + t_{0,2} \).

Comparing \((0,0)\)-entries, we get \( b_2 \equiv t_{0,1} - b_1 \). Substituting \( b_1 \) we obtain \( -t_{0,2} - b_3d_2 \).

Comparing \((1,1)\)-entries, we get \( b_3d_1 \equiv s_{1,2} \).
Comparing $(2,0)$-entries, we get $-b_2 + b_3d_2 \equiv s_{2,1}$. Substituting $b_2$ we obtain $s_{2,1} \equiv t_{0,2} - b_3d_2$.

We see that $\partial_2x \in W_2$ if and only if

$$
t_{1,1} \equiv 0 \quad (B1)
$$

$$
b_{3}d_{3} \equiv 0 \quad (B2)
$$

$$
b_{3}d_{4} \equiv t_{0,2} \quad (B3)
$$

$$
b_{3}d_{1} \equiv s_{1,2} \quad (B4)
$$

$$
s_{2,1} \equiv t_{0,2} - b_{3}d_{2} \quad (B5).
$$

**Case 5.2.2.1.1** Let $m_3 = v_2$. Thus

$$
m_3 = \begin{bmatrix}
0 & 1 & 0 \\
0 & 0 & 0 \\
0 & 0 & 0
\end{bmatrix}.
$$

Let $W_2 = < W_1, m_3 > \in \mathcal{L}_2$. The number of subgroups $W_2$ of this type is 1. Since $m_3 \notin W_1$ and $3m_1 \in W_1$ we have $|W_2/W_1| = 3$. Since $|W_1| = 3^5$ it follows that $|W_2| = 3^6$. Note that $\partial^{-1}W_1/W_2 \cong \mathbb{Z}_3 \times \mathbb{Z}_3 \times \mathbb{Z}_3$ and $\text{rank}(\partial^{-1}W_1/W_2) = 3$.

We now calculate the pullback of $\partial^{-1}W_2$. We observed in Case 5.2.2.1 that $\partial^{-1}W_2$ is contained in the pattern subgroup

$$
\begin{bmatrix}
2 & 2 & 2 \\
2 & 2 & 1 \\
2 & 1 & 0
\end{bmatrix}.
$$
Let

\[
x = \begin{bmatrix}
3s_{0,0} + t_{0,0} & 3s_{0,1} + t_{0,1} & 3s_{0,2} + t_{0,2} \\
3s_{1,0} + t_{1,0} & 3s_{1,1} + t_{1,1} & 3s_{1,2} \\
3s_{2,0} + t_{2,0} & 3s_{2,1} & 0 \\
\end{bmatrix}
\in\begin{bmatrix}
2 & 2 & 2 \\
2 & 2 & 1 \\
2 & 1 & 0 \\
\end{bmatrix}.
\]

The variables that are in play are those appearing in the matrix

\[
\begin{bmatrix}
0 & t_{0,1} & t_{0,2} \\
t_{1,0} & t_{1,1} & 3s_{1,2} \\
t_{2,0} & 3s_{2,1} & 0 \\
\end{bmatrix}.
\]

In the notation of Case 5.2.2.1, we are taking \(d_1 = 0, \ d_2 = 0, \ d_3 = 0,\) and \(d_4 = 1.\) We wish to identify values \(a_1, a_2, a_3 \in \mathbb{Z}_9\) such that \(\partial_1 x \equiv a_1m_1 + a_2m_2 + a_3m_3 \pmod{I}.\) We see that \(\partial_1 x \in W_2\) if and only if

\[
t_{1,1} \equiv 0 \quad (A1)
\]
\[
a_3 \equiv 0 \quad (A2)
\]
\[
0 \equiv t_{2,0} \quad (A3)
\]
\[
0 \equiv s_{2,1} \quad (A4)
\]
\[
s_{1,2} \equiv 0 \quad (A5).
\]
We wish to identify values $b_1, b_2, b_3 \in \mathbb{Z}_9$ such that $\partial_1 x \equiv b_1 m_1 + b_2 m_2 + b_3 m_3 \pmod{I}$. We see that $\partial_2 x \in W_2$ if and only if

$$t_{1,1} \equiv 0 \quad (\text{B1})$$

$$b_3 \equiv t_{0,2} \quad (\text{B2})$$

$$0 \equiv s_{1,2} \quad (\text{B4})$$

$$s_{2,1} \equiv t_{0,2} \quad (\text{B5}).$$

(B1) is redundant with (A1) so we may ignore it. (A4) tells us (B5) holds and substituting (B4) into (A5) we obtain $t_{0,2} \equiv 0$.

Hence $x \in \partial^{-1}W_2$ if and only if

$$t_{1,1} \equiv 0$$

$$t_{2,0} \equiv 0$$

$$t_{0,2} \equiv 0$$

$$s_{2,1} \equiv 0$$

$$s_{1,2} \equiv 0.$$

So $\partial^{-1}W_2 = \partial^{-1}W_1$. Thus $\text{rank}(\partial^{-1}W_2/W_2) = \text{rank}(\partial^{-1}W_1/W_2)$, $W_2$ is terminal, and $W_2 \not\in \hat{L}_2$. 
Case 5.2.2.1.2 We fix arbitrary value $c_4 \in \{0, 1, 2\}$. There are 3 ways to choose the value $c_4$. Let $m_3 = v_1 + c_4 v_2$. Thus

$$
m_3 = \begin{bmatrix}
0 & c_4 & 0 \\
1 & 0 & 0 \\
0 & 0 & 0
\end{bmatrix}.
$$

Let $W_2 = < W_1, m_3 > \in \mathcal{L}_2$. The number of subgroups $W_2$ of this type is 3. Since $m_3 \notin W_1$ and $3m_1 \notin W_1$ we have $|W_2/W_1| = 3$. Since $|W_1| = 3^5$ it follows that $|W_2| = 3^6$. Note that $\partial^{-1}W_1/W_2 \cong \mathbb{Z}_3 \times \mathbb{Z}_3 \times \mathbb{Z}_3$ and $\text{rank}(\partial^{-1}W_1/W_2) = 3$. We now calculate the pullback of $\partial^{-1}W_2$. We observed in Case 5.2.2.1 that $\partial^{-1}W_2$ is contained in the pattern subgroup

$$
\begin{bmatrix}
2 & 2 & 2 \\
2 & 2 & 1 \\
2 & 1 & 0
\end{bmatrix}.
$$

Let

$$
x = \begin{bmatrix}
3s_{0,0} + t_{0,0} & 3s_{0,1} + t_{0,1} & 3s_{0,2} + t_{0,2} \\
3s_{1,0} + t_{1,0} & 3s_{1,1} + t_{1,1} & 3s_{1,2} \\
3s_{2,0} + t_{2,0} & 3s_{2,1} & 0
\end{bmatrix} \in \begin{bmatrix}
2 & 2 & 2 \\
2 & 2 & 1 \\
2 & 1 & 0
\end{bmatrix}.
$$

The variables that are in play are those appearing in the matrix

$$
\begin{bmatrix}
0 & t_{0,1} & t_{0,2} \\
t_{1,0} & t_{1,1} & 3s_{1,2} \\
t_{2,0} & 3s_{2,1} & 0
\end{bmatrix}.
$$
In the notation of Case 5.2.2.1, we are taking \( d_1 = 0, \ d_2 = 0, \ d_3 = 1, \) and \( d_4 = c_4. \) We wish to identify values \( a_1, a_2, a_3 \in \mathbb{Z}_9 \) such that \( \partial_1 x \equiv a_1 m_1 + a_2 m_2 + a_3 m_3 \pmod{I}. \)

We see that \( \partial_1 x \in W_2 \) if and only if

\[
\begin{align*}
  t_{1,1} &\equiv 0 \quad (A1) \\
  a_3 c_4 &\equiv 0 \quad (A2) \\
  a_3 &\equiv t_{2,0} \quad (A3) \\
  0 &\equiv s_{2,1} \quad (A4) \\
  s_{1,2} &\equiv t_{2,0} \quad (A5). 
\end{align*}
\]

We wish to identify values \( b_1, b_2, b_3 \in \mathbb{Z}_9 \) such that \( \partial_2 x \equiv b_1 m_1 + b_2 m_2 + b_3 m_3 \pmod{I}. \)

We see that \( \partial_2 x \in W_2 \) if and only if

\[
\begin{align*}
  t_{1,1} &\equiv 0 \quad (B1) \\
  b_3 &\equiv 0 \quad (B2) \\
  b_3 c_4 &\equiv t_{0,2} \quad (B3) \\
  0 &\equiv s_{1,2} \quad (B4) \\
  s_{2,1} &\equiv t_{0,2} \quad (B5). 
\end{align*}
\]

(B1) is redundant with (A1) so we may ignore it. Substituting \( b_3 \) into (B3) we obtain \( 0 \equiv t_{0,2}. \) Substituting (B3) and (A4) we see that (B5) holds. Substituting (B4) into (A5) we obtain \( t_{2,0} \equiv 0. \)
Hence $x \in \partial^{-1}W_2$ if and only if

\begin{align*}
t_{1,1} &\equiv 0 \\
t_{2,0} &\equiv 0 \\
t_{0,2} &\equiv 0 \\
s_{1,2} &\equiv 0 \\
s_{2,1} &\equiv 0.
\end{align*}

Thus $\partial^{-1}W_2 = \partial^{-1}W_1$. Thus $\text{rank}(\partial^{-1}W_2/W_2) = \text{rank}(\partial^{-1}W_1/W_2)$, $W_2$ is terminal, and $W_2 \not\in \hat{L}_2$.

**Case 5.2.2.1.3** We fix arbitrary values $c_4, c_5 \in \{0, 1, 2\}, (c_4, c_5) \neq (0, 0)$. There are 8 ways to choose the values $c_4, c_5$. Let $m_3 = y_1 + y_3 + c_4v_1 + c_5v_2$. Thus

$$m_3 = \begin{bmatrix} 0 & c_5 & 3 \\ c_4 & 0 & 0 \\ 3 & 0 & 0 \end{bmatrix}.$$ 

Let $W_2 = \langle W_1, m_3 \rangle \in \mathcal{L}_2$. The number of subgroups $W_2$ of this type is 8. Since $m_3 \not\in W_1$ and $3m_1 \in W_1$ we have $|W_2/W_1| = 3$. Since $|W_1| = 3^5$ it follows that $|W_2| = 3^6$. Note that $\partial^{-1}W_1/W_2 \cong \mathbb{Z}_3 \times \mathbb{Z}_3 \times \mathbb{Z}_3$ and $\text{rank}(\partial^{-1}W_1/W_2) = 3$. We now calculate the pullback of $\partial^{-1}W_2$. We observed in Case 5.2.2.1 that $\partial^{-1}W_2$ is
contained in the pattern subgroup

\[
\begin{bmatrix}
  2 & 2 & 2 \\
  2 & 2 & 1 \\
  2 & 1 & 0
\end{bmatrix}.
\]

Let

\[
x = \begin{bmatrix}
  3s_{0,0} + t_{0,0} & 3s_{0,1} + t_{0,1} & 3s_{0,2} + t_{0,2} \\
  3s_{1,0} + t_{1,0} & 3s_{1,1} + t_{1,1} & 3s_{1,2} \\
  3s_{2,0} + t_{2,0} & 3s_{2,1} & 0
\end{bmatrix}
\in \begin{bmatrix}
  2 & 2 & 2 \\
  2 & 2 & 1 \\
  2 & 1 & 0
\end{bmatrix}.
\]

The variables that are in play are those appearing in the matrix

\[
\begin{bmatrix}
  0 & t_{0,1} & t_{0,2} \\
  t_{1,0} & t_{1,1} & 3s_{1,2} \\
  t_{2,0} & 3s_{2,1} & 0
\end{bmatrix}.
\]

In the notation of Case 5.2.2.1, we are taking \(d_1 = 0, d_2 = 1, d_3 = c_4,\) and \(d_4 = c_5.\)

We wish to identify values \(a_1, a_2, a_3 \in \mathbb{Z}_9\) such that \(\partial_1 x \equiv a_1 m_1 + a_2 m_2 + a_3 m_3 \pmod{I}.\)

We see that \(\partial_1 x \in W_2\) if and only if

\[
\begin{align*}
t_{1,1} & \equiv 0 \quad (A1) \\
a_3 c_5 & \equiv 0 \quad (A2) \\
a_3 c_4 & \equiv t_{2,0} \quad (A3) \\
0 & \equiv s_{2,1} \quad (A4) \\
s_{1,2} & \equiv t_{2,0} - a_3 \quad (A5).
\end{align*}
\]
We wish to identify values $b_1, b_2, b_3 \in \mathbb{Z}_9$ such that $\partial_2 x \equiv b_1 m_1 + b_2 m_2 + b_3 m_3 (\text{mod } I)$. We see that $\partial_2 x \in W_2$ if and only if

$$
t_{1,1} \equiv 0 \quad (B1)
$$

$$
b_3 c_4 \equiv 0 \quad (B2)
$$

$$
b_3 c_5 \equiv t_{0,2} \quad (B3)
$$

$$
0 \equiv s_{1,2} \quad (B4)
$$

$$
s_{2,1} \equiv t_{0,2} - b_3 \quad (B5). 
$$

(B1) is redundant with (A1) so we may ignore it.

Hence $x \in \partial^{-1} W_2$ if and only if

$$
t_{1,1} \equiv 0 \quad (A1)
$$

$$
a_3 c_5 \equiv 0 \quad (A2)
$$

$$
a_3 c_4 \equiv t_{2,0} \quad (A3)
$$

$$
s_{1,2} \equiv a_3 - t_{2,0} \quad (A4)
$$

$$
s_{1,2} \equiv t_{2,0} - a_3 \quad (A5)
$$

$$
b_3 c_4 \equiv 0 \quad (B2)
$$

$$
b_3 c_5 \equiv t_{0,2} \quad (B3)
$$

$$
0 \equiv s_{1,2} \quad (B4)
$$

$$
s_{2,1} \equiv t_{0,2} - b_3 \quad (B5). 
$$

It is convenient to consider the cases $c_4 = 0$ and $c_4 \neq 0$ separately.
Case 5.2.1.2.1.3.1 First we consider the case \( c_4 = 0 \) then \( c_5 \neq 0 \). Then from (A2) we obtain \( a_3 \equiv 0 \). Substituting \( a_3 \) into (A3) and (A4) we obtain \( t_{2,0} \equiv 0 \) and \( s_{1,2} \equiv 0 \). (A5) becomes \( s_{1,2} \equiv 0 \). (B2) is automatic and (B3) becomes \( b_3 \equiv c_5 t_{0,2} \). Substituting \( b_3 \) into (B5) we obtain \( s_{2,1} \equiv (1 - c_5) t_{0,2} \). Combining (A4) and (B5) we obtain \( 0 \equiv (1 - c_5) t_{0,2} \) which we will denote as our new (B5).

Hence \( x \in \partial^{-1} W_2 \) if and only if

\[
\begin{align*}
t_{1,1} &\equiv 0 \quad (A1) \\
t_{2,0} &\equiv 0 \quad (A3) \\
s_{2,1} &\equiv 0 \quad (A4) \\
s_{1,2} &\equiv 0 \quad (A5) \\
0 &\equiv (1 - c_5) t_{0,2} \quad (B5).
\end{align*}
\]

If \( 1 + c_5 \neq 0 \) then \( t_{0,2} \equiv 0 \). Hence \( \partial^{-1} W_2 = \partial^{-1} W_1 \). Thus \( \text{rank}(\partial^{-1} W_2/W_2) = \text{rank}(\partial^{-1} W_1/W_2) \), \( W_2 \) is terminal, and \( W_2 \notin \hat{\mathcal{L}}_2 \).

If \( 1 + c_5 = 0 \) then \( t_{0,2} \) is a free variable. Taking \( t_{0,2} \equiv 1 \), \( t_{0,1} = 0 \), and \( t_{1,0} = 0 \), the matrix \( x \) becomes

\[
v_3 = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}.
\]

We see that neither \( v_3 \) nor \( 3v_3 \) is contained in \( W_2 \) and that \( \partial^{-1} W_2 = \partial^{-1} W_0, v_1, v_2, v_3 \). We know that \( v_3 \in \partial^{-1} W_2 \) and since \( 3v_3 \notin W_2 \) then \( v_3 + W_2 \) is an element of the group \( \partial^{-1} W_2/W_2 \) whose order is larger than 3. Recall that
$|\partial^{-1}W_2/W_2| = 3^4$ and that $\partial^{-1}W_1/W_2$ has rank 3. Since $\partial^{-1}W_2$ has an element of order larger than 3 then rank($\partial^{-1}W_2/W_2$) is not greater than 3 since $\partial^{-1}W_2/W_2 \cong \mathbb{Z}_9 \times \mathbb{Z}_3 \times \mathbb{Z}_3$ and therefore $W_2$ is terminal and $W_2 \notin \hat{L}_2$.

Case 5.2.1.2.1.3.2 Now we consider the case $c_4 \neq 0$. Then $c_4^3 \equiv 1$. (A3) becomes $a_3 \equiv c_4t_{2,0}$. Substituting $a_3, s_{1,2}$ into (A2) and (A5) we obtain $c_4c_5t_{2,0} \equiv 0$ and $0 \equiv (1 - c_4)t_{2,0}$. Combining (A2) and (A5) we obtain $0 \equiv (c_4c_5 - 1 + c_4)t_{2,0}$. (B2) becomes $b_3 \equiv 0$. Substituting $b_3$ into (B3) we obtain $t_{0,2} \equiv 0$. (B5) holds automatically.

If $c_4c_5 - 1 + c_4 \neq 0$ then $t_{2,0} \equiv 0$. Hence $\partial^{-1}W_2 = \partial^{-1}W_1$. Thus

$\text{rank}(\partial^{-1}W_2/W_2) = \text{rank}(\partial^{-1}W_1/W_2)$, $W_2$ is terminal, and $W_2 \notin \hat{L}_2$.

If $c_4c_5 - 1 + c_4 \equiv 0$ then $c_5 \equiv c_4 - 1$ and $t_{2,0}$ is a free variable. Taking $t_{2,0} \equiv 1$, $t_{0,1} = 0$, and $t_{1,0} = 0$, the matrix $x$ becomes

$$v_3 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix}.$$

We see that neither $v_3$ nor $3v_3$ is contained in $W_2$ and that $\partial^{-1}W_2 = \langle \partial^{-1}W_0, v_1, v_2, v_3 \rangle$. We know that $v_3 \in \partial^{-1}W_2$ and since $3v_3 \notin W_2$ then $v_3 + W_2$ is an element of the group $\partial^{-1}W_2/W_2$ whose order is larger than 3. Recall that $|\partial^{-1}W_2/W_2| = 3^4$ and that $\partial^{-1}W_1/W_2$ has rank 3. Since $\partial^{-1}W_2$ has an element of order larger than 3 then rank($\partial^{-1}W_2/W_2$) is not greater than 3 since $\partial^{-1}W_2/W_2 \cong \mathbb{Z}_9 \times \mathbb{Z}_3 \times \mathbb{Z}_3$ and therefore $W_2$ is terminal and $W_2 \notin \hat{L}_2$. 

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Case 5.2.2.1.4 We fix arbitrary values $c_4, c_5, c_6 \in \{0, 1, 2\}, (c_5, c_6) \neq (0, 0)$. There are 24 ways to choose the values $c_4, c_5, c_6$. Let $m_3 = y_2 + c_4(y_1 + y_3) + c_5 v_1 + c_6 v_2$.

Thus

$$m_3 = \begin{bmatrix}
0 & c_6 & 3c_4 \\
c_5 & 3 & 0 \\
3c_4 & 0 & 0
\end{bmatrix}.$$ 

Let $W_2 = \langle W_1, m_3 \rangle \in L_2$. The number of subgroups $W_2$ of this type is 24. Since $m_3 \notin W_1$ and $3m_1 \in W_1$ we have $|W_2/W_1| = 3$. Since $|W_1| = 3^5$ it follows that $|W_2| = 3^6$. Note that $\partial^{-1}W_1/W_2 \cong \mathbb{Z}_3 \times \mathbb{Z}_3 \times \mathbb{Z}_3$ and rank($\partial^{-1}W_1/W_2$) = 3. We now calculate the pullback of $\partial^{-1}W_2$. We observed in Case 5.2.2.1 that $\partial^{-1}W_2$ is contained in the pattern subgroup

$$\begin{bmatrix}
2 & 2 & 2 \\
2 & 2 & 1 \\
2 & 1 & 0
\end{bmatrix},$$

Let

$$x = \begin{bmatrix}
3s_{0,0} + t_{0,0} & 3s_{0,1} + t_{0,1} & 3s_{0,2} + t_{0,2} \\
3s_{1,0} + t_{1,0} & 3s_{1,1} + t_{1,1} & 3s_{1,2} \\
3s_{2,0} + t_{2,0} & 3s_{2,1} & 0
\end{bmatrix} \in \begin{bmatrix}
2 & 2 & 2 \\
2 & 2 & 1 \\
2 & 1 & 0
\end{bmatrix}.$$ 

The variables that are in play are those appearing in the matrix

$$\begin{bmatrix}
0 & t_{0,1} & t_{0,2} \\
t_{1,0} & t_{1,1} & 3s_{1,2} \\
t_{2,0} & 3s_{2,1} & 0
\end{bmatrix}.$$
In the notation of Case 5.2.2.1, we are taking \( d_1 = 1, \ d_2 = c_3, \ d_3 = c_4, \) and \( d_4 = c_5. \)

We wish to identify values \( a_1, a_2, a_3 \in \mathbb{Z}_9 \) such that \( \partial_1 x \equiv a_1 m_1 + a_2 m_2 + a_3 m_3 \) (mod \( I \)).

We see that \( \partial_1 x \in W_2 \) if and only if

\[
\begin{align*}
    t_{1,1} &\equiv 0 \quad \text{(A1)} \\
    a_3 c_6 &\equiv 0 \quad \text{(A2)} \\
    a_3 c_5 &\equiv t_{2,0} \quad \text{(A3)} \\
    a_3 &\equiv s_{2,1} \quad \text{(A4)} \\
    s_{1,2} &\equiv t_{2,0} - a_3 c_4 \quad \text{(A5)}. 
\end{align*}
\]

We wish to identify values \( b_1, b_2, b_3 \in \mathbb{Z}_9 \) such that \( \partial_2 x \equiv b_1 m_1 + b_2 m_2 + b_3 m_3 \) (mod \( I \)).

We see that \( \partial_2 x \in W_2 \) if and only if

\[
\begin{align*}
    t_{1,1} &\equiv 0 \quad \text{(B1)} \\
    b_3 c_5 &\equiv 0 \quad \text{(B2)} \\
    c_6 s_{1,2} &\equiv t_{0,2} \quad \text{(B3)} \\
    b_3 &\equiv s_{1,2} \quad \text{(B4)} \\
    s_{2,1} &\equiv t_{0,2} - b_3 c_4 \quad \text{(B5)}. 
\end{align*}
\]

(B1) is redundant with (A1) so we may ignore it. Substituting \( a_3 \) into (A2), (A3), and (A5) we obtain \( c_6 s_{2,1} \equiv 0, \ c_5 s_{2,1} = t_{2,0}, \) and \( s_{1,2} = t_{2,0} - c_4 s_{2,1}. \) Substituting
(A3) into (A5) we obtain \( s_{1,2} \equiv (c_5 - c_4)s_{2,1} \). Substituting \( b_3 \) into (B2), (B3), and (B5) we obtain \( c_5s_{1,2} \equiv 0 \), \( c_6s_{1,2} \equiv t_{0,2} \), and \( s_{2,1} \equiv (c_6 - c_4)s_{1,2} \).

Hence \( x \in \partial^{-1}W_2 \) if and only if

\[
\begin{align*}
t_{1,1} &\equiv 0 \quad (A1) \\
c_6s_{2,1} &\equiv 0 \quad (A2) \\
c_5s_{2,1} &\equiv t_{2,0} \quad (A3) \\
s_{1,2} &\equiv (c_5 - c_4)s_{2,1} \quad (A5) \\
c_5s_{1,2} &\equiv 0 \quad (B2) \\
c_6s_{1,2} &\equiv t_{0,2} \quad (B3) \\
s_{2,1} &\equiv (c_6 - c_4)s_{1,2} \quad (B4).
\end{align*}
\]

It is convenient to consider the cases \( c_5 = 0 \) and \( c_5 \neq 0 \) separately.

**Case 5.2.1.2.1.4.1** Suppose \( c_5 = 0 \). Then \( c_6 \neq 0 \) and \( c_6^2 \equiv 1 \). Then from (A2) we obtain \( s_{2,1} \equiv 0 \). Thus \( t_{2,0} \equiv 0 \) and \( s_{2,1} \equiv 0 \). (B2) is automatic. From (B3) and (B4) we obtain \( t_{0,2} \equiv 0 \) and \( s_{2,1} \equiv 0 \). Hence \( \partial^{-1}W_2 = \partial^{-1}W_1 \). Thus \( \text{rank}(\partial^{-1}W_2/W_2) = \text{rank}(\partial^{-1}W_1/W_2) \) if \( W_2 \) is terminal, and \( W_2 \notin \hat{\mathcal{L}}_2 \).

**Case 5.2.1.2.1.4.2** Suppose \( c_5 \neq 0 \). Then \( c_5^2 \equiv 1 \). From (B2) we obtain \( s_{1,2} \equiv 0 \). Thus (B3) and (B5) become \( t_{0,2} \equiv 0 \) and \( s_{2,1} \equiv 0 \). (A2) and (A5) now hold automatically. (A3) becomes \( t_{2,0} \equiv 0 \). Hence \( \partial^{-1}W_2 = \partial^{-1}W_1 \). Thus \( \text{rank}(\partial^{-1}W_2/W_2) = \text{rank}(\partial^{-1}W_1/W_2) \) if \( W_2 \) is terminal, and \( W_2 \notin \hat{\mathcal{L}}_2 \).
Case 5.2.2.2

We consider the 2-dimensional subspaces $W_2/W_1$. Let $d_1, d_2, d_3, d_4, e_1, e_2, e_3, e_4$ be unspecified variables. Let $m_3 = d_1y_2 + d_2(y_1 + y_3) + d_3v_1 + d_4v_2$ and $m_4 = e_1y_2 + e_2(y_1 + y_3) + e_3v_1 + e_4v_2$. In all the cases we consider the value of $e_1 = 0$ therefore we may exclude it from our expression of $m_4$. A formal expression for $m_3$ is

$$m_3 = d_1 \begin{bmatrix} 0 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 0 \end{bmatrix} + d_2 \begin{bmatrix} 0 & 0 & 3 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} + d_3 \begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} + d_4 \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & d_4 & 3d_2 \\ d_3 & 3d_1 & 0 \\ 3d_2 & 0 & 0 \end{bmatrix}.$$ 

A formal expression for $m_4$ is

$$m_4 = e_2 \begin{bmatrix} 0 & 0 & 3 \\ 0 & 0 & 0 \\ 3 & 0 & 0 \end{bmatrix} + e_3 \begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} + e_4 \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & e_4 & 3e_2 \\ e_3 & 0 & 0 \\ 3e_2 & 0 & 0 \end{bmatrix}.$$
Let $W_2 = <W_1, m_3, m_4> \in L_2$. We now calculate the pullback $\partial^{-1}W_2$. The subgroup $W_2$ is contained in the pattern subgroup

\[
\begin{bmatrix}
2 & 2 & 1 \\
2 & 1 & 0 \\
1 & 0 & 0
\end{bmatrix}
\]

Thus the pullback $\partial^{-1}W_2$ is contained in the pattern subgroup

\[
\begin{bmatrix}
2 & 2 & 2 \\
2 & 2 & 1 \\
2 & 1 & 0
\end{bmatrix}
\cap
\begin{bmatrix}
2 & 2 & 2 \\
2 & 2 & 1 \\
2 & 1 & 0
\end{bmatrix}
= \begin{bmatrix}
2 & 2 & 2 \\
2 & 2 & 1 \\
2 & 1 & 0
\end{bmatrix}
\]

Let

\[
x = \begin{bmatrix}
3s_{0,0} + t_{0,0} & 3s_{0,1} + t_{0,1} & 3s_{0,2} + t_{0,2} \\
3s_{1,0} + t_{1,0} & 3s_{1,1} + t_{1,1} & 3s_{1,2} \\
3s_{2,0} + t_{2,0} & 3s_{2,1} & 0
\end{bmatrix}
\in \begin{bmatrix}
2 & 2 & 2 \\
2 & 2 & 1 \\
2 & 1 & 0
\end{bmatrix}
\]

Thus

\[
\partial_1 x = \begin{bmatrix}
3s_{1,0} + t_{1,0} & 3s_{1,1} + t_{1,1} & 3s_{1,2} \\
3s_{2,0} + t_{2,0} & 3s_{2,1} & 0 \\
-3(t_{1,0} + t_{2,0}) & -3t_{1,1} & 0
\end{bmatrix}
\text{ and }
\partial_2 x = \begin{bmatrix}
3s_{0,1} + t_{0,1} & 3s_{0,2} + t_{0,2} & -3(t_{0,1} + t_{0,2}) \\
3s_{1,1} + t_{1,1} & 3s_{1,2} & -3t_{1,1} \\
3s_{2,1} & 0 & 0
\end{bmatrix}
\]
The variables that are in play are those appearing in the matrix
\[
\begin{bmatrix}
0 & t_{0,1} & t_{0,2} \\
t_{1,0} & t_{1,1} & 3s_{1,2} \\
t_{2,0} & 3s_{2,1} & 0
\end{bmatrix}.
\]

We wish to identify values \(a_1, a_2, a_3 \in \mathbb{Z}_9\) such that \(\partial_1 x \equiv a_1 m_1 + a_2 m_2 + a_3 m_3 \pmod{I}\). A formal expression for \(a_1 m_1 + a_2 m_2 + a_3 m_3 + a_4 m_4\) is
\[
\begin{bmatrix}
1 & 0 & -3 \\
0 & 0 & 0 \\
0 & 0 & 0
\end{bmatrix} a_1
+ \begin{bmatrix}
1 & 0 & 0 \\
0 & 0 & 0 \\
3d_2 & 0 & 0
\end{bmatrix} a_2
+ \begin{bmatrix}
0 & d_4 & 3d_2 \\
d_3 & 3d_1 & 0 \\
3d_2 & 0 & 0
\end{bmatrix} a_3
+ \begin{bmatrix}
0 & e_4 & 3e_2 \\
e_3 & 0 & 0 \\
3e_2 & 0 & 0
\end{bmatrix} a_4
= \begin{bmatrix}
0 & e_4 & 3e_2 \\
e_3 & 0 & 0 \\
3e_2 & 0 & 0
\end{bmatrix}.
\]

Comparing \((2,1)\)-entries, we get \(t_{1,1} \equiv 0\).

Comparing \((0,1)\)-entries, we get \(a_3 d_4 + a_4 e_4 \equiv t_{1,1}\). Since \(t_{1,1} \equiv 0\) this becomes \(a_3 d_4 + a_4 e_4 \equiv 0\).

Comparing \((1,0)\)-entries, we get \(a_3 d_3 + a_4 e_3 \equiv t_{2,0}\).

Comparing \((0,2)\)-entries, we get \(a_1 \equiv a_3 d_2 + a_4 e_2 - s_{1,2}\).

Comparing \((0,0)\)-entries, we get \(a_2 \equiv t_{1,0} - a_1\). Substituting \(a_1\) we obtain \(a_2 \equiv t_{1,0} - a_3 d_2 - a_4 e_2 + s_{1,2}\).

Comparing \((1,1)\)-entries, we get \(a_3 d_1 \equiv s_{2,1}\).
Comparing (2, 0)-entries, we get $-a_2 + a_3 d_2 + a_4 e_2 \equiv -t_{1,0} - t_{2,0}$. Substituting $a_2$ we obtain $s_{1,2} \equiv t_{2,0} - a_3 d_2 - a_4 e_2$.

We see that $\partial_1 x \in W_2$ if and only if

\[ t_{1,1} \equiv 0 \quad \text{(A1)} \]
\[ a_3 d_4 + a_4 e_4 \equiv 0 \quad \text{(A2)} \]
\[ a_3 d_3 + a_4 e_3 \equiv t_{2,0} \quad \text{(A3)} \]
\[ a_3 d_1 \equiv s_{2,1} \quad \text{(A4)} \]
\[ s_{1,2} \equiv t_{2,0} - a_3 d_2 - a_4 e_2 \quad \text{(A5)} \]

We wish to identify values $b_1, b_2, b_3 \in \mathbb{Z}_9$ such that $\partial_2 x \equiv b_1 m_1 + b_2 m_2 + b_3 m_3 \text{ (mod } I)$. A formal expression for $b_1 m_1 + b_2 m_2 + b_3 m_3$ is

\[
\begin{bmatrix}
1 & 0 & -3 \\
0 & 0 & 0
\end{bmatrix} b_1 + \begin{bmatrix}
1 & 0 & 0 \\
0 & 0 & 0
\end{bmatrix} b_2 + \begin{bmatrix}
0 & d_4 & 3d_2 \\
d_3 & 3d_1 & 0
\end{bmatrix} b_3 + \begin{bmatrix}
0 & e_4 & 3e_2 \\
e_3 & 0 & 0
\end{bmatrix} b_4 = \begin{bmatrix}
b_1 + b_2 & b_3 d_4 + b_4 e_4 & 3(-b_1 + b_3 d_2 + b_4 e_2) \\
b_3 d_4 + b_4 e_3 & 3b_3 d_1 & 0 \\
3(-b_2 + b_3 d_2 + b_4 e_2) & 0 & 0
\end{bmatrix}.
\]

Comparing (1, 2)-entries, we get $t_{1,1} \equiv 0$.

Comparing (1, 0)-entries, we get $b_3 d_3 + b_4 e_3 \equiv t_{1,1}$. Since $t_{1,1} \equiv 0$ this becomes $b_3 d_3 + b_4 e_3 \equiv 0$.

Comparing (0, 1)-entries, we get $b_3 d_4 + b_4 e_4 \equiv t_{0,2}$. 

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Comparing (0, 2)-entries, we get $b_1 \equiv b_3 d_2 + b_4 e_2 + t_{0,1} + t_{0,2}$.

Comparing (0, 0)-entries, we get $b_2 \equiv t_{0,1} - b_1$. Substituting $b_1$ we obtain $b_2 \equiv -b_3 d_2 - b_4 e_2 - t_{0,2}$.

Comparing (1, 1)-entries, we get $b_3 \equiv s_{1,2}$.

Comparing (2, 0)-entries, we get $-b_2 + b_3 d_2 + b_4 e_2 \equiv s_{2,1}$. Substituting $b_2$ we obtain $s_{2,1} \equiv t_{0,2} - b_3 d_2 - b_4 e_2$.

We see that $\partial_2 x \in W_2$ if and only if

\[
\begin{align*}
t_{1,1} &\equiv 0 \quad (B1) \\
 b_3 d_3 + b_4 e_3 &\equiv 0 \quad (B2) \\
 b_3 d_4 + b_4 e_4 &\equiv t_{0,2} \quad (B3) \\
 s_{1,2} &\equiv b_3 d_1 \quad (B4) \\
 s_{2,1} &\equiv t_{0,2} - b_3 d_2 - b_4 e_2 \quad (B5).
\end{align*}
\]

**Case 5.2.2.2.1** Let $m_3 = v_1$ and $m_4 = v_2$. Thus

\[
m_3 = \begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \quad \text{and} \quad m_4 = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}.
\]

Let $W_2 = \langle W_1, m_3, m_4 \rangle \in \mathcal{L}_2$. The number of subgroups $W_2$ of this type is 1. Since $m_3, m_4 \notin W_1$ and $3m_1, 3m_4 \in W_1$ we have $|W_2/W_1| = 3^2$. Since $|W_1| = 3^5$ it follows that $|W_2| = 3^7$. Note that $\partial^{-1}W_1/W_2 \cong \mathbb{Z}_3 \times \mathbb{Z}_3$ and $\text{rank}(\partial^{-1}W_1/W_2) = 2$.

We now calculate the pullback of $\partial^{-1}W_2$. We observed in Case 5.2.2.2 that $\partial^{-1}W_2$ is
contained in the pattern subgroup

\[
\begin{bmatrix}
2 & 2 & 2 \\
2 & 2 & 1 \\
2 & 1 & 0
\end{bmatrix}.
\]

Let

\[
x = \begin{bmatrix}
3s_{0,0} + t_{0,0} & 3s_{0,1} + t_{0,1} & 3s_{0,2} + t_{0,2} \\
3s_{1,0} + t_{1,0} & 3s_{1,1} + t_{1,1} & 0 \\
3s_{2,0} + t_{2,0} & 0 & 0
\end{bmatrix} \in \begin{bmatrix}
2 & 2 & 2 \\
2 & 2 & 1 \\
2 & 1 & 0
\end{bmatrix}.
\]

The variables that are in play are those appearing in the matrix

\[
\begin{bmatrix}
0 & t_{0,1} & t_{0,2} \\
t_{1,0} & t_{1,1} & 0 \\
t_{2,0} & 0 & 0
\end{bmatrix}.
\]

In the notation of Case 5.2.2.2, we are taking \(d_1 = 0, \ d_2 = 0, \ d_3 = 1, \ d_4 = 0, \ e_2 = 0, \ e_3 = 0, \) and \(e_4 = 1.\) We wish to identify values \(a_1, a_2, a_3, a_4 \in \mathbb{Z}_9\) such that

\[\partial_1 x \equiv a_1 m_1 + a_2 m_2 + a_3 m_3 + a_4 m_4 \pmod{I}.\]

We see that \(\partial_1 x \in W_2\) if and only if

\[t_{1,1} \equiv 0\quad (A1)\]
\[a_4 \equiv 0\quad (A2)\]
\[a_3 \equiv t_{2,0}\quad (A3)\]
\[s_{2,1} \equiv 0\quad (A4)\]
\[s_{1,2} \equiv t_{2,0}\quad (A5).\]
We wish to identify values $b_1, b_2, b_3, b_4 \in \mathbb{Z}_9$ such that $\partial_1 x \equiv b_1 m_1 + b_2 m_2 + b_3 m_3 + b_4 m_4 \pmod{I}$. We see that $\partial_2 x \in W_2$ if and only if

$$t_{1,1} \equiv 0 \quad (B1)$$

$$b_3 \equiv 0 \quad (B2)$$

$$b_4 \equiv t_{0,2} \quad (B3)$$

$$s_{1,2} \equiv 0 \quad (B4)$$

$$s_{2,1} \equiv t_{0,2} \quad (B5).$$

(B1) is redundant with (A1) so we may ignore it. Combining (A4) and (B5) we obtain $t_{0,2} \equiv 0$. Combining (B4) and (A5) we obtain $t_{2,0} \equiv 0$.

Hence $x \in \partial^{-1}W_2$ if and only if

$$t_{1,1} \equiv 0$$

$$s_{2,1} \equiv 0$$

$$s_{1,2} \equiv 0$$

$$t_{0,2} \equiv 0$$

$$t_{2,0} \equiv 0.$$

Hence $\partial^{-1}W_2 = \partial^{-1}W_1$. Thus $\text{rank}(\partial^{-1}W_2/W_2) = \text{rank}(\partial^{-1}W_1/W_2)$, $W_2$ is terminal, and $W_2 \notin \hat{\mathcal{L}}_2$. 

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Case 5.2.2.2 We fix arbitrary value $c_4 \in \{1, 2\}$. There are 2 ways to choose the value $c_4$. Let $m_3 = y_1 + y_3 + c_4v_2$ and $m_4 = v_1$. Thus

$$m_3 = \begin{bmatrix} 0 & 0 & 3 \\ c_4 & 0 & 0 \\ 3 & 0 & 0 \end{bmatrix}$$

and

$$m_4 = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

Let $W_2 = \langle W_1, m_3, m_4 \rangle \in \mathcal{L}_2$. The number of subgroups $W_2$ of this type is 2. Since $m_3, m_4 \notin W_1$ and $3m_1, 3m_4 \in W_1$ we have $|W_2/W_1| = 3^2$. Since $|W_1| = 3^6$ it follows that $|W_2| = 3^7$. Note that $\partial^{-1}W_1/W_2 \cong \mathbb{Z}_3 \times \mathbb{Z}_3$ and $\text{rank}(\partial^{-1}W_1/W_2) = 2$.

We now calculate the pullback of $\partial^{-1}W_2$. We observed in Case 5.2.2.2 that $\partial^{-1}W_2$ is contained in the pattern subgroup

$$\begin{bmatrix} 2 & 2 & 2 \\ 2 & 2 & 1 \\ 2 & 1 & 0 \end{bmatrix}.$$

Let

$$x = \begin{bmatrix} 3s_{0,0} + t_{0,0} & 3s_{0,1} + t_{0,1} & 3s_{0,2} + t_{0,2} \\ 3s_{1,0} + t_{1,0} & 3s_{1,1} + t_{1,1} & 3s_{1,2} \\ 3s_{2,0} + t_{2,0} & 3s_{2,1} & 0 \end{bmatrix} \in \begin{bmatrix} 2 & 2 & 2 \\ 2 & 2 & 1 \\ 2 & 1 & 0 \end{bmatrix}.$$

The variables that are in play are those appearing in the matrix

$$\begin{bmatrix} 0 & t_{0,1} & t_{0,2} \\ t_{1,0} & t_{1,1} & 3s_{1,2} \\ t_{2,0} & 3s_{2,1} & 0 \end{bmatrix}.$$
In the notation of Case 5.2.2.2, we are taking $d_1 = 0$, $d_2 = 1$, $d_3 = c_3, d_4 = 0$, $e_2 = 0$, $e_3 = 0$, and $e_4 = 1$. We wish to identify values $a_1, a_2, a_3, a_4 \in \mathbb{Z}_9$ such that

$$\partial_1 x \equiv a_1 m_1 + a_2 m_2 + a_3 m_3 + a_4 m_4 \pmod{I}.$$

We see that $\partial_1 x \in W_2$ if and only if

$$t_{1,1} \equiv 0 \quad (A1)$$

$$a_4 \equiv 0 \quad (A2)$$

$$a_3 c_4 \equiv t_{2,0} \quad (A3)$$

$$s_{2,1} \equiv 0 \quad (A4)$$

$$s_{1,2} \equiv t_{2,0} - a_3 \quad (A5).$$

We wish to identify values $b_1, b_2, b_3, b_4 \in \mathbb{Z}_9$ such that $\partial_2 x \equiv b_1 m_1 + b_2 m_2 + b_3 m_3 + b_4 m_4 \pmod{I}.$

We see that $\partial_2 x \in W_2$ if and only if

$$t_{1,1} \equiv 0 \quad (B1)$$

$$b_3 c_4 \equiv 0 \quad (B2)$$

$$b_4 \equiv t_{0,2} \quad (B3)$$

$$s_{1,2} \equiv 0 \quad (B4)$$

$$s_{2,1} \equiv t_{0,2} - b_3 \quad (B5).$$

(B1) is redundant with (A1) so we may ignore it. Since $c_4 \neq 0$ then $c_4^2 \equiv 1$. (A3) becomes $a_3 \equiv c_4 t_{2,0}$ and (B2) becomes $b_3 \equiv 0$. (A5) becomes $s_{1,2} \equiv (1 - c_4) t_{2,0}$. 

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Substituting (B4) into (A5) we obtain 0 \equiv (1 - c_4)t_{2,0}. (B5) becomes s_{2,1} \equiv t_{0,2}.

Substituting (A4) into (B5) we obtain \( t_{0,2} \equiv 0 \).

Hence \( x \in \partial^{-1}W_2 \) if and only if

\[
\begin{align*}
t_{1,1} &\equiv 0 \quad \text{(A1)} \\
s_{2,1} &\equiv 0 \quad \text{(A4)} \\
0 &\equiv (1 - c_4)t_{2,0} \quad \text{(A5)} \\
s_{1,2} &\equiv 0 \quad \text{(B4)} \\
t_{0,2} &\equiv 0 \quad \text{(B5)}. 
\end{align*}
\]

We consider the cases \( c_4 = 2 \) and \( c_4 = 1 \) separately. Suppose \( c_4 = 2 \).

Then \( 1 - c_4 \neq 0 \) so \( t_{2,0} \equiv 0 \). Hence \( \partial^{-1}W_2 = \partial^{-1}W_1 \). Thus \( \text{rank}(\partial^{-1}W_2/W_2) = \text{rank}(\partial^{-1}W_1/W_2) \), \( W_2 \) is terminal, and \( W_2 \notin \hat{\mathcal{L}}_2 \).

Suppose \( c_4 = 1 \). Then \( 1 - c_4 = 0 \) and \( t_{2,0} \) is a free variable. Taking \( t_{2,0} \equiv 1 \), \( t_{0,1} = 0 \), and \( t_{1,0} = 0 \), the matrix \( x \) becomes

\[
v_3 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix}.
\]

We see that neither \( v_3 \) nor \( 3v_3 \) is contained in \( W_2 \) and that \( \partial^{-1}W_2 =< \partial^{-1}W_0, v_1, v_2, v_3 > \). We know that \( v_3 \in \partial^{-1}W_2 \) and since \( 3v_3 \notin W_2 \) then \( v_3 + W_2 \) is an element of the group \( \partial^{-1}W_2/W_2 \) whose order is larger than 3.

Recall that \( |\partial^{-1}W_2/W_2| = 3^3 \) and that \( \partial^{-1}W_1/W_2 \) has rank 2. Since \( \partial^{-1}W_2 \) has
an element of order larger than 3 then \( \text{rank}(\partial^{-1}W_2/W_2) \) is not greater than 2 since 
\[ \partial^{-1}W_2/W_2 \cong \mathbb{Z}_9 \times \mathbb{Z}_3 \] and therefore \( W_2 \) is terminal and \( W_2 \notin \hat{\mathcal{L}}_2 \).

**Case 5.2.2.2.3** We fix arbitrary values \( c_4 \in \{1, 2\} \) and \( c_5 \in \{0, 1, 2\} \). There are 6 ways to choose the values \( c_4 \) and \( c_5 \). Let \( m_3 = y_1 + y_3 + c_4v_2 \) and \( m_4 = v_1 + c_5v_2 \). Thus 
\[ m_3 = \begin{bmatrix} 0 & c_4 & 3 \\ 0 & 0 & 0 \\ 3 & 0 & 0 \end{bmatrix} \quad \text{and} \quad m_4 = \begin{bmatrix} 0 & c_5 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}. \]

Let \( W_2 = < W_1, m_3, m_4 > \in \mathcal{L}_2 \). The number of subgroups \( W_2 \) of this type is 6. Since \( m_3, m_4 \notin W_1 \) and \( 3m_3, 3m_4 \in W_1 \), we have \( |W_2/W_1| = 3^2 \). Since \( |W_1| = 3^5 \) it follows that \( |W_2| = 3^7 \). Note that \( \partial^{-1}W_1/W_2 \cong \mathbb{Z}_3 \times \mathbb{Z}_3 \) and \( \text{rank}(\partial^{-1}W_1/W_2) = 2 \).

We now calculate the pullback of \( \partial^{-1}W_2 \). We observed in Case 5.2.2.2 that \( \partial^{-1}W_2 \) is contained in the pattern subgroup 
\[ \begin{bmatrix} 2 & 2 & 2 \\ 2 & 2 & 1 \\ 2 & 1 & 0 \end{bmatrix}. \]

Let 
\[ x = \begin{bmatrix} 3s_{0,0} + t_{0,0} & 3s_{0,1} + t_{0,1} & 3s_{0,2} + t_{0,2} \\ 3s_{1,0} + t_{1,0} & 3s_{1,1} + t_{1,1} & 3s_{1,2} \\ 3s_{2,0} + t_{2,0} & 3s_{2,1} & 0 \end{bmatrix} \in \begin{bmatrix} 2 & 2 & 2 \\ 2 & 2 & 1 \\ 2 & 1 & 0 \end{bmatrix}. \]
The variables that are in play are those appearing in the matrix

\[
\begin{bmatrix}
0 & t_{0,1} & t_{0,2} \\
t_{1,0} & t_{1,1} & 3s_{1,2} \\
t_{2,0} & 3s_{2,1} & 0
\end{bmatrix}.
\]

In the notation of Case 5.2.2.2, we are taking \(d_1 = 0, d_2 = 1, d_3 = 0, d_4 = c_4, e_2 = 0, e_3 = 1, \) and \(e_4 = c_5\). We wish to identify values \(a_1, a_2, a_3, a_4 \in \mathbb{Z}_9\) such that \(\partial_1 x \equiv a_1 m_1 + a_2 m_2 + a_3 m_3 + a_4 m_4 \pmod{I}\).

We see that \(\partial_1 x \in W_2\) if and only if

\[
t_{1,1} \equiv 0 \quad (A1)
\]

\[
a_3 c_4 + a_4 c_5 \equiv 0 \quad (A2)
\]

\[
a_4 \equiv t_{2,0} \quad (A3)
\]

\[
s_{2,1} \equiv 0 \quad (A4)
\]

\[
s_{1,2} \equiv t_{2,0} - a_3 \quad (A5).
\]

We wish to identify values \(b_1, b_2, b_3, b_4 \in \mathbb{Z}_9\) such that \(\partial_2 x \equiv b_1 m_1 + b_2 m_2 + b_3 m_3 + b_4 m_4 \pmod{I}\).
We see that $\partial_2 x \in W_2$ if and only if

\[
\begin{align*}
t_{1,1} &\equiv 0 \quad (B1) \\
b_4 &\equiv 0 \quad (B2) \\
b_3c_4 + b_4c_5 &\equiv t_{0,2} \quad (B3) \\
s_{1,2} &\equiv 0 \quad (B4) \\
s_{2,1} &\equiv t_{0,2} - b_3 \quad (B5).
\end{align*}
\]

(B1) is redundant with (A1) so we may ignore it. Substituting $b_4$, (B3) becomes $b_3c_4 \equiv t_{0,2}$. Since $c_4 \not\equiv 0$ then (B3) becomes $b_3 \equiv c_4t_{0,2}$. Substituting $a_4$ into (A2) we obtain $a_3 \equiv -c_4c_5t_{2,0}$. Substituting $a_3, s_{1,2}$ into (A5) we obtain $0 \equiv (1 + c_4c_5)t_{2,0}$. Substituting $b_3, s_{2,1}$ into (B5) we obtain $0 \equiv (1 - c_4)t_{0,2}$.

Hence $x \in \partial^{-1}W_2$ if and only if

\[
\begin{align*}
t_{1,1} &\equiv 0 \quad (A1) \\
s_{2,1} &\equiv 0 \quad (A4) \\
0 &\equiv (c_4c_5 + 1)t_{2,0} \quad (A5) \\
s_{1,2} &\equiv 0 \quad (B4) \\
0 &\equiv (1 - c_4)t_{0,2} \quad (B5).
\end{align*}
\]

It is convenient to consider the cases $c_4 = 2$ and $c_4 = 1$ separately.

**Case 5.2.2.2.3.1** Suppose $c_4 = 2$. Then (B5) becomes $t_{0,2} \equiv 0$ and (A5) becomes $0 \equiv (-c_5 + 1)t_{2,0}$. 

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If \(-c_5 + 1 \neq 0\) then \(t_{2,0} \equiv 0\). Hence \(\partial^{-1}W_2 = \partial^{-1}W_1\). Thus \(\text{rank}(\partial^{-1}W_2/W_2) = \text{rank}(\partial^{-1}W_1/W_2)\), \(W_2\) is terminal, and \(W_2 \notin \hat{L}_2\).

If \(-c_5 + 1 \equiv 0\) then \(c_5 = 1\) and \(t_{2,0}\) is a free variable. Taking \(t_{2,0} \equiv 1\), \(t_{0,1} = 0\), and \(t_{1,0} = 0\), the matrix \(x\) becomes

\[
v_3 = \begin{bmatrix}
0 & 0 & 0 \\
0 & 0 & 0 \\
1 & 0 & 0
\end{bmatrix}.
\]

We see that neither \(v_3\) nor \(3v_3\) is contained in \(W_2\) and that \(\partial^{-1}W_2 = < \partial^{-1}W_0, v_1, v_2, v_3 >\). We know that \(v_3 \in \partial^{-1}W_2\) and since \(3v_3 \notin W_2\) then \(v_3 + W_2\) is an element of the group \(\partial^{-1}W_2/W_2\) whose order is larger than 3.

Recall that \(|\partial^{-1}W_2/W_2| = 3^3\) and that \(\partial^{-1}W_1/W_2\) has rank 2. Since \(\partial^{-1}W_2\) has an element of order larger than 2 then \(\text{rank}(\partial^{-1}W_2/W_2)\) is not greater than 2 since \(\partial^{-1}W_2/W_2 \cong \mathbb{Z}_9 \times \mathbb{Z}_3\) and therefore \(W_2\) is terminal and \(W_2 \notin \hat{L}_2\).

**Case 5.2.2.3.2** Suppose \(c_4 = 1\). Then (A5) becomes \((c_5 + 1)t_{2,0} \equiv 0\) and (B3) holds automatically. Therefore \(t_{0,2}\) is a free variable.

Suppose \(c_5 + 1 \neq 0\) then \(t_{2,0} \equiv 0\). Taking \(t_{0,2} \equiv 1\), \(t_{0,1} = 0\), and \(t_{1,0} = 0\), the matrix \(x\) becomes

\[
v_3 = \begin{bmatrix}
0 & 0 & 1 \\
0 & 0 & 0 \\
0 & 0 & 0
\end{bmatrix}.
\]
We see that neither $v_3$ nor $3v_3$ is contained in $W_2$ and that

$\partial^{-1}W_2 =< \partial^{-1}W_0, v_1, v_2, v_3 >$. We know that $v_3 \in \partial^{-1}W_2$ and since $3v_3 \notin W_2$ then $v_3 + W_2$ is an element of the group $\partial^{-1}W_2/W_2$ whose order is larger than 3.

Recall that $|\partial^{-1}W_2/W_2| = 3^3$ and that $\partial^{-1}W_1/W_2$ has rank 2. Since $\partial^{-1}W_2$ has an element of order larger than 2 then rank($\partial^{-1}W_2/W_2$) is not greater than 3 since $\partial^{-1}W_2/W_2 \cong \mathbb{Z}_9 \times \mathbb{Z}_3$ and therefore $W_2$ is terminal and $W_2 \notin \hat{L}_2$.

Suppose $c_5 + 1 \equiv 0$ then $c_5 \equiv -1$ and $t_{2,0}$ is also a free variable. Taking $t_{2,0} \equiv 1$, $t_{0,1} = 0$, $t_{1,0} = 0$, and $t_{0,2} = 0$, the matrix $x$ becomes

$$v_3 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix}.$$  

Taking $t_{0,2} \equiv 1$, $t_{0,1} = 0$, and $t_{1,0} = 0$, the matrix $x$ becomes

$$v_4 = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$  

We see that $\partial^{-1}W_2 =< \partial^{-1}W_0, v_1, v_2, v_3, v_4 >$. Note $3v_3 \notin W_2$ so $v_3 + W_2$ is an element of order 9 in $\partial^{-1}W_2/W_2$. Note $v_3 - v_4 + W_2 \in \partial^{-1}W_2/W_2$ and

$$3(v_3 - v_4) = y_1 - y_3 = \begin{bmatrix} 0 & 0 & -3 \\ 0 & 0 & 0 \\ 3 & 0 & 0 \end{bmatrix} \in W_2.$$  

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Hence \( v_3 - v_4 + W_2 \) is an element of order 3. Thus \( |\partial^{-1}W_2/W_2| = 3^4 \) and \( \partial^{-1}W_2/W_2 \cong \mathbb{Z}_9 \times \mathbb{Z}_3 \times \mathbb{Z}_3 \). Therefore \( \operatorname{rank}(\partial^{-1}W_1/W_2) < \operatorname{rank}(\partial^{-1}W_2/W_2) \), \( W_2 \) is nonterminal, and \( W_2 \in \hat{\mathcal{L}}_2 \).

Let \( c_6, c_7 \in \{0, 1, 2\} \). Let \( m_5 = c_6 y_2 + c_7 (y_1 + y_3) + v_3 - v_4 \). Thus

\[
m_5 = \begin{bmatrix}
0 & 0 & 3c_7 - 1 \\
0 & 3c_6 & 0 \\
3c_7 + 1 & 0 & 0
\end{bmatrix}.
\]

Let \( W_3 = \langle W_2, m_5 \rangle \). The number of subgroups of this type is 9. Since \( m_5 \notin W_2 \) while \( 3m_5 \in W_2 \) we have \( |W_3/W_2| = 3 \). Recalling \( |W_2| = 3^7 \) we get \( |W_3| = 3^8 \). Since \( |W_3/W_2| = 3 \) and the antidiagonal of \( m_5 \) has an entry not divisible by 3, by the Modified Terminal Lemma we conclude that \( \partial^{-1}W_3 = \partial^{-1}W_2 \). Thus \( \operatorname{rank}(\partial^{-1}W_3/W_3) = \operatorname{rank}(\partial^{-1}W_2/W_3) \), \( W_3 \) is terminal, and \( W_3 \notin \hat{\mathcal{L}}_3 \).

**Case 5.2.2.2.4** We fix arbitrary values \( c_4 \in \{0, 1, 2\} \) and \( c_5 \in \{1, 2\} \). There are 6 ways to choose the values \( c_4 \) and \( c_5 \). Let \( m_3 = y_2 + c_4 (y_1 + y_3) + c_5 v_1 \) and \( m_4 = v_2 \).

Thus

\[
m_3 = \begin{bmatrix}
0 & 0 & 3c_4 \\
c_5 & 3 & 0 \\
3c_4 & 0 & 0
\end{bmatrix} \quad \text{and} \quad m_4 = \begin{bmatrix}
0 & 1 & 0 \\
0 & 0 & 0 \\
0 & 0 & 0
\end{bmatrix}.
\]

Let \( W_2 = \langle W_1, m_3, m_4 \rangle \in \mathcal{L}_2 \). The number of subgroups \( W_2 \) of this type is 6.

Since \( m_3, m_4 \notin W_1 \) and \( 3m_1, 3m_4 \in W_1 \) we have \( |W_2/W_1| = 3^2 \). Since \( |W_1| = 3^5 \) it follows that \( |W_2| = 3^7 \). Note that \( \partial^{-1}W_1/W_2 \cong \mathbb{Z}_3 \times \mathbb{Z}_3 \) and \( \operatorname{rank}(\partial^{-1}W_1/W_2) = 2 \).
We now calculate the pullback of $\partial^{-1}W_2$. We observed in Case 5.2.2.2 that $\partial^{-1}W_2$ is contained in the pattern subgroup

$$\begin{bmatrix}
2 & 2 & 2 \\
2 & 2 & 1 \\
2 & 1 & 0
\end{bmatrix}.$$  

Let

$$x = \begin{bmatrix}
3s_{0,0} + t_{0,0} & 3s_{0,1} + t_{0,1} & 3s_{0,2} + t_{0,2} \\
3s_{1,0} + t_{1,0} & 3s_{1,1} + t_{1,1} & 3s_{1,2} \\
3s_{2,0} + t_{2,0} & 3s_{2,1} & 0
\end{bmatrix} \in \begin{bmatrix}
2 & 2 & 2 \\
2 & 2 & 1 \\
2 & 1 & 0
\end{bmatrix}.$$

The variables that are in play are those appearing in the matrix

$$\begin{bmatrix}
0 & t_{0,1} & t_{0,2} \\
t_{1,0} & t_{1,1} & 3s_{1,2} \\
t_{2,0} & 3s_{2,1} & 0
\end{bmatrix}.$$  

In the notation of Case 5.2.2.2, we are taking $d_1 = 1$, $d_2 = c_4$, $d_3 = c_5$, $d_4 = 0$, $e_2 = 0$, $e_3 = 0$, and $e_4 = 1$. We wish to identify values $a_1, a_2, a_3, a_4 \in \mathbb{Z}_9$ such that $\partial_1 x \equiv a_1 m_1 + a_2 m_2 + a_3 m_3 + a_4 m_4 \pmod{I}$. 

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We see that $\partial_1 x \in W_2$ if and only if

$$t_{1,1} \equiv 0 \quad (A1)$$

$$a_4 \equiv 0 \quad (A2)$$

$$a_3 c_5 \equiv t_{2,0} \quad (A3)$$

$$a_3 \equiv s_{2,1} \quad (A4)$$

$$s_{1,2} \equiv t_{2,0} - a_3 c_4 \quad (A5).$$

We wish to identify values $b_1, b_2, b_3, b_4 \in \mathbb{Z}_9$ such that $\partial_2 x \equiv b_1 m_1 + b_2 m_2 + b_3 m_3 + b_4 m_4 \pmod{I}$.

We see that $\partial_2 x \in W_2$ if and only if

$$t_{1,1} \equiv 0 \quad (B1)$$

$$b_3 c_5 \equiv 0 \quad (B2)$$

$$b_4 \equiv t_{0,2} \quad (B3)$$

$$s_{1,2} \equiv 0 \quad (B4)$$

$$s_{2,1} \equiv t_{0,2} - b_3 c_4 \quad (B5).$$

(B1) is redundant with (A1) so we may ignore it. Since $c_5 \neq 0$ then $c_5^2 \equiv 1$.

Therefore $a_3 \equiv c_5 t_{2,0}$ and $b_3 \equiv 0$. Substituting $a_3, s_{1,2}$ into (A5) we obtain $0 \equiv (1 - c_4 c_5) t_{2,0}$. Substituting $b_3, b_4$ into (B5) we obtain $t_{0,2} \equiv c_5 t_{2,0}$. 

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Hence \( x \in \partial^{-1}W_2 \) if and only if

\[
\begin{align*}
t_{1,1} & \equiv 0 \quad (A1) \\
s_{2,1} & \equiv c_5 t_{2,0} \quad (A3) \\
0 & \equiv (1 - c_4 c_5) t_{2,0} \quad (A5) \\
s_{1,2} & \equiv 0 \quad (B4) \\
t_{0,2} & \equiv c_5 t_{2,0} \quad (B5).
\end{align*}
\]

If \( 1 - c_4 c_5 \neq 0 \) then \( t_{2,0} \equiv 0 \). Hence \( \partial^{-1}W_2 = \partial^{-1}W_1 \). Thus \( \text{rank}(\partial^{-1}W_2/W_2) = \text{rank}(\partial^{-1}W_1/W_2) \), \( W_2 \) is terminal, and \( W_2 \not\in \hat{L}_2 \).

If \( 1 - c_4 c_5 = 0 \) then \( c_4 \equiv c_5 \) and \( t_{2,0} \) is a free variable. Taking \( t_{2,0} \equiv 1, t_{0,1} = 0, \) and \( t_{1,0} = 0 \), the matrix \( x \) becomes

\[
v_3 = \begin{bmatrix}
0 & 0 & c_5 \\
0 & 0 & 0 \\
1 & 3c_5 & 0
\end{bmatrix}.
\]

We see that neither \( v_3 \) nor \( 3v_3 \) is contained in \( W_2 \) and that \( \partial^{-1}W_2 = < \partial^{-1}W_0, v_1, v_2, v_3 > \). We know that \( v_3 \in \partial^{-1}W_2 \) and since \( 3v_3 \not\in W_2 \) then \( v_3 + W_2 \) is an element of the group \( \partial^{-1}W_2/W_2 \) whose order is larger than 3. Recall that \( |\partial^{-1}W_2/W_2| = 3^3 \) and that \( \partial^{-1}W_1/W_2 \) has rank 2. Since \( \partial^{-1}W_2 \) has an element of order larger than 2 then \( \text{rank}(\partial^{-1}W_2/W_2) \) is not greater than 3 since \( \partial^{-1}W_2/W_2 \cong \mathbb{Z}_9 \times \mathbb{Z}_3 \) and therefore \( W_2 \) is terminal and \( W_2 \not\in \hat{L}_2 \).
Case 5.2.2.2.5 We fix arbitrary values $c_4, c_6 \in \{0, 1, 2\}$ and $c_5 \in \{1, 2\}$. There are 18 ways to choose the values $c_4, c_5, c_6$. Let $m_3 = y_2 + c_4(y_1 + y_3) + c_5v_2$ and $m_4 = v_1 + c_6v_2$. Thus

$$m_3 = \begin{bmatrix} 0 & c_5 & 3c_4 \\ 0 & 3 & 0 \\ 3c_4 & 0 & 0 \end{bmatrix} \quad \text{and} \quad m_4 = \begin{bmatrix} 0 & c_6 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$ 

Let $W_2 = \langle W_1, m_3, m_4 \rangle \in \mathcal{L}_2$. The number of subgroups $W_2$ of this type is 18. Since $m_3, m_4 \notin W_1$ and $3m_1, 3m_4 \in W_1$ we have $|W_2/W_1| = 3^2$. Since $|W_1| = 3^5$ it follows that $|W_2| = 3^7$. Note that $\partial^{-1}W_1/W_2 \cong \mathbb{Z}_3 \times \mathbb{Z}_3$ and $\text{rank}(\partial^{-1}W_1/W_2) = 2$.

We now calculate the pullback of $\partial^{-1}W_2$. We observed in Case 5.2.2.2 that $\partial^{-1}W_2$ is contained in the pattern subgroup

$$\begin{bmatrix} 2 & 2 & 2 \\ 2 & 2 & 1 \\ 2 & 1 & 0 \end{bmatrix}.$$ 

Let

$$x = \begin{bmatrix} 3s_{0,0} + t_{0,0} & 3s_{0,1} + t_{0,1} & 3s_{0,2} + t_{0,2} \\ 3s_{1,0} + t_{1,0} & 3s_{1,1} + t_{1,1} & 3s_{1,2} \\ 3s_{2,0} + t_{2,0} & 3s_{2,1} & 0 \end{bmatrix} \in \begin{bmatrix} 2 & 2 & 2 \\ 2 & 2 & 1 \\ 2 & 1 & 0 \end{bmatrix}.$$ 

The variables that are in play are those appearing in the matrix

$$\begin{bmatrix} 0 & t_{0,1} & t_{0,2} \\ t_{1,0} & t_{1,1} & 3s_{1,2} \\ t_{2,0} & 3s_{2,1} & 0 \end{bmatrix}.$$ 

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In the notation of Case 5.2.2.2, we are taking $d_1 = 1$, $d_2 = c_4$, $d_3 = 0$, $d_4 = c_5$, $e_2 = 0$, $e_3 = 1$, and $e_4 = c_6$. We wish to identify values $a_1, a_2, a_3, a_4 \in \mathbb{Z}_9$ such that
\[ \partial_1 x \equiv a_1 m_1 + a_2 m_2 + a_3 m_3 + a_4 m_4 \pmod{I}. \]

We see that $\partial_1 x \in W_2$ if and only if
\[ t_{1,1} \equiv 0 \quad (A1) \]
\[ a_3 c_5 + a_4 c_6 \equiv 0 \quad (A2) \]
\[ a_4 \equiv t_{2,0} \quad (A3) \]
\[ a_3 \equiv s_{2,1} \quad (A4) \]
\[ s_{1,2} \equiv t_{2,0} - a_3 c_4 \quad (A5). \]

We wish to identify values $b_1, b_2, b_3, b_4 \in \mathbb{Z}_9$ such that $\partial_2 x \equiv b_1 m_1 + b_2 m_2 + b_3 m_3 + b_4 m_4 \pmod{I}$.

We see that $\partial_2 x \in W_2$ if and only if
\[ t_{1,1} \equiv 0 \quad (B1) \]
\[ b_4 \equiv 0 \quad (B2) \]
\[ b_3 c_5 + b_4 c_6 \equiv t_{0,2} \quad (B3) \]
\[ s_{1,2} \equiv b_3 \quad (B4) \]
\[ s_{2,1} \equiv t_{0,2} - b_3 c_4 \quad (B5). \]

(B1) is redundant with (A1) so we may ignore it. (B3) becomes $b_3 \equiv c_5 t_{0,2}$ and (A2) becomes $a_3 \equiv -c_5 c_6 t_{2,0}$. Substituting $b_3, a_3$ into (B4) and (A4) we obtain
\( s_{1,2} \equiv c_5 t_{0,2} \) and \( s_{2,1} \equiv -c_5 c_6 t_{2,0} \). Substituting \( a_3, s_{1,2} \) into (A5) we obtain \( c_5 t_{0,2} \equiv (c_4 c_5 c_6 + 1) t_{2,0} \). Substituting \( b_3, s_{2,1} \) into (B5) we obtain \( -c_3 c_6 t_{2,0} \equiv (1 - c_4 c_5) t_{0,2} \).

Hence \( x \in \partial^{-1} W_2 \) if and only if

\[
\begin{align*}
t_{1,1} & \equiv 0 \quad \text{(A1)} \\
s_{2,1} & \equiv -c_5 c_6 t_{2,0} \quad \text{(A4)} \\
c_5 t_{0,2} & \equiv (c_4 c_5 c_6 + 1) t_{2,0} \quad \text{(A5)} \\
s_{1,2} & \equiv c_5 t_{0,2} \quad \text{(B4)} \\
-c_5 c_6 t_{2,0} & \equiv (1 - c_4 c_5) t_{0,2} \quad \text{(B5)}.
\end{align*}
\]

Since \( c_5 \neq 0 \) then (A5) becomes \( t_{0,2} \equiv (c_4 c_6 + c_5) t_{2,0} \). Substituting (A5) into (B4) and (B5) we obtain \( s_{1,2} \equiv (c_4 c_5 c_6 + 1) t_{2,0} \) and \( 0 \equiv (c_4 c_6 + c_5 - c_4^2 c_5 c_6 + c_5 c_6 - c_4) t_{2,0} \).

For convenience let \( q = c_4 c_6 + c_5 - c_4^2 c_5 c_6 + c_5 c_6 - c_4 \). Thus (B5) becomes \( 0 \equiv qt_{2,0} \).

Hence \( x \in \partial^{-1} W_2 \) if and only if

\[
\begin{align*}
t_{1,1} & \equiv 0 \quad \text{(A1)} \\
s_{2,1} & \equiv -c_5 c_6 t_{2,0} \quad \text{(A4)} \\
t_{0,2} & \equiv (c_4 c_6 + c_5) t_{2,0} \quad \text{(A5)} \\
s_{1,2} & \equiv (c_4 c_5 c_6 + 1) t_{2,0} \quad \text{(B4)} \\
0 & \equiv qt_{2,0} \quad \text{(B5)}.
\end{align*}
\]

If \( q \neq 0 \) then \( t_{2,0} \equiv 0 \). Hence \( \partial^{-1} W_2 = \partial^{-1} W_1 \). Thus \( \text{rank}(\partial^{-1} W_2/W_2) = \text{rank}(\partial^{-1} W_1/W_2) \), \( W_2 \) is terminal, and \( W_2 \notin \hat{L}_2 \).
If \( q \equiv 0 \), then \( t_{2,0} \) is a new free variable. Taking \( t_{2,0} \equiv 1, t_{1,0} = 0, \) and \( t_{0,1} = 0 \), the matrix \( x \) becomes

\[
\begin{bmatrix}
0 & 0 & c_4c_6 + c_5 \\
0 & 0 & 3(c_4c_5c_6 + 1) \\
1 & -3c_5c_6 & 0
\end{bmatrix}
\]

We see that neither \( v_3 \) nor \( 3v_3 \) is contained in \( W_2 \) and that \( \partial^{-1}W_2 = <\partial^{-1}W_0, v_1, v_2, v_3> \). We know that \( v_3 \in \partial^{-1}W_2 \) and since \( 3v_3 \notin W_2 \) then \( v_3 + W_2 \) is an element of the group \( \partial^{-1}W_2/W_2 \) whose order is larger than 3. Recall that \( |\partial^{-1}W_2/W_2| = 3^3 \) and that \( \partial^{-1}W_1/W_2 \) has rank 2. Since \( \partial^{-1}W_2 \) has an element of order larger than 2 then \( \text{rank}(\partial^{-1}W_2/W_2) \) is not greater than 3 since \( \partial^{-1}W_2/W_2 \cong \mathbb{Z}_9 \times \mathbb{Z}_3 \) and therefore \( W_2 \) is terminal and \( W_2 \notin \mathcal{L}_2 \).

**Case 5.2.2.2.6** We fix arbitrary values \( c_4, c_5, c_6, c_7 \) where \( \{(c_4, c_5), (c_6, c_7)\} \) is an ordered pair of linearly independent vectors in \( \mathbb{Z}_3 \times \mathbb{Z}_3 \). There are 48 ways to choose the values these ordered pairs. Let \( m_3 = y_2 + c_4v_1 + c_5v_2 \) and \( m_4 = y_1 + y_3 + c_6v_1 + c_7v_2 \). Thus

\[
m_3 = \begin{bmatrix}
0 & c_5 & 0 \\
c_4 & 3 & 0 \\
0 & 0 & 0
\end{bmatrix}\quad\text{and}\quad m_4 = \begin{bmatrix}
0 & c_7 & 3 \\
c_6 & 0 & 0 \\
3 & 0 & 0
\end{bmatrix}.
\]

Let \( W_2 = <W_1, m_3, m_4> \in \mathcal{L}_2 \). The number of subgroups \( W_2 \) of this type is 48.

Since \( m_3, m_4 \notin W_1 \) and \( 3m_1, 3m_4 \in W_1 \) we have \( |W_2/W_1| = 3^2 \). Since \( |W_1| = 3^5 \) it follows that \( |W_2| = 3^7 \). Note that \( \partial^{-1}W_1/W_2 \cong \mathbb{Z}_3 \times \mathbb{Z}_3 \) and \( \text{rank}(\partial^{-1}W_1/W_2) = 2 \).
We now calculate the pullback of $\partial^{-1}W_2$. We observed in Case 5.2.2.2 that $\partial^{-1}W_2$ is contained in the pattern subgroup

$$
\begin{bmatrix}
2 & 2 & 2 \\
2 & 2 & 1 \\
2 & 1 & 0
\end{bmatrix}.
$$

Let

$$
x = \begin{bmatrix}
3s_{0,0} + t_{0,0} & 3s_{0,1} + t_{0,1} & 3s_{0,2} + t_{0,2} \\
3s_{1,0} + t_{1,0} & 3s_{1,1} + t_{1,1} & 3s_{1,2} \\
3s_{2,0} + t_{2,0} & 3s_{2,1} & 0
\end{bmatrix} \in \begin{bmatrix}
2 & 2 & 2 \\
2 & 2 & 1 \\
2 & 1 & 0
\end{bmatrix}.
$$

The variables that are in play are those appearing in the matrix

$$
\begin{bmatrix}
0 & t_{0,1} & t_{0,2} \\
t_{1,0} & t_{1,1} & 3s_{1,2} \\
t_{2,0} & 3s_{2,1} & 0
\end{bmatrix}.
$$

In the notation of Case 5.2.2.2, we are taking $d_1 = 1$, $d_2 = 0$, $d_3 = c_4$,$d_4 = c_5$, $e_2 = 1$, $e_3 = c_6$, and $e_4 = c_7$. We wish to identify values $a_1, a_2, a_3, a_4 \in \mathbb{Z}_9$ such that $\partial_1 x \equiv a_1m_1 + a_2m_2 + a_3m_3 + a_4m_4 \pmod{I}$. 

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We see that $\partial_1 x \in W_2$ if and only if

$$t_{1,1} \equiv 0 \quad (A1)$$

$$a_3 c_5 + a_4 c_7 \equiv 0 \quad (A2)$$

$$a_3 c_4 + a_4 c_6 \equiv t_{2,0} \quad (A3)$$

$$a_3 \equiv s_{2,1} \quad (A4)$$

$$s_{1,2} \equiv t_{2,0} - a_4 \quad (A5).$$

We wish to identify values $b_1, b_2, b_3, b_4 \in \mathbb{Z}_9$ such that $\partial_2 x \equiv b_1 m_1 + b_2 m_2 + b_3 m_3 + b_4 m_4 (\text{mod } I)$.

We see that $\partial_2 x \in W_2$ if and only if

$$t_{1,1} \equiv 0 \quad (B1)$$

$$b_3 c_4 + b_4 c_6 \equiv 0 \quad (B2)$$

$$b_3 c_5 + b_4 c_7 \equiv t_{0,2} \quad (B3)$$

$$s_{1,2} \equiv b_3 \quad (B4)$$

$$s_{2,1} \equiv t_{0,2} - b_4 \quad (B5).$$

(B1) is redundant with (A1) so we may ignore it. Substituting $a_3, a_4$ into (A2) and (A3) we obtain $c_5 s_{2,1} + c_7 (t_{2,0} - s_{1,2}) \equiv 0$ and $c_4 s_{2,1} + c_6 (t_{2,0} - s_{1,2}) \equiv t_{2,0}$. Substituting $b_3, b_4$ into (B2) and (B3) we obtain $c_4 s_{1,2} + c_6 (t_{0,2} - s_{2,1}) \equiv 0$ and $c_5 s_{1,2} + c_7 (t_{0,2} - s_{2,1}) \equiv t_{0,2}$. 

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Hence $x \in \partial^{-1}W_2$ if and only if

\[ t_{1,1} \equiv 0 \quad (A1) \]

\[ c_5 s_{2,1} + c_7 (t_{2,0} - s_{1,2}) \equiv 0 \quad (A2) \]

\[ c_4 s_{2,1} + c_6 (t_{0,2} - s_{2,1}) \equiv 0 \quad (A3) \]

\[ c_4 s_{1,2} + c_6 (t_{0,2} - s_{2,1}) \equiv 0 \quad (B2) \]

\[ c_5 s_{1,2} + c_7 (t_{0,2} - s_{2,1}) \equiv t_{0,2} \quad (B3). \]

Let $\{(c_4, c_5), (c_6, c_7)\}$ be a pair of linearly independent vectors in $\mathbb{Z}_3 \times \mathbb{Z}_3$.

Hence

\[
\begin{bmatrix}
  c_4 & c_5 \\
  c_6 & c_7 \\
\end{bmatrix} \in GL(2, 3).
\]

Thus the determinant is not equivalent to zero, $c_4 c_7 - c_5 c_6 \neq 0$. There are three cases where this holds. The first case is $c_4 c_7 \neq 0$ and $c_5 c_6 = 0$ which has 20 possibilities.

The second case is $c_4 c_7 \equiv 0$ and $c_5 c_6 \neq 0$ which has 20 possibilities. The third case is $0 \neq c_4 c_7 \neq c_5 c_6 \neq 0$ which has 8 possibilities.

We shall now discuss how the notion of transpose-symmetry enables us to use the work done in Case 5.2.2.2.6.1 in order to handle Case 5.2.2.2.6.2 with almost no additional effort. The transpose map sends the subgroup $W_0$ to itself and also interchanges the pair of elements

\[
m_1 = y_1 + y_4 = \begin{bmatrix} 1 & 0 & -3 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \] and \[
m_2 = y_3 + y_4 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ -3 & 0 & 0 \end{bmatrix}.
\]
Since $W_1 = \langle W_0, m_1, m_2 \rangle$, it follows that the subgroup $W_1$ is invariant under the transpose map. Each subgroup $W_2$ of Case 5.2.2.6 is of the form $W_2 = \langle W_1, m_3, m_4 \rangle$ where $m_3 = y_2 + c_4 v_1 + c_5 v_2$ and $m_4 = y_1 + y_3 + c_6 v_1 + c_7 v_2$ for some choice of values $c_4, c_5, c_6, c_7$. The transpose map sends each of $y_2$ and $y_1 + y_3$ to itself and interchanges the pair of elements

$$v_1 = \begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \quad \text{and} \quad v_2 = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$ 

Hence the transpose map sends $m_3$ to $y_2 + c_5 v_1 + c_4 v_2$ and sends $m_4$ to $y_1 + y_3 c_7 v_1 + c_6 v_2$.

From this we see that the transpose map sends the subgroup $W_2$ corresponding to $(c_4, c_5, c_6, c_7)$ to the subgroup $W_2$ corresponding to $(c_5, c_4, c_7, c_6)$. In Case 5.2.2.6.1 we consider the subgroups $W_2$ for which the conditions $c_4 c_7 \not\equiv 0$ and $c_5 c_6 \equiv 0$ hold, and in Case 5.2.2.6.2 we consider the subgroups $W_2$ for which the conditions $c_4 c_7 \equiv 0$ and $c_5 c_6 \not\equiv 0$ hold. It is clear that the transpose map gives us a bijection between the subgroups $W_2$ appearing in Case 5.2.2.6.1 and those appearing in Case 5.2.2.6.2.

**Case 5.2.2.6.1** Suppose $c_4 c_7 \not\equiv 0$ and $c_5 c_6 \equiv 0$. It is convenient to consider the cases $c_6 \equiv 0$ and $c_6 \not\equiv 0$ separately.

**Case 5.2.2.6.1.1** Suppose $c_6 \equiv 0$. Then (B2) yields $s_{1,2} \equiv 0$. (A3) says $c_4 s_{2,1} \equiv t_{2,0}$ which says $s_{2,1} \equiv c_4 t_{2,0}$. Next we substitute for $s_{1,2}$ and $s_{2,1}$ in (A2) and (B3). (A2) becomes $(c_4 c_5 + c_7) t_{2,0} \equiv 0$ and (B3) becomes $(c_7 - 1) t_{0,2} \equiv c_4 c_7 t_{2,0}$. We now examine
the cases $c_7 = 1$ and $c_7 = 2$ separately.

**Case 5.2.2.6.1.1.1** Suppose $c_7 = 1$. Then $(c_4, c_5, c_6, c_7)$ is one of the following:

$(1, 1, 0, 1)$, $(1, 2, 0, 1)$, $(1, 2, 0, 1)$, and $(2, 2, 0, 1)$. Thus (B3) becomes $c_4 t_{2,0} \equiv 0$, and since $c_4 \neq 0$ this is equivalent to $t_{2,0} \equiv 0$. Hence (A3) yields $s_{2,1} \equiv 0$. So $t_{0,2}$ is the only new free variable. Taking $t_{0,2} \equiv 1$, $t_{0,1} = 0$, and $t_{1,0} = 0$, the matrix $x$ becomes

$$v_3 = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$ 

From Lemma 8.0.4, we see that $v_3 + W_2$ has order 9. Then $\partial^{-1}W_2/W_2 \cong \mathbb{Z}_9 \times \mathbb{Z}_3$ Therefore $W_2$ is terminal and $W_2 \not\in \hat{L}_2$.

**Case 5.2.2.6.1.1.2** Suppose $c_7 = 2$. Thus (B3) becomes $t_{0,2} \equiv -c_4 t_{2,0}$ and (A2) becomes $(c_4 c_5 - 1)t_{2,0} \equiv 0$. Since $c_4 \neq 0$, the condition $c_4 c_5 - 1 \equiv 0$ is equivalent to $c_4 \equiv c_5$. We now examine the cases $c_4 \neq c_5$ and $c_4 \equiv c_5$.

**Case 5.2.2.6.1.1.2.1** Suppose $c_4 \neq c_5$. Thus (A2) yields $t_{2,0} \equiv 0$. Then (B3) and (A3) yield $t_{0,2} \equiv 0$ and $s_{2,1} \equiv 0$ respectively. Thus $\partial^{-1}W_2 = \partial^{-1}W_1$. Therefore $W_2$ is terminal and $W_2 \not\in \hat{L}_2$.

**Case 5.2.2.6.1.1.2.2** Suppose $c_4 \equiv c_5$. Then $(c_4, c_5, c_6, c_7)$ is one of the following:

$(1, 1, 0, 1)$, $(1, 1, 0, 2)$, $(2, 2, 0, 1)$, $(2, 2, 0, 2)$. Thus (A2) is automatic and so $t_{2,0}$ is
the unique new free variable. Taking \( t_{2,0} \equiv 1 \), \( t_{0,1} = 0 \), and \( t_{1,0} = 0 \), the matrix \( x \) becomes

\[
v_3 = \begin{bmatrix}
0 & 0 & -c_4 \\
0 & 0 & 0 \\
1 & 3c_4 & 0
\end{bmatrix}.
\]

Now we must determine the order of \( v_3 + W_2 \). In order to do so, we will consider the cases \( c_4 = 2 \) and \( c_4 = 1 \) separately.

**Case 5.2.2.6.1.1.2.2.2** Suppose \( c_4 = 2 \). Then

\[
v_3 = \begin{bmatrix}
0 & 0 & -2 \\
0 & 0 & 0 \\
1 & -3 & 0
\end{bmatrix}
\]

and

\[
3v_3 = \begin{bmatrix}
0 & 0 & 3 \\
0 & 0 & 0 \\
3 & 0 & 0
\end{bmatrix} \not\in W_2
\]

by Lemma 8.0.4. Hence the element \( v_3 + W_2 \) has order 9. Therefore \( \partial^{-1}W_2/W_2 \cong Z_9 \times Z_3 \), \( W_2 \) is terminal, and \( W_2 \not\in \hat{L}_2 \).

**Case 5.2.2.6.1.1.2.2.2** Suppose \( c_4 = 1 \). Then \((c_4, c_5, c_6, c_7)\) is either \((1, 1, 0, 1)\) or \((1, 1, 0, 2)\) and

\[
v_3 = \begin{bmatrix}
0 & 0 & -1 \\
0 & 0 & 0 \\
1 & 3 & 0
\end{bmatrix}
\]

and

\[
3v_3 = \begin{bmatrix}
0 & 0 & -3 \\
0 & 0 & 0 \\
3 & 0 & 0
\end{bmatrix} = y_1 - y_3 = m_1 - m_2 \in W_1 \subseteq W_2
\]

by Lemma 8.0.4. Hence the element \( v_3 + W_2 \) has order 3. Thus \( \partial^{-1}W_2/W_2 \cong Z_3 \times Z_3 \times Z_3 \). Since \( \partial^{-1}W_1/W_2 \cong Z_3 \times Z_3 \) we see that \( W_2 \) is nonterminal. Let
\[c_8, c_9 \in \{0, 1, 2\} \text{ and } m_5 = c_8y_2 + c_9(y_1 + y_3) + v_3. \text{ Thus}\]

\[m_5 = \begin{bmatrix}
0 & 0 & 3c_9 - 1 \\
0 & 3c_8 & 0 \\
3c_9 + 1 & 3 & 0
\end{bmatrix}.
\]

Let \(W_3 = \langle W_2, m_5 \rangle.\) The number of subgroups of this type is 9. Since \(m_5 \not\in W_2\) while \(3m_5 \in W_2\) we have \(|W_3/W_2| = 3\). Recalling \(|W_2| = 3^7\) we get \(|W_3| = 3^8.\) Since \(|W_3/W_2| = 3\) and the antidiagonal of \(m_5\) has an entry not divisible by 3, by the Modified Terminal Lemma we conclude that \(\partial^{-1}W_3 = \partial^{-1}W_2.\) Thus \(\text{rank}(\partial^{-1}W_3/W_3) = \text{rank}(\partial^{-1}W_2/W_3),\) \(W_3\) is terminal, and \(W_3 \not\in \hat{L}_3.\)

**Case 5.2.2.2.6.1.2** Suppose \(c_6 \neq 0.\) Since \(c_5c_6 \equiv 0,\) it follows that \(c_5 \equiv 0.\) (A2) yields \(t_{2,0} - s_{1,2} \equiv 0,\) which says \(s_{1,2} \equiv t_{2,0}.\) (A3) the yields \(c_4s_{2,1} \equiv t_{2,0},\) which says \(s_{2,1} \equiv c_4t_{2,0}.\) Next we substitute for \(s_{1,2}\) and \(s_{2,1}\) in (B2) and (B3). (B2) becomes \(c_4t_{2,0} + c_6(t_{0,2} - c_4t_{2,0}) \equiv 0,\) which then becomes \(t_{0,2} \equiv (1 - c_6)c_4t_{2,0}.\) (B3) becomes \(c_7(t_{0,2} - c_4t_{2,0}) \equiv t_{0,2},\) which becomes \((c_7 - 1)t_{0,2} \equiv c_4c_7t_{2,0}.\) We consider the cases \(c_7 = 1\) and \(c_7 = 2\) separately.

**Case 5.2.2.2.6.1.2.1** Suppose \(c_7 = 1.\) Thus (B3) becomes \(c_4t_{2,0} \equiv 0,\) and since \(c_4 \neq 0\) this yields \(t_{2,0} \equiv 0.\) Hence (A2), (A3), and (B2) yield \(s_{1,2} \equiv s_{2,1} \equiv t_{0,2} \equiv 0.\) Thus \(\partial^{-1}W_2 = \partial^{-1}W_1, W_2\) is terminal, and \(W_2 \not\in \hat{L}_2.\)
Case 5.2.2.6.1.2.2  Suppose \( c_7 = 2 \). Thus (B3) becomes \( t_{0,2} \equiv -c_4 t_{2,0} \). Now use (B2) to substitute for \( t_{0,2} \) in (B3), so that (B3) becomes \( (1-c_6)c_4 t_{2,0} \equiv -c_4 t_{2,0} \), which becomes \( (c_6 + 1)t_{2,0} \equiv 0 \). We now examine the cases \( c_6 = 1 \) and \( c_6 = 2 \) separately.

Case 5.2.2.6.1.2.2.1  Suppose \( c_6 = 1 \). Thus (B3) is equivalent to \( t_{2,0} \equiv 0 \). Hence (A2), (A3), and (B2) yield \( s_{1,2} \equiv s_{2,1} \equiv t_{0,2} \equiv 0 \). Thus \( \partial^{-1}W_2 = \partial^{-1}W_1 \), \( W_2 \) is terminal, and \( W_2 \not\in \hat{L}_2 \).

Case 5.2.2.6.1.2.2.2  Suppose \( c_6 = 2 \). Then \( (c_4, c_5, c_6, c_7) \) is either \( (1, 0, 2, 2) \) or \( (2, 0, 2, 20) \). So (B2) becomes \( t_{0,2} \equiv -c_4 t_{2,0} \). Thus (B3) is automatic and so \( t_{2,0} \) is the unique new free variable. Taking \( t_{2,0} \equiv 1 \), \( t_{0,1} = 0 \), and \( t_{1,0} = 0 \), the matrix \( x \) becomes

\[
v_3 = \begin{bmatrix} 0 & 0 & -c_4 \\ 0 & 0 & 3 \\ 1 & 3c_4 & 0 \end{bmatrix}.
\]

Now we must determine the order of \( v_3 + W_2 \). In order to do so, we will consider the cases \( c_4 = 2 \) and \( c_4 = 1 \) separately.

Case 5.2.2.6.1.2.2.2.1  Suppose \( c_4 = 2 \). Then

\[
v_3 = \begin{bmatrix} 0 & 0 & -2 \\ 0 & 0 & 3 \\ 1 & -3 & 0 \end{bmatrix} \quad \text{and} \quad 3v_3 = \begin{bmatrix} 0 & 0 & 3 \\ 0 & 0 & 0 \\ 3 & 0 & 0 \end{bmatrix} \not\in W_2.
\]

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by Lemma 8.0.4. Hence the element $v_3 + W_2$ has order 9. Therefore $\partial^{-1}W_2/W_2 \cong \mathbb{Z}_9 \times \mathbb{Z}_3$, $W_2$ is terminal, and $W_2 \notin \hat{\mathcal{L}}_2$.

**Case 5.2.2.2.6.1.2.2.2** Suppose $c_4 = 1$. Then

$$v_3 = \begin{bmatrix} 0 & 0 & -1 \\ 0 & 0 & 0 \\ 1 & 3 & 0 \end{bmatrix} \quad \text{and} \quad 3v_3 = \begin{bmatrix} 0 & 0 & -3 \\ 0 & 0 & 0 \\ 3 & 0 & 0 \end{bmatrix} = y_1 - y_3 = m_1 - m_2 \in W_1 \subseteq W_2$$

by Lemma 8.0.4. Hence the element $v_3 + W_2$ has order 3. Thus $\partial^{-1}W_2/W_2 \cong \mathbb{Z}_3 \times \mathbb{Z}_3 \times \mathbb{Z}_3$. Since $\partial^{-1}W_1/W_2 \cong \mathbb{Z}_3 \times \mathbb{Z}_3$ we see that $W_2$ is nonterminal. Let $c_8, c_9 \in \{0, 1, 2\}$ and $m_5 = c_8y_2 + c_9(y_1 + y_3) + v_3$. Thus

$$m_5 = \begin{bmatrix} 0 & 0 & 3c_9 - 1 \\ 0 & 3c_8 & 0 \\ 3c_9 + 1 & 3 & 0 \end{bmatrix}$$

Let $W_3 = < W_2, m_5 >$. The number of subgroups of this type is 9. Since $m_5 \notin W_2$ while $3m_5 \in W_2$ we have $|W_3/W_2| = 3$. Recalling $|W_2| = 3^7$ we get $|W_3| = 3^8$. Since $|W_3/W_2| = 3$ and the antidiagonal of $m_5$ has an entry not divisible by 3, by the Modified Terminal Lemma we conclude that $\partial^{-1}W_3 = \partial^{-1}W_2$. Thus $\text{rank}(\partial^{-1}W_3/W_3) = \text{rank}(\partial^{-1}W_2/W_3)$, $W_3$ is terminal, and $W_3 \notin \hat{\mathcal{L}}_3$.

**Case 5.2.2.2.6.2** Suppose $c_4c_7 \equiv 0$ and $c_5c_6 \neq 0$. By previous work, we know that the transpose map gives us a bijection between the subgroups we obtain in this case
the ones that we obtained in Case 5.2.2.6.1. Therefore this case is finished.

**Case 5.2.2.6.3** Suppose $c_4 \neq 0$, $c_5 \neq 0$, $c_6 \neq 0$, and $c_7 \neq 0$. Let $D = c_4 c_7 - c_5 c_6$ and note that $D$ is the determinant of the 2-by-2 matrix whose top row if $(c_4, c_5)$ and whose bottom row is $(c_6, c_7)$. Because these two row vectors are linearly independent, either $D = 1$ or $D = 2$. It follows that \( \{c_4 c_7, c_5 c_6\} = \{1, 2\} \). Since each of $c_4, c_5, c_6, c_7$ is either 1 or 2, it follows that either exactly one or exactly three of $c_4, c_5, c_6, c_7$ has value 2, a fact that we shall use later.

We subtract $c_6$ times (A2) from $c_7$ times (A3) to obtain $D s_{2,1} \equiv c_7 t_{2,0}$, and since $D^2 \equiv 1$ this last congruence is equivalent to $s_{2,1} \equiv D c_7 t_{2,0}$, which we will denote as ($\star$).

We subtract $c_6$ times (B3) from $c_7$ times (B2) to obtain $D s_{1,2} \equiv -c_6 t_{0,2}$, and since $D^2 \equiv 1$, this last congruence is equivalent to $s_{1,2} \equiv -D c_6 t_{0,2}$, which we will denote as ($\star$). We shall now argue that $t_{2,0} \equiv t_{0,2} \equiv s_{2,1} \equiv s_{1,2} \equiv 0$, which of course would tell us that $\partial^{-1} W_2 = \partial^{-1} W_1$ and that $W_2$ is terminal. In view of the two congruences that we just obtained, it suffices to show that $t_{2,0} \equiv 0$ and $t_{0,2} \equiv 0$.

We now use ($\star$) and ($\star$) to substitute for $s_{2,1}$ and $s_{1,2}$ in the congruences (A2) and (B2). In this manner (after algebraic simplifications) (A2) becomes $t_{0,2} \equiv -c_6 D(c_5 D + 1) t_{2,0}$ and (B2) becomes $t_{2,0} \equiv -c_7 D(c_4 D - 1) t_{0,2}$. Now, to achieve our goal of establishing that $t_{2,0} \equiv 0$ and $t_{0,2} \equiv 0$, it suffices to show merely that either $t_{2,0} \equiv 0$ or $t_{0,2} \equiv 0$. We now examine three subcases separately. The first subcase is when $c_6 = 2$. The second subcase is when $c_7 = 2$. The third subcase is when
\(c_6 = c_7 = 1.\)

**Case 5.2.2.6.3.1** Suppose \(c_6 = 2.\) Thus either two or none of \(c_4, c_5, c_7\) has value 2, and it follows that \(c_4c_5c_7 \equiv 1.\) By definition of \(D\) we obtain \(c_5D + 1 \equiv c_4c_5c_7 - c_6 + 1.\) We deduce that \(c_5D + 1 \equiv 0.\) Hence by (A2) we obtain \(t_{0,2} \equiv 0,\) as desired and \(W_2\) is terminal.

**Case 5.2.2.6.3.2** Suppose \(c_7 = 2.\) Thus either two or none of \(c_4, c_5, c_6\) has value 2, and it follows that \(c_4c_5c_6 \equiv 1.\) By definition of \(D\) we obtain \(c_4D - 1 \equiv c_7 - c_4c_5c_6 - 1.\) We deduce that \(c_4D - 1 \equiv 0.\) Hence by (B2) we obtain \(t_{2,0} \equiv 0,\) as desired and \(W_2\) is terminal.

**Case 5.2.2.6.3.3** Suppose \(c_6 = c_7 = 1.\) We use (A2) to substitute for \(t_{0,2}\) in (B2) to obtain \(t_{2,0} \equiv (c_4D - 1)(c_5D + 1)t_{2,0}.\) We shall now argue that \((c_4D - 1)(c_5D + 1) \equiv -1,\) which in turn would imply that \(t_{2,0},\) as desired. Because \(c_6 = 1,\) exactly one or three of \(c_4, c_5, c_7\) has value 2, and so \(c_4c_5c_7 \equiv -1.\) Using \(c_5D + 1 \equiv c_4c_5c_7 - c_6 + 1\) we deduce that \(c_5D + 1 \equiv -1.\) Because \(c_7 = 1,\) exactly one or three of \(c_4, c_5, c_6\) has value 2, and so \(c_4c_5c_6 \equiv -1.\) Using \(c_4D - 1 \equiv c_7 - c_4c_5c_6 - 1,\) we deduce that \(c_4D - 1 \equiv 1.\) Hence \((c_4D - 1)(c_5D + 1) \equiv (1)(-1),\) as desired and \(W_2\) is terminal.

In Case 5 we found 16 subgroups \(W_1 \in \mathcal{L}_1,\) all satisfying \(|W_1| = 3^5.\) Exactly 11 of these 16 members of \(\mathcal{L}_1\) are nonterminal. 10 of these 11 nonterminal members of \(\mathcal{L}_1\) are contained in 9 members of \(\mathcal{L}_2.\) The 11th nonterminal member of \(\mathcal{L}_1\) is
contained in 36 members of $L_2$. Thus we found 216 subgroups $W_2 \in L_2$, 125 of which satisfy $|W_2| = 3^6$ and 81 of which satisfy $|W_2| = 3^7$. Every member of $L_2$ that satisfies $|W_2| = 3^6$ is terminal. Exactly 5 members of $L_2$ that satisfies $|W_2| = 3^7$ are nonterminal. Each of these 5 members of $L_2$ is contained in 9 members of $L_3$. Thus we found 45 subgroups $W_3 \in L_3$, all satisfying $|W_3| = 3^8$. Every member of $L_3$ is terminal. So in Case 5 we found a total of $16 + 216 + 45 = 277$ subgroups.
We fix arbitrary values $c_1, c_2, c_3, c_4 \in \{0, 1, 2\}$ such that $(c_1, c_2) \neq (0, 0)$ and $(c_3, c_4) \neq (0, 0)$. Let $m_1 = y_1 + c_1y_3 + c_2y_4$ and $m_2 = y_2 + c_3y_3 + c_4y_4$. Thus

$$m_1 = \begin{bmatrix} c_2 & 0 & 3(c_1 - c_2) \\ 0 & 0 & 0 \\ 3(1 - c_2) & 0 & 0 \end{bmatrix} \quad \text{and} \quad m_2 = \begin{bmatrix} c_4 & 0 & 3(c_3 - c_4) \\ 0 & 3 & 0 \\ -3c_4 & 0 & 0 \end{bmatrix}.$$ 

Let $W_1 = < W_0, m_1, m_2 > \in \mathcal{L}_1$. The number of subgroups of this type is 64.

Note that $|W_1| = 3^5$ and $\partial^{-1}W_0/W_1 \cong \mathbb{Z}_3 \times \mathbb{Z}_3$ and rank $(\partial^{-1}W_0/W_1) = 2$. We now calculate the pullback $\partial^{-1}W_1$. The subgroup $W_1$ is contained in the pattern subgroup

$$\begin{bmatrix} 2 & 1 & 1 \\ 1 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}.$$ 

Thus the pullback $\partial^{-1}W_1$ is contained in the pattern subgroup

$$\begin{bmatrix} 2 & 2 & 2 \\ 2 & 1 & 1 \\ 2 & 1 & 0 \end{bmatrix} \cap \begin{bmatrix} 2 & 2 & 1 \\ 2 & 1 & 1 \\ 2 & 1 & 0 \end{bmatrix} = \begin{bmatrix} 2 & 2 \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} 2 & 2 \end{bmatrix} \cap \begin{bmatrix} 2 & 1 \end{bmatrix} = \begin{bmatrix} 2 & 1 \end{bmatrix}.$$
Let
\[
x = \begin{bmatrix}
3s_{0,0} + t_{0,0} & 3s_{0,1} + t_{0,1} & 3s_{0,2} \\
3s_{1,0} + t_{1,0} & 3s_{1,1} & 3s_{1,2} \\
3s_{2,0} & 3s_{2,1} & 0
\end{bmatrix} \in \begin{bmatrix}
2 & 2 & 1 \\
2 & 1 & 1 \\
1 & 1 & 0
\end{bmatrix}.
\]

Thus
\[
\partial_1 x = \begin{bmatrix}
3s_{1,0} + t_{1,0} & 3s_{1,1} & 3s_{1,2} \\
3s_{2,0} & 3s_{2,1} & 0 \\
-3t_{1,0} & 0 & 0
\end{bmatrix} \quad \text{and} \quad \partial_2 x = \begin{bmatrix}
3s_{0,1} + t_{0,1} & 3s_{0,2} & -3t_{0,1} \\
3s_{1,1} & 3s_{1,2} & 0 \\
3s_{2,1} & 0 & 0
\end{bmatrix}.
\]

The variables that are in play are those appearing in the matrix
\[
\begin{bmatrix}
0 & t_{0,1} & 0 \\
t_{1,0} & 0 & 3s_{1,2} \\
0 & 3s_{2,1} & 0
\end{bmatrix}.
\]

We want to identify a value \(a_1, a_2 \in \mathbb{Z}_9\) such that \(\partial_1 x \equiv a_1 m_1 + a_2 m_2 (\text{mod } I)\). A formal expression for \(a_1 m_1 + a_2 m_2\) is
\[
a_1 \begin{bmatrix}
c_2 & 0 & 3(c_1 - c_2) \\
0 & 0 & 0 \\
3(1 - c_2) & 0 & 0
\end{bmatrix} + a_2 \begin{bmatrix}
c_4 & 0 & 3(c_3 - c_4) \\
0 & 3 & 0 \\
-3c_4 & 0 & 0
\end{bmatrix} + \begin{bmatrix}
a_1 c_2 + a_2 c_4 & 0 & 3((c_1 - c_2)a_1 + (c_3 - c_4)a_2) \\
0 & 3a_2 & 0 \\
3(a_1 - a_1 c_2 - a_2 c_4) & 0 & 0
\end{bmatrix}.
\]
Comparing \((0,1)\) - entries, we get \(0 \equiv 3s_{1,1}\) which gives no information.

Comparing \((1,0)\) - entries, we get \(0 \equiv s_{2,0}\) which gives no information.

Comparing \((1,1)\)-entries, we get \(a_2 \equiv s_{2,1}\).

Comparing \((0,0)\) - entries, we get \(t_{1,0} \equiv a_1c_2 + a_2c_4\). Substituting \(a_2\) we obtain \(a_1c_2 \equiv t_{1,0} - c_4s_{2,1}\).

Comparing \((2,0)\) - entries, we get \(a_1 - a_1c_2 - a_2c_4 \equiv -t_{1,0}\). Substituting \(a_1c_2\) and \(a_2\) we obtain \(a_1 \equiv 0\). Therefore \(a_1c_2 \equiv t_{1,0} - c_4s_{2,1}\) becomes \(t_{1,0} \equiv c_4s_{2,1}\).

Comparing \((0,2)\) - entries, we get \(a_1c_1 + a_1c_2 + a_2c_3 - a_2c_4 \equiv s_{1,2}\). Substituting \(a_1, a_2\) we obtain \(s_{1,2} \equiv (c_3 - c_4)s_{2,1}\).

We see that \(\partial_1 x \in W_2\) if and only if

\[
t_{1,0} \equiv c_4s_{2,1} \quad (A1)
\]

\[
s_{1,2} \equiv (c_3 - c_4)s_{2,1} \quad (A2).
\]

We want to identify a value \(b_1, b_2 \in \mathbb{Z}_9\) such that \(\partial_2 x \equiv b_1m_1 + b_2m_2\) (mod \(I\)). A formal expression for \(b_1m_1 + b_2m_2\) is

\[
\begin{bmatrix}
  c_2 & 0 & 3(c_1 - c_2) \\
  0 & 0 & 0 \\
 3(1 - c_2) & 0 & 0
\end{bmatrix}
+ b_2
\begin{bmatrix}
 c_4 & 0 & 3(c_3 - c_4) \\
 0 & 3 & 0 \\
-3c_4 & 0 & 0
\end{bmatrix}
\]

\[
= \begin{bmatrix}
  b_1c_2 + b_2c_4 & 0 & 3((c_1 - c_2)b_1 + (c_3 - c_4)b_2) \\
  0 & 3b_2 & 0 \\
 3(b_1 - b_1c_2 - b_2c_4) & 0 & 0
\end{bmatrix}
\]

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Comparing \((0, 1)\) - entries, we get \(0 \equiv 3s_{0,2}\) which gives no information.

Comparing \((1, 0)\) - entries, we get \(0 \equiv s_{1,1}\) which gives no information.

Comparing \((1, 1)\)-entries, we get \(b_2 \equiv s_{1,2}\).

Comparing \((0, 0)\) - entries, we get \(b_1c_2 + c_4b_2 \equiv t_{0,1}\). Substituting \(b_2\) we obtain \(b_1c_2 \equiv t_{0,1} - c_4s_{1,2}\).

Comparing \((2, 0)\) - entries, we get \(b_1 - b_1c_2 - b_2c_4 \equiv s_{2,1}\). Substituting \(b_1c_2, b_2\) we obtain \(b_1 \equiv s_{2,1} + t_{0,1}\). Therefore \(b_1c_2 \equiv t_{0,1} - c_4s_{1,2}\) becomes \((1 - c_2)t_{0,1} \equiv c_2s_{2,1} + c_4s_{1,2}\).

Comparing \((0, 2)\)-entries, we get \(b_1c_1 - b_1c_2 + b_2c_3 - b_2c_4 \equiv -t_{0,1}\). Substituting \(b_1, b_2\) we obtain \((c_1 + 1 - c_2)t_{0,1} \equiv (c_2 - c_1)s_{2,1} + (c_4 - c_3)s_{1,2}\).

We see that \(\partial_1 x \in W_2\) if and only if

\[
(1 - c_2)t_{0,1} \equiv c_2s_{2,1} + c_4s_{1,2} \quad \text{(B1)}
\]

\[
(c_1 + 1 - c_2)t_{0,1} \equiv (c_2 - c_1)s_{2,1} + (c_4 - c_3)s_{1,2} \quad \text{(B2)}.
\]

We now use (A2) to substitute for \(s_{2,1}\) in each of (B1) and (B2). (B1) becomes

\[
(1 - c_2)t_{0,1} \equiv (c_2 + c_4(c_3 - c_4))s_{2,1}. \quad \text{For convenience let } q_1 = c_2 + c_4(c_3 - c_4). \quad \text{Hence}
\]

\[
(1 - c_2)t_{0,1} \equiv q_1s_{2,1}. \quad \text{(B2) becomes } (c_1 + 1 - c_2)t_{0,1} \equiv (c_2 - c_1 - (c_3 - c_4)^2)s_{2,1}. \quad \text{For}
\]

convenience let \(q_2 \equiv c_2 - c_1 - (c_3 - c_4)^2\). Hence \((c_1 + 1 - c_2)t_{0,1} \equiv q_2s_{2,1}\).
Hence \( x \in \partial^{-1}W_2 \) if and only if

\[
\begin{align*}
t_{1,0} &\equiv c_4 s_{2,1} \quad \text{(A1)} \\
s_{1,2} &\equiv (c_3 - c_4) s_{2,1} \quad \text{(A2)} \\
(1 - c_2)t_{0,1} &\equiv q_1 s_{2,1} \quad \text{(B1)} \\
(c_1 + 1 - c_2)t_{0,1} &\equiv q_2 s_{2,1} \quad \text{(B2).}
\end{align*}
\]

It is convenient to consider the cases \( c_2 \neq 1 \) and \( c_2 = 1 \) separately.

9.1 Case 6.1

Suppose \( c_2 \neq 1 \). Thus \( 1 - c_2 \neq 0 \) and we multiply (B1) by \( 1 - c_2 \) to obtain \( t_{0,1} \equiv (1 - c_2)q_1 s_{2,1} \). Now we use this last congruence to substitute for \( t_{0,1} \) in (B2), which becomes \( (c_1 - c_2 + 1)(1 - c_2)q_1 s_{2,1} \equiv q_2 s_{2,1} \). Since \( (1 - c_2)^2 \equiv 1 \), we multiply this last congruence by \( 1 - c_2 \) and we obtain \( (c_1 - c_2 + 1)q_1 s_{2,1} \equiv (1 - c_2)q_2 s_{2,1} \). Let \( q = (c_1 - c_2 + 1)q_1 - (1 - c_2)q_2 \). Thus (B2) becomes \( q s_{2,1} \equiv 0 \).

Hence \( x \in \partial^{-1}W_2 \) if and only if

\[
\begin{align*}
t_{1,0} &\equiv c_4 s_{2,1} \quad \text{(A1)} \\
s_{1,2} &\equiv (c_3 - c_4) s_{2,1} \quad \text{(A2)} \\
t_{0,1} &\equiv (1 - c_2)q_1 s_{2,1} \quad \text{(B1)} \\
q s_{2,1} &\equiv 0 \quad \text{(B2).}
\end{align*}
\]

It is convenient to consider the cases \( q \neq 0 \) and \( q \equiv 0 \) separately.
9.1.1 Case 6.1.1

Suppose $q \not\equiv 0$. Then (B2) is equivalent to $s_{2,1} \equiv 0$. It follows that $t_{1,0} \equiv s_{1,2} \equiv t_{0,1} \equiv 0$. Hence $\partial^{-1}W_1 = \partial^{-1}W_0$. Thus $\text{rank}(\partial^{-1}W_1/W_1) = \text{rank}(\partial^{-1}W_0/W_1)$ and $W_1$ is terminal and $W_1 \notin \hat{L}_1$.

9.1.2 Case 6.1.2

Suppose $q \equiv 0$. Thus (B2) is automatic and $s_{2,1}$ is free. Taking $s_{2,1} \equiv 1$, the matrix $x$ becomes

$$
v_1 = \begin{bmatrix}
0 & (1 - c_2)q_1 & 0 \\
c_4 & 0 & 3(c_3 - c_4) \\
0 & 3 & 0
\end{bmatrix}.
$$

Note $\partial^{-1}W_1 = < \partial^{-1}W_0, v_1 >$. Since $v_1 \notin \partial^{-1}W_0$ and $3v_1 \in \partial^{-1}W_0$, we know $|\partial^{-1}W_1/\partial^{-1}W_0| = 3$. Since $|\partial^{-1}W_0| = 3^7$ then $|\partial^{-1}W_1| = 3^8$. Because $|W_1| = 3^5$ then $|\partial^{-1}W_1/W_1| = 3^3$. Clearly $\partial^{-1}W_1/W_1 \cong \mathbb{Z}_3 \times \mathbb{Z}_3 \times \mathbb{Z}_3$ while $\partial^{-1}W_0/W_1 \cong \mathbb{Z}_3 \times \mathbb{Z}_3$ so $W_1$ is nonterminal. To determine a convenient ordered basis for vector space $\partial^{-1}W_0/W_1$ we must consider cases $c_4 = 0$ and $c_4 \in \{1, 2\}$ separately.

Case 6.1.2.1

Suppose $c_4 = 0$. Recall $c_2 \not\equiv 1$. Now we consider cases $c_2 = 0$ and $c_2 = 2$ separately.

Case 6.1.2.1.1 Suppose $c_2 = 0$. One can show this forces $(c_1, c_2, c_3, c_4)$ to be either $(2, 0, 1, 0)$ or $(2, 0, 2, 0)$. Recalling that $q_1 = c_2 + c_4(c_3 - c_4)$, the assumptions $c_2 = 0$ and
$c_4 = 0$ yield $q_1 = 0$. The matrix $v_1$ becomes

$$v_1 = \begin{bmatrix}
0 & 0 & 0 \\
0 & 0 & 3c_3 \\
0 & 3 & 0
\end{bmatrix}. $$

We see that $\partial^{-1}W_1 = < \partial^{-1}W_0, v_1 >$. Since $v_1 \notin \partial^{-1}W_0$ and $3v_1 \in \partial^{-1}W_0$, we know $|\partial^{-1}W_1/\partial^{-1}W_0| = 3$. Since $|\partial^{-1}W_0| = 3^7$ then $|\partial^{-1}W_1| = 3^8$. Because $|W_1| = 3^5$ then $|\partial^{-1}W_1/W_1| = 3^3$. Also, $v_1, y_3, v_4 \in \partial^{-1}W_1$ but are not contained in $W_1$. Each of $3v_1, 3y_3, 3y_4$ is contained in $W_1$. So $v_1 + W_1, y_3 + W_1, y_4 + W_1$ are elements of order 3 in the group $\partial^{-1}W_1/W_1$. These three elements form a generating set for the group $\partial^{-1}W_1/W_1$. Recall that $|\partial^{-1}W_1/W_1| = 3^3$, we obtain $\partial^{-1}W_1/W_1 \cong \mathbb{Z}_3 \times \mathbb{Z}_3 \times \mathbb{Z}_3$. Since rank($\partial^{-1}W_1/W_1$) = 3 and rank($\partial^{-1}W_0/W_1$) = 2, then rank($\partial^{-1}W_0/W_1$) < rank($\partial^{-1}W_1/W_1$). Hence $W_1$ is nonterminal and $W_1 \in \hat{L}_1$. A basis for $\partial^{-1}W_0/W_1$ is $v_1 + W_1, y_3 + W_1, y_4 + W_1$. A basis for $\partial^{-1}W_0/W_1$ is $y_3 + W_1, y_4 + W_1$.

We fix arbitrary values $c_5, c_6 \in \{0, 1, 2\}$. There are $3^2 = 9$ ways to choose these values. Let $m_3 = c_5y_3 + c_6y_4 + v_1$. Thus

$$m_3 = \begin{bmatrix}
c_6 & 0 & 3(c_5 - c_6) \\
0 & 0 & 3c_3 \\
-3c_6 & 3 & 0
\end{bmatrix}. $$

Let $W_2 = < W_1, m_3 >$. There are 9 subgroups of $W_2$ of this type. Since $m_3 \notin W_1$ and $3m_3 \in W_1$ then $|W_2/W_1| = 3$. Also, since $|W_1| = 3^5$ then $|W_2| = 3^6$. Recall $|\partial^{-1}W_1| = 3^8$, then $\partial^{-1}W_1/W_2 \cong \mathbb{Z}_3 \times \mathbb{Z}_3$ and rank($\partial^{-1}W_1/W_2$) = 2.
The subgroup $W_2$ is contained in the pattern subgroup
\[
\begin{bmatrix}
2 & 1 & 1 \\
1 & 1 & 1 \\
1 & 1 & 0
\end{bmatrix}.
\]

The pullback $\partial^{-1}W_2$ is contained in the pattern subgroup
\[
\begin{bmatrix}
2 & 2 & 2 \\
2 & 1 & 1 \\
1 & 1 & 1
\end{bmatrix} \cap \begin{bmatrix}
2 & 2 & 1 \\
2 & 1 & 1 \\
2 & 1 & 1
\end{bmatrix} = \begin{bmatrix}
2 & 2 & 1 \\
2 & 1 & 1 \\
1 & 1 & 1
\end{bmatrix}.
\]

Let
\[
x = \begin{bmatrix}
3s_{0,0} + t_{0,0} & 3s_{0,1} + t_{0,1} & 3s_{0,2} \\
3s_{1,0} + t_{1,0} & 3s_{1,1} & 3s_{1,2} \\
3s_{2,0} & 3s_{2,1} & 3s_{2,2}
\end{bmatrix} \in \begin{bmatrix}
2 & 2 & 1 \\
2 & 1 & 1 \\
1 & 1 & 1
\end{bmatrix}.
\]

Thus
\[
\partial_1 x = \begin{bmatrix}
3s_{1,0} + t_{1,0} & 3s_{1,1} & 3s_{1,2} \\
3s_{2,0} & 3s_{2,1} & 3s_{2,2} \\
-3t_{1,0} & 0 & 0
\end{bmatrix} \quad \text{and} \quad \partial_2 x = \begin{bmatrix}
3s_{0,1} + t_{0,1} & 3s_{0,2} & -3t_{0,1} \\
3s_{1,1} & 3s_{1,2} & 0 \\
3s_{2,1} & 3s_{2,2} & 0
\end{bmatrix}.
\]

The variables that are in play are those appearing in the matrix
\[
\begin{bmatrix}
0 & t_{0,1} & 0 \\
t_{1,0} & 0 & 3s_{1,2} \\
0 & 3s_{2,1} & 3s_{2,2}
\end{bmatrix}.
\]
We want to identify a value \(a_1, a_2, a_3 \in \mathbb{Z}_9\) such that \(\partial_1 x \equiv a_1m_1 + a_2m_2 + a_3m_3 \pmod{I}\). A formal expression for \(a_1m_1 + a_2m_2 + a_3m_3\) is

\[
\begin{bmatrix}
0 & 0 & -3 \\
0 & 0 & 0 \\
3 & 0 & 0 \\
\end{bmatrix} + a_2
\begin{bmatrix}
0 & 0 & 3c_3 \\
0 & 3 & 0 \\
0 & 0 & 0 \\
\end{bmatrix} + a_3
\begin{bmatrix}
c_6 & 0 & 3(c_5 - c_6) \\
0 & 0 & 3c_3 \\
-3c_6 & 3 & 0 \\
\end{bmatrix}
\]

\[
= \begin{bmatrix}
3(a_3c_6) & 0 & 3(-a_1 + a_2c_3 + a_3c_5 - a_3c_6) \\
0 & 3a_2 & 3a_3c_3 \\
3(a_1 - a_3c_6) & 3a_3 & 0 \\
\end{bmatrix}.
\]

Comparing \((0, 1)\)-entries, we get \(3s_{1,1} \equiv 0\) which gives no information.

Comparing \((1, 0)\)-entries, we get \(0 \equiv 3s_{2,0}\) which gives no information.

Comparing \((2, 1)\)-entries, we get \(a_3 \equiv 0\).

Comparing \((0, 0)\)-entries, we get \(a_3c_6 \equiv t_{1,0}\). Substituting \(a_3\) we obtain \(t_{1,0} \equiv 0\).

Comparing \((1, 2)\)-entries, we get \(a_3c_3 \equiv s_{2,2}\). Substituting \(a_3\) we obtain \(s_{2,2} \equiv 0\).

Comparing \((1, 1)\)-entries, we get \(a_2 \equiv s_{2,1}\).

Comparing \((2, 0)\)-entries, we get \(a_1 \equiv -t_{1,0} + a_3c_6\). Substituting \(a_3\) we obtain \(a_1 \equiv 0\).

Comparing \((0, 2)\)-entries, we get \(-a_1 + a_2c_3 + a_3(c_5 - c_6) \equiv s_{1,2}\). Substituting \(a_1, a_2, a_3\) we obtain \(s_{1,2} \equiv c_3s_{2,1}\).
We see that $\partial_1 x \in W_2$ if and only if

$$t_{1,0} \equiv 0 \quad (A1)$$

$$s_{2,2} \equiv 0 \quad (A2)$$

$$s_{1,2} \equiv c_3 s_{2,1} \quad (A3).$$

We want to identify a value $b_1, b_2, b_3 \in \mathbb{Z}_9$ such that $\partial_2 x \equiv b_1 m_1 + b_2 m_2 + b_3 m_3 \pmod{I}$. A formal expression for $b_1 m_1 + b_2 m_2 + b_3 m_3$ is

$$b_1 \begin{bmatrix} 0 & 0 & -3 \\ 0 & 0 & 0 \\ 3 & 0 & 0 \end{bmatrix} + b_2 \begin{bmatrix} 0 & 0 & 3c_3 \\ 0 & 3 & 0 \\ 0 & 0 & 0 \end{bmatrix} + b_3 \begin{bmatrix} c_6 & 0 & 3(c_5 - c_6) \\ 0 & 0 & 3c_3 \\ -3c_6 & 3 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} b_3c_6 & 0 & 3(-b_1 + b_2c_3 + b_3c_5 - b_3c_6) \\ 0 & 3b_2 & 3b_3c_3 \\ 3(b_1 - b_3c_6) & 3b_3 & 0 \end{bmatrix}.$$

Comparing $(1,0)$-entries, we get $0 \equiv 3s_{1,1}$ which gives no information.

Comparing $(0,1)$-entries, we get $0 \equiv 3s_{0,2}$ which gives no information.

Comparing $(1,2)$-entries we get $b_3 c_3 \equiv 0$. Since $c_3 \neq 0$ then $b_3 \equiv 0$.

Comparing $(2,1)$-entries, we get $b_3 \equiv s_{2,2}$. Hence $s_{2,2} \equiv 0$.

Comparing $(0,0)$-entries, we get $b_3 c_6 \equiv t_{0,1}$. Hence $t_{0,1} \equiv 0$.

Comparing $(1,1)$-entries, we get $b_2 \equiv s_{1,2}$.

Comparing $(2,0)$-entries, we get $b_1 \equiv s_{2,1} + b_3 c_6$. Substituting $b_3$ we obtain $b_1 \equiv s_{2,1}$.  

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Comparing $(0,2)$-entries, we get $-b_1 + b_2 c_3 + b_3 (c_5 - c_6) \equiv -t_{0,1}$. Substituting $b_1, b_2, b_3$ we obtain $s_{2,1} \equiv c_3 s_{1,2}$.

We see that $\partial_2 x \in W_2$ if and only if

\begin{align*}
    s_{2,2} & \equiv 0 \quad \text{(B1)} \\
    t_{0,1} & \equiv 0 \quad \text{(B2)} \\
    s_{2,1} & \equiv c_3 s_{1,2} \quad \text{(B3)}. \\
\end{align*}

(B1) and (B3) are redundant with (A2) and (A3) respectively so we may ignore them.

Hence $x \in \partial^{-1} W_2$ if and only if

\begin{align*}
    t_{1,0} & \equiv 0 \\
    s_{2,2} & \equiv 0 \\
    s_{1,2} & \equiv c_3 s_{2,1} \\
    t_{0,1} & \equiv 0. \\
\end{align*}

Hence $\partial^{-1} W_2 = \partial^{-1} W_1$. Thus rank($\partial^{-1} W_2 / W_2$) = rank ($\partial^{-1} W_1 / W_2$) and $W_2$ is terminal and $W_2 \notin \hat{\mathcal{L}}_2$.

**Case 6.1.2.1.2** Suppose $c_2 = 2$. One can show this forces $(c_1, c_2, c_3, c_4)$ to be either $(1,2,1,0)$ or $(1,2,2,0)$. Recalling $q_1 = c_2 + c_4 (c_3 - c_4)$ the assumptions $c_4 = 0$ and
\(c_2 = 2\) yield \(q_1 = 2\). The matrix \(v_1\) becomes
\[
\begin{bmatrix}
0 & -2 & 0 \\
0 & 0 & 3c_3 \\
0 & 3 & 0
\end{bmatrix}
\]
We see that \(\partial^{-1}W_1 = < \partial^{-1}W_0, v_1 >\). Since \(v_1 \notin \partial^{-1}W_0\) and \(3v_1 \in \partial^{-1}W_0\), we know \(|\partial^{-1}W_1/\partial^{-1}W_0| = 3\). Since \(|\partial^{-1}W_0| = 3^7\) then \(|\partial^{-1}W_1| = 3^8\). Because \(|W_1| = 3^5\) then \(|\partial^{-1}W_1/W_1| = 3^3\). Also, \(v_1, y_1, v_3 \in \partial^{-1}W_1\) but are not contained in \(W_1\). Each of \(3v_1, 3y_1, 3y_3\) is contained in \(W_1\). So \(v_1 + W_1, y_1 + W_1, y_3 + W_1\) are elements of order \(3\) in the group \(\partial^{-1}W_1/W_1\). These three elements form a generating set for the group \(\partial^{-1}W_1/W_1\). Recall that \(|\partial^{-1}W_1/W_1| = 3^8\), we obtain \(\partial^{-1}W_1/W_1 \cong \mathbb{Z}_3 \times \mathbb{Z}_3 \times \mathbb{Z}_3\).

Since \(\text{rank}(\partial^{-1}W_1/W_1) = 3\) and \(\text{rank}(\partial^{-1}W_0/W_1) = 2\), then \(\text{rank}(\partial^{-1}W_0/W_1) < \text{rank}(\partial^{-1}W_1/W_1)\). Hence \(W_1\) is nonterminal and \(W_1 \in \hat{L}_1\). A basis for \(\partial^{-1}W_1/W_1\) is \(v_1 + W_1, y_1 + W_1, y_3 + W_1\). A basis for \(\partial^{-1}W_0/W_1\) is \(y_1 + W_1, y_3 + W_1\).

We fix arbitrary values \(c_5, c_6 \in \{0, 1, 2\}\). There are \(3^2 = 9\) ways to choose these values. Let \(m_3 = c_5y_1 + c_6y_3 + v_1\). Thus
\[
m_3 = \begin{bmatrix}
0 & 1 & 3c_6 \\
0 & 0 & 3c_3 \\
3c_5 & 3 & 0
\end{bmatrix}
\]
Let \(W_2 = < W_1, m_3 >\). There are 9 subgroups of \(W_2\) of this type. Since \(m_3 \notin W_1\) and \(3m_3 \notin W_1\) then \(|W_2/W_1| = 3\). Also, since \(|W_1| = 3^5\) then \(|W_2| = 3^6\).
Recall \(|\partial^{-1}W_1| = 3^8\), then \(\partial^{-1}W_1/W_2 \cong \mathbb{Z}_3 \times \mathbb{Z}_3\) and \(\text{rank}(\partial^{-1}W_1/W_2) = 2\).
The subgroup $W_2$ is contained in the pattern subgroup
\[
\begin{bmatrix}
2 & 2 & 1 \\
1 & 1 & 1 \\
1 & 1 & 0
\end{bmatrix}.
\]

The pullback $\partial^{-1}W_2$ is contained in the pattern subgroup
\[
\begin{bmatrix}
2 & 2 & 2 \\
2 & 2 & 1 \\
1 & 1 & 1
\end{bmatrix} \cap \begin{bmatrix}
2 & 2 & 2 \\
2 & 1 & 1 \\
1 & 1 & 1
\end{bmatrix} = \begin{bmatrix}
2 & 2 & 2 \\
2 & 1 & 1 \\
1 & 1 & 1
\end{bmatrix}.
\]

Let
\[
x = \begin{bmatrix}
3s_{0,0} + t_{0,0} & 3s_{0,1} + t_{0,1} & 3s_{0,2} + t_{0,2} \\
3s_{1,0} + t_{1,0} & 3s_{1,1} & 3s_{1,2} \\
3s_{2,0} & 3s_{2,1} & 3s_{2,2}
\end{bmatrix} \in \begin{bmatrix}
2 & 2 & 2 \\
2 & 1 & 1 \\
1 & 1 & 1
\end{bmatrix}.
\]

Thus
\[
\partial_1 x = \begin{bmatrix}
3s_{1,0} + t_{1,0} & 3s_{1,1} & 3s_{1,2} \\
3s_{2,0} & 3s_{2,1} & 3s_{2,2} \\
-3t_{1,0} & 0 & 0
\end{bmatrix}
\]
and
\[
\partial_2 x = \begin{bmatrix}
3s_{0,1} + t_{0,1} & 3s_{0,2} + t_{0,2} & -3(t_{0,1} + t_{0,2}) \\
3s_{1,1} & 3s_{1,2} & 0 \\
3s_{2,1} & 3s_{2,2} & 0
\end{bmatrix}.
\]

The variables that are in play are those appearing in the matrix
\[
\begin{bmatrix}
0 & t_{0,1} & t_{0,2} \\
t_{1,0} & 0 & 3s_{1,2} \\
0 & 3s_{2,1} & 3s_{2,2}
\end{bmatrix}.
\]

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We want to identify a value $a_1, a_2, a_3 \in \mathbb{Z}_9$ such that $\partial_1 x \equiv a_1 m_1 + a_2 m_2 + a_3 m_3 \pmod{I}$. A formal expression for $a_1 m_1 + a_2 m_2 + a_3 m_3$ is

$$
\begin{bmatrix}
2 & 0 & -3 \\
0 & 0 & 0 \\
-3 & 0 & 0
\end{bmatrix}
+ a_1
\begin{bmatrix}
0 & 0 & 3c_3 \\
0 & 3 & 0 \\
0 & 0 & 0
\end{bmatrix}
+ a_3
\begin{bmatrix}
0 & 0 & 3c_6 \\
0 & 0 & 3c_3 \\
3c_5 & 3 & 0
\end{bmatrix}
= 
\begin{bmatrix}
-a_1 & a_3 & 3(-a_1 + a_2 c_3 + a_3 c_6) \\
0 & 3a_2 & 3a_3 c_3 \\
3(-a_1 + a_2 c_5) & 3a_3 & 0
\end{bmatrix}.
$$

Comparing $(1,0)$-entries, we get $0 \equiv 3s_{2,0}$ which gives no information.

Comparing $(0,1)$-entries, we get $a_3 \equiv s_{1,1}$.

Comparing $(2,1)$-entries, we get $a_3 \equiv 0$.

Comparing $(1,2)$-entries, we get $a_3 c_3 \equiv s_{2,2}$. Hence $s_{2,2} \equiv 0$.

Comparing $(1,1)$-entries, we get $a_2 \equiv s_{2,1}$.

Comparing $(0,0)$-entries, we get $a_1 \equiv -t_{1,0}$.

Comparing $(0,2)$-entries, we get $-a_1 \equiv -t_{1,0} - a_3 c_5$. Hence $a_1 \equiv t_{1,0}$. This implies that $t_{1,0} \equiv 0$.

Comparing $(2,0)$-entries, we get $-a_1 + a_2 c_3 + a_3 c_6 \equiv s_{1,2}$. Substituting $a_1, a_2, a_3$ we obtain $s_{1,2} \equiv c_3 s_{2,1}$.
We see that \( \partial_1 x \in W_2 \) if and only if

\[
\begin{align*}
    s_{2,2} &\equiv 0 \quad (A1) \\
    t_{1,0} &\equiv 0 \quad (A2) \\
    s_{1,2} &\equiv c_3s_{2,1} \quad (A3).
\end{align*}
\]

We want to identify a value \( b_1, b_2, b_3 \in Z_9 \) such that \( \partial_2 x \equiv b_1m_1 + b_2m_2 + b_3m_3 \pmod{I} \). A formal expression for \( b_1m_1 + b_2m_2 + b_3m_3 \) is

\[
\begin{bmatrix}
    0 & 0 & -3 \\
    0 & 0 & 0 \\
    3 & 0 & 0
\end{bmatrix}
\begin{bmatrix}
    c_6 & 0 & 3(c_5 - c_6) \\
    0 & 3 & 0 \\
    -3c_6 & 3 & 0
\end{bmatrix}
\]

\[
= \begin{bmatrix}
    -b_1 & b_3 & 3(-b_1 + b_2c_3 + b_3c_6) \\
    0 & 3b_2 & 3b_3c_3 \\
    3(-b_1b_3c_5) & 3b_3 & 0
\end{bmatrix}.
\]

Comparing \((1,0)\)-entries, we get \( 0 \equiv 3s_{1,1} \) which gives no information.

Comparing \((1,2)\)-entries we get \( b_3c_3 \equiv 0 \). Since \( c_3 \neq 0 \) then \( b_3 \equiv 0 \).

Comparing \((0,1)\)-entries, we get \( b_3 \equiv t_{0,2} \). Hence \( t_{0,2} \equiv 0 \).

Comparing \((2,1)\)-entries, we get \( b_3 \equiv s_{2,2} \). Hence \( s_{2,2} \equiv 0 \).

Comparing \((0,0)\)-entries, we get \( b_1 \equiv -t_{0,1} \).

Comparing \((1,1)\)-entries, we get \( b_2 \equiv s_{2,1} \).

Comparing \((2,0)\)-entries, we get \( b_1 \equiv -s_{2,1} + b_3c_5 \). Substituting \( b_1, b_3 \) we obtain \( t_{0,1} \equiv s_{2,1} \).
Comparing $(0,2)$-entries, we get $-b_1 + b_2 c_3 + b_3 c_6 \equiv -t_{0,1} - t_{0,2}$. Substituting $b_1, b_2, b_3$ we obtain $c_3 s_{2,1} \equiv t_{0,1}$. Substituting $t_{0,1}$ we obtain $(c_3 - 1)s_{2,1} \equiv 0$.

We see that $\partial_2 x \in W_2$ if and only if

$$t_{0,2} \equiv 0 \quad \text{(B1)}$$

$$t_{0,1} \equiv 0 \quad \text{(B2)}$$

$$(c_3 - 1)s_{2,1} \equiv 0 \quad \text{(B3)}.$$  

If $c_3 = 2$ then by (B3) $s_{2,1} \equiv 0$.

Hence $x \in \partial^{-1} W_2$ if and only if

$$t_{1,0} \equiv 0$$

$$s_{2,2} \equiv 0$$

$$s_{1,2} \equiv 0$$

$$t_{0,1} \equiv 0$$

$$t_{0,2} \equiv 0$$

$$s_{2,1} \equiv 0.$$  

Hence $\partial^{-1} W_2 = \partial^{-1} W_1$. Thus $\text{rank}(\partial^{-1} W_2/W_2) = \text{rank}(\partial^{-1} W_1/W_2)$ and $W_2$ is terminal and $W_2 \not\in \hat{L}_2$.

If $c_3 = 1$ then $s_{2,1}$ is a free variable. This is the same as in Case 6.1.2. Hence $\partial^{-1} W_2 = \partial^{-1} W_1$. Thus $\text{rank}(\partial^{-1} W_2/W_2) = \text{rank}(\partial^{-1} W_1/W_2)$ and $W_2$ is terminal and $W_2 \not\in \hat{L}_2$. 

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Case 6.1.2.2

Suppose $c_4 \in \{1, 2\}$. There are exactly 16 possibilities for $(c_1, c_2, c_3, c_4)$ for this case.

The matrix $v_1$ becomes

$$v_1 = \begin{bmatrix}
0 & (1 - c_2)q_1 & 0 \\
c_4 & 0 & 3(c_3 - c_4) \\
0 & 3 & 0
\end{bmatrix}.$$

We see that $\partial^{-1}W_1 = < \partial^{-1}W_0, v_1 >$. Since $v_1 \notin \partial^{-1}W_0$ and $3v_1 \in \partial^{-1}W_0$, we know $|\partial^{-1}W_1/\partial^{-1}W_0| = 3$. Since $|\partial^{-1}W_0| = 3^7$ it follows $|\partial^{-1}W_1| = 3^8$. Because $|W_1| = 3^5$ then $|\partial^{-1}W_1/W_1| = 3^3$. Also, $v_1, y_2, v_3 \in \partial^{-1}W_1$ but are not contained in $W_1$. Each of $3v_1, 3y_2, 3y_3$ is contained in $W_1$. So $v_1 + W_1, y_2 + W_1, y_3 + W_1$ are elements of order 3 in the group $\partial^{-1}W_1/W_1$. These three elements form a generating set for the group $\partial^{-1}W_1/W_1$. Recall that $|\partial^{-1}W_1/W_1| = 3^3$, we obtain $\partial^{-1}W_1/W_1 \cong \mathbb{Z}_3 \times \mathbb{Z}_3 \times \mathbb{Z}_3$. Since rank($\partial^{-1}W_1/W_1$) = 3 and rank($\partial^{-1}W_0/W_1$) = 2, then rank($\partial^{-1}W_0/W_1$) < rank($\partial^{-1}W_1/W_1$). Hence $W_1$ is nonterminal and $W_1 \in \hat{L}_1$. A basis for $\partial^{-1}W_1/W_1$ is $v_1 + W_1, y_2 + W_1, y_3 + W_1$. A basis for $\partial^{-1}W_0/W_1$ is $y_2 + W_1, y_3 + W_1$.

We fix arbitrary values $c_5, c_6 \in \{0, 1, 2\}$. There are $3^2 = 9$ ways to choose these values. Let $m_3 = c_5y_2 + c_6y_3 + v_1$.

Thus

$$m_3 = \begin{bmatrix}
0 & (1 - c_2)q_1 & 3c_6 \\
c_4 & 3c_5 & 3(c_3 - c_4) \\
0 & 3 & 0
\end{bmatrix}.$$
Let $W_2 = \langle W_1, m_3 \rangle$. There are 9 subgroups of $W_2$ of this type. Since $m_3 \notin W_1$ and $3m_3 \in W_1$ then $|W_2/W_1| = 3$. Also, since $|W_1| = 3^5$ then $|W_2| = 3^6$. Recall $|\partial^{-1}W_1| = 3^8$, then $\partial^{-1}W_1/W_2 \cong \mathbb{Z}_3 \times \mathbb{Z}_3$ and $\text{rank}(\partial^{-1}W_1/W_2) = 2$.

The subgroup $W_2$ is contained in the pattern subgroup

$$\begin{bmatrix} 2 & 2 & 1 \\ 2 & 1 & 1 \\ 1 & 1 & 0 \end{bmatrix}.$$ 

The pullback $\partial^{-1}W_2$ is contained in the pattern subgroup

$$\begin{bmatrix} 2 & 2 & 2 \\ 2 & 2 & 1 \\ 2 & 1 & 1 \end{bmatrix} \cap \begin{bmatrix} 2 & 2 & 2 \\ 2 & 2 & 1 \\ 2 & 1 & 1 \end{bmatrix} = \begin{bmatrix} 2 & 2 & 2 \\ 2 & 2 & 1 \\ 2 & 1 & 1 \end{bmatrix}.$$

Let

$$x = \begin{bmatrix} 3s_{0,0} + t_{0,0} & 3s_{0,1} + t_{0,1} & 3s_{0,2} + t_{0,2} \\ 3s_{1,0} + t_{1,0} & 3s_{1,1} + t_{1,1} & 3s_{1,2} \\ 3s_{2,0} + t_{2,0} & 3s_{2,1} & 3s_{2,2} \end{bmatrix} \in \begin{bmatrix} 2 & 2 & 2 \\ 2 & 2 & 1 \\ 2 & 1 & 1 \end{bmatrix}.$$

Thus

$$\partial_1 x = \begin{bmatrix} 3s_{1,0} + t_{1,0} & 3s_{1,1} + t_{1,1} & 3s_{1,2} \\ 3s_{2,0} + t_{2,0} & 3s_{2,1} & 3s_{2,2} \\ -3(t_{1,0} + t_{2,0}) & -3t_{1,1} & 0 \end{bmatrix}$$

and

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\[ \partial_2 x = \begin{bmatrix} 3s_{0,1} + t_{0,1} & 3s_{0,2} + t_{0,2} & -3(t_{0,1} + t_{0,2}) \\ 3s_{1,1} + t_{1,1} & 3s_{1,2} & -3t_{1,1} \\ 3s_{2,1} & 3s_{2,2} & 0 \end{bmatrix}. \]

The variables that are in play are those appearing in the matrix

\[
\begin{bmatrix}
0 & t_{0,1} & t_{0,2} \\
t_{1,0} & t_{1,1} & 3s_{1,2} \\
t_{2,0} & 3s_{2,1} & 3s_{2,2}
\end{bmatrix}
\]

We want to identify values \( a_1, a_2, a_3 \in \mathbb{Z}_9 \) such that \( \partial_1 x \equiv a_1 m_1 + a_2 m_2 + a_3 m_3 (\text{mod } I) \). A formal expression for \( a_1 m_1 + a_2 m_2 + a_3 m_3 \) is

\[
a_1 \begin{bmatrix} c_2 & 0 & 3(c_1 - c_2) \\ 0 & 0 & 0 \\ 3(1 - c_2) & 0 & 0 \end{bmatrix} + a_2 \begin{bmatrix} c_4 & 0 & 3(c_3 - c_4) \\ 0 & 3 & 0 \\ -3c_4 & 0 & 0 \end{bmatrix}
\]

\[
+ a_3 \begin{bmatrix} 0 & (1 - c_2)q_1 & 3c_6 \\ c_4 & 3c_5 & 3(c_3 - c_4) \\ 0 & 3 & 0 \end{bmatrix}
\]

\[
= \begin{bmatrix} a_1 c_2 + a_2 c_4 & a_3 (1 - c_2)q_1 & 3[a_1(c_1 - c_2) + a_2(c_3 - c_4) + a_3 c_6] \\ a_3 c_4 & 3(a_2 + a_3 c_5) & 3a_3(c_3 - c_4) \\ 3(a_1 - a_1 c_2 - a_2 c_4) & 3a_3 & 0 \end{bmatrix}.
\]

Comparing (2,1)-entries, we get \( a_3 \equiv -t_{1,1} \).

Comparing (1,0)-entries, we get \( a_3 c_4 \equiv t_{2,0} \). Substituting \( a_3 \) we obtain \( t_{2,0} \equiv -c_4 t_{1,1} \).
Comparing $(0, 1)$-entries, we get $a_3(1 - c_2)q_1 \equiv t_{1,1}$. Substituting $a_3$ we obtain $0 \equiv (1 + q_1 - c_2q_1)t_{1,1}$.

Comparing $(1, 2)$-entries, we get $a_3(c_3 - c_4) \equiv s_{2,2}$. Substituting $a_3$ we obtain $s_{2,2} \equiv -(c_3 - c_4)t_{1,1}$.

Comparing $(1, 1)$-entries, we get $a_2 \equiv s_{2,1} - a_3c_5$. Substituting $a_3$ we obtain $a_2 \equiv s_{2,1} + c_5t_{1,1}$.

Comparing $(2, 0)$-entries, we get $t_{1,0} + t_{2,0} \equiv a_1c_1 + a_2c_2 - a_1$. We use $t_{2,0} \equiv -c_4t_{1,1}$, to substitute here for $t_{2,0}$, and then we rearrange to obtain $a_1 \equiv a_1c_2 + a_2c_4 - t_{1,0} + c_4t_{1,1}$.

Comparing $(0, 0)$-entries, we get $t_{1,0} \equiv a_1c_2 + a_2c_4$. We use this to substitute for $t_{1,0}$ in the expression for $a_1$ above to obtain $a_1 \equiv c_4t_{1,1}$. Using this along with earlier congruence $a_2 \equiv s_{2,1} + c_5t_{1,1}$ we obtain $t_{1,0} \equiv c_4s_{2,1} + c_4(c_2 + c_5)t_{1,1}$.

Comparing $(0, 2)$-entries, we get $s_{1,2} \equiv (c_1 - c_2)a_1 + (c_3 - c_4)a_2 + c_6a_3$. Thus $s_{1,2} \equiv (c_1 - c_2)c_4t_{1,1} + (c_3 - c_4)(s_{2,1} + c_5t_{1,1}) - c_6t_{1,1}$, which we simplify to obtain $s_{1,2} \equiv (c_3 - c_4)s_{2,1} + Rt_{1,1}$ where $R = (c_1 - c_2)c_4 + (c_3 - c_4)c_5 - c_6$.

We see that $\partial_1 x \in W_2$ if and only if

\begin{align*}
t_{2,0} & \equiv -c_4t_{1,1} \quad \text{(A1)} \\
0 & \equiv (1 + q_1 - c_2q_1)t_{1,1} \quad \text{(A2)} \\
s_{2,2} & \equiv -(c_3 - c_4)t_{1,1} \quad \text{(A3)} \\
t_{1,0} & \equiv c_4s_{2,1} + c_4(c_2 + c_5)t_{1,1} \quad \text{(A4)} \\
s_{1,2} & \equiv (c_3 - c_4)s_{2,1} + Rt_{1,1} \quad \text{(A5)}.
\end{align*}
We want to identify a value \( b_1, b_2, b_3 \in \mathbb{Z}_9 \) such that \( \partial_2 x \equiv b_1 m_1 + b_2 m_2 + b_3 m_3 \pmod{I} \). A formal expression for \( b_1 m_1 + b_2 m_2 + b_3 m_3 \) is

\[
\begin{bmatrix}
    c_2 & 0 & 3(c_1 - c_2) \\
    0 & 0 & 0 \\
    3(1 - c_2) & 0 & 0
\end{bmatrix} + b_2
\begin{bmatrix}
    c_4 & 0 & 3(c_3 - c_4) \\
    0 & 3 & 0 \\
    -3c_4 & 0 & 0
\end{bmatrix}
\]

\[
+b_3
\begin{bmatrix}
    0 & (1 - c_2)q_1 & 3c_6 \\
    c_4 & 3c_5 & 3(c_3 - c_4) \\
    0 & 3 & 0
\end{bmatrix}
\]

\[
= \begin{bmatrix}
    b_1c_2 + b_2c_4 & b_3(1 - c_2)q_1 & 3[b_1(c_1 - c_2) + b_2(c_3 - c_4) + b_3c_6] \\
    b_3c_4 & 3(b_2 + b_3c_5) & 3b_3(c_3 - c_4) \\
    3(b_1(1 - c_2) - b_2c_4) & 3b_3 & 0
\end{bmatrix}
\]

Comparing (2,1)-entries, we get \( b_3 \equiv s_{2,2} \).

Comparing (1,0)-entries, we get \( b_3c_4 \equiv t_{1,1} \). Substituting \( b_3 \) we obtain \( s_{2,2} \equiv c_4 t_{1,1} \).

Comparing (1,2)-entries we get, \( b_3(c_3 - c_4) \equiv -t_{1,1} \). Substituting \( b_3 \) we obtain \( c_3c_4t_{1,1} \equiv 0 \).

Comparing (0,1)-entries, we get \( b_3(1 - c_2)q_1 \equiv t_{0,2} \). Substituting \( b_3 \) we obtain \( t_{0,2} \equiv (c_4q_1 - c_2c_4q_1)t_{1,1} \).

Comparing (1,1)-entries, we get \( b_2 \equiv s_{1,2} - b_3c_5 \). Substituting \( b_3 \) we obtain \( b_2 \equiv s_{1,2} - c_4c_5t_{1,1} \).

Comparing (2,0)-entries, we get \( s_{2,1} \equiv b_1 - (b_1c_2 + b_2c_4) \). Hence \( b_1 \equiv s_{2,1} + (b_1c_2 + b_2c_4) \).
Comparing \((0, 0)-\)entries, we get \(t_{0,1} \equiv b_1c_2 + b_2c_4\). We use this to substitute for \(b_1c_2 + b_2c_4\) in the expression for \(b_1\) above to get \(b_1 \equiv s_{2,1} + t_{0,1}\). Using this along with congruence \(b_2 \equiv s_{1,2} - c_4c_5t_{1,1}\) obtained earlier, we get \(t_{0,1} \equiv c_2s_{2,1} + c_2t_{0,1} + c_4s_{1,2} - c_5t_{1,1}\), which we rearrange to get \((1 - c_2)t_{0,1} \equiv c_2s_{2,1} + c_4s_{1,2} - c_5t_{1,1}\).

Comparing \((0, 2)-\)entries, we get \(t_{0,1} + t_{0,2} \equiv (c_2 - c_1)b_1 + (c_4 - c_3)b_2 - c_6b_3\). Hence, substituting \(b_2, b_3, t_{0,2}\) we obtain \((c_1 + 1 - c_2)t_{0,1} \equiv (c_2 - c_1)s_{2,1} + (c_4 - c_3)s_{1,2} + Qt_{1,1}\) where \(Q = -c_4q_1(1 - c_2) + (c_3c_4 - 1)c_5 - c_4c_6\).

We see that \(\partial_2 x \in W_2\) if and only if

\[
\begin{align*}
  s_{2,2} & \equiv c_4t_{1,1} \quad \text{(B1)} \\
  c_3c_4t_{1,1} & \equiv 0 \quad \text{(B2)} \\
  t_{0,2} & \equiv (c_4q_1 - c_2c_4q_1)t_{1,1} \quad \text{(B3)} \\
  (1 - c_2)t_{0,1} & \equiv c_2s_{2,1} + c_4s_{1,2} - c_5t_{1,1} \quad \text{(B4)} \\
  (c_1 + 1 - c_2)t_{0,1} & \equiv (c_2 - c_1)s_{2,1} + (c_4 - c_3)s_{1,2} + Qt_{1,1} \quad \text{(B5)}.
\end{align*}
\]

Recalling \(c_2 \in \{0, 2\}\) we see \(1 - c_2 \neq 0\) so \((1 - c_2)^2 \equiv 1\). Multiplying (B4) be \(a - c_2\) we obtain \(t_{0,1} \equiv (1 - c_2)[c_2s_{2,1} + c_4s_{1,2} - c_5t_{1,1}]\). Combining (A3) and (B1) we obtain \(0 \equiv c_3t_{1,1}\) which we call the new \(\text{(A3)}\). Note (B2) is a consequence of \(c_3t_{1,1} \equiv 0\), so we ignore (B2).
Hence $x \in \partial^{-1}W_2$ if and only if

$$t_{2,0} \equiv -c_4 t_{1,1} \quad (A1)$$

$$(1 + q_1 - c_2 q_1) t_{1,1} \equiv 0 \quad (A2)$$

$$c_3 t_{1,1} \equiv 0 \quad (A3)$$

$$t_{1,0} \equiv c_4 s_{2,1} + c_4 (c_2 + c_5) t_{1,1} \quad (A4)$$

$$s_{1,2} \equiv (c_3 - c_4) s_{2,1} + R t_{1,1} \quad (A5)$$

$$s_{2,2} \equiv c_4 t_{1,1} \quad (B1)$$

$$t_{0,2} \equiv c_4 (1 - c_2) q_1 t_{1,1} \quad (B3)$$

$$t_{0,1} \equiv (1 - c_2) [c_2 s_{2,1} + c_4 s_{1,2} - c_5 t_{1,1}] \quad (B4)$$

$$(c_1 + 1 - c_2) t_{0,1} \equiv (c_2 - c_1) s_{2,1} + (c_4 - c_3) s_{1,2} + Q t_{1,1} \quad (B5).$$

Suppose $t_{1,1}$ is not a free variable. Then $t_{1,1} \equiv 0$ which would force

$$\partial^{-1}W_2 = \partial^{-1}W_1.$$ Hence $W_2$ is terminal.

Now suppose $t_{1,1}$ is a free variable. Let $v_2$ denote the matrix $x$ in case we take $s_{2,1} = 0$, $t_{1,1} = 1$. By (A1) we get $t_{2,0} \equiv -c_4$. By (B3) we get $t_{0,2} \equiv c_4 (1 - c_2) q_1$.

Thus

$$3v_2 = \begin{bmatrix} 0 & 0 & 3c_4(1 - c_2) q_1 \\ 0 & 3 & 0 \\ -3c_4 & 0 & 0 \end{bmatrix} \pmod{I}.$$

We consider the order of the element $v_2 + W_2$ in $\partial^{-1}W_2/W_2$. This element has order 3 if and only if $3v_2 \in W_2$. Since $3v_2 \in \partial^{-1}W_0$ and $W_1 = \partial^{-1}W_0 \cap W_2$, we have $3v_2 \in W_2$ if and only if $3v_2 \in W_1$. Take $r_1 = -c_4$, $r_2 = 1$, and $r_3 = c_4 (1 - c_2) q_1$. 278
Recall $3v_2 \in W_1$ if and only if both $c_2 r_1 + c_4 r_2 \equiv 0$ and $r_3 \equiv c_1 r_1 + c_3 r_2$ hold. Observe that $c_2 r_1 + c_4 r_2 \equiv -c_2 c_4 + c_4 \equiv c_4 (1 - c_2)$. Recall we are assuming $c_2 \not\equiv 1$ and $c_4 \not\equiv 0$. Hence we have $c_2 r_1 + c_4 r_2 \not\equiv 0$. Hence $v_2 + W_2$ has order 9. Therefore $\partial^{-1} W_1/W_2 \cong \mathbb{Z}_9 \times \mathbb{Z}_3$. Recall we have $\partial^{-1} W_1/W_2 \cong \mathbb{Z}_3 \times \mathbb{Z}_3$. Therefore $\text{rank}(\partial^{-1} W_1/W_2) = \text{rank}(\partial^{-1} W_2/W_2)$ and $W_2$ is terminal.

9.2 Case 6.2

Suppose $c_2 = 1$. Thus (B1) becomes $q_1 s_{2,1} \equiv 0$. It also follows that $c_1 - c_2 + 1 \equiv c_1$ and so (B2) becomes $c_1 t_{0,1} \equiv q_2 s_{2,1}$. Consider case $c_1 = 0$ and $c_1 \in \{1, 2\}$ separately.

9.2.1 Case 6.2.1

Suppose $c_1 = 0$. Thus (B2) becomes $q_2 s_{2,1} \equiv 0$. Now consider cases $c_3 \in \{1, 2\}$ and $c_3 = 0$ separately.

Case 6.2.1.1

Suppose $c_3 \in \{1, 2\}$. One can show $q_1 \not\equiv 0$ thus $s_{2,1} \equiv 0$. It follows $s_{2,1} \equiv s_{1,2} \equiv t_{1,0} \equiv 0$ but $t_{0,1}$ is a free variable. Taking $t_{0,1} \equiv 1$, the matrix $x$ becomes

$$
\begin{bmatrix}
0 & 1 & 0 \\
0 & 0 & 0 \\
0 & 0 & 0
\end{bmatrix}
$$

We see that $\partial^{-1} W_1 = \langle \partial^{-1} W_0, v_1 \rangle$. Since $v_1 \not\in \partial^{-1} W_0$ and $3v_1 \in \partial^{-1} W_0$, we know $|\partial^{-1} W_1/\partial^{-1} W_0| = 3$. Since $|\partial^{-1} W_0| = 3^7$ then $|\partial^{-1} W_1| = 3^8$. Because $|W_1| = 3^5$ then $|\partial^{-1} W_1/W_1| = 3^3$. Also, $v_1, y_1, y_3 \in \partial^{-1} W_1$ but are not contained in $W_1$. Each
of $3v_1, 3y_1, 3y_3$ is contained in $W_1$. So $v_1 + W_1, y_1 + W_1, y_3 + W_1$ are elements of order 3 in the group $\partial^{-1}W_1/W_1$. These three elements form a generating set for the group $\partial^{-1}W_1/W_1$. Recall that $|\partial^{-1}W_1/W_1| = 3^3$, we obtain $\partial^{-1}W_1/W_1 \cong \mathbb{Z}_3 \times \mathbb{Z}_3 \times \mathbb{Z}_3$. Since rank($\partial^{-1}W_1/W_1$) = 3 and rank($\partial^{-1}W_0/W_1$) = 2, then rank($\partial^{-1}W_0/W_1$) $< \text{rank}(\partial^{-1}W_1/W_1)$. Hence $W_1$ is nonterminal and $W_1 \in \mathcal{L}_1$. A basis for $\partial^{-1}W_1/W_1$ is $v_1 + W_1, y_1 + W_1, y_3 + W_1$. A basis for $\partial^{-1}W_0/W_1$ is $y_1 + W_1, y_3 + W_1$.

We fix arbitrary values $c_5, c_6 \in \{0, 1, 2\}$. There are $3^2 = 9$ ways to choose these values. Let $m_3 = c_5y_1 + c_6y_3 + v_1$. Thus

$$m_3 = \begin{bmatrix} 0 & 1 & 3c_6 \\ 0 & 0 & 0 \\ 3c_5 & 0 & 0 \end{bmatrix}.$$

Let $W_2 =< W_1, m_3 >$. There are 9 subgroups of $W_2$ of this type. Since $m_3 \notin W_1$ and $3m_3 \in W_1$ then $|W_2/W_1| = 3$. Also, since $|W_1| = 3^5$ then $|W_2| = 3^6$. Recall $|\partial^{-1}W_1| = 3^8$, then $\partial^{-1}W_1/W_2 \cong \mathbb{Z}_3 \times \mathbb{Z}_3$ and rank($\partial^{-1}W_1/W_2$) = 2.

The subgroup $W_2$ is contained in the pattern subgroup

$$\begin{bmatrix} 2 & 2 & 1 \\ 1 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}.$$

The pullback $\partial^{-1}W_2$ is contained in the pattern subgroup

$$\begin{bmatrix} 2 & 2 & 2 \\ 2 & 1 & 1 \\ 1 & 1 & 0 \end{bmatrix} \cap \begin{bmatrix} 2 & 2 & 2 \\ 2 & 1 & 1 \\ 2 & 1 & 0 \end{bmatrix} = \begin{bmatrix} 2 & 2 \\ 2 & 1 \\ 1 & 1 \end{bmatrix}.$$
Let
\[
x = \begin{bmatrix}
3s_{0,0} + t_{0,0} & 3s_{0,1} + t_{0,1} & 3s_{0,2} + t_{0,2} \\
3s_{1,0} + t_{1,0} & 3s_{1,1} & 3s_{1,2} \\
3s_{2,0} & 3s_{2,1} & 0
\end{bmatrix}
\in \begin{bmatrix}
2 & 2 & 2 \\
2 & 1 & 1 \\
1 & 1 & 0
\end{bmatrix}.
\]

Thus
\[
\partial_1 x = \begin{bmatrix}
3s_{1,0} + t_{1,0} & 3s_{1,1} & 3s_{1,2} \\
3s_{2,0} & 3s_{2,1} & 0 \\
-3t_{1,0} & 0 & 0
\end{bmatrix}
\text{ and }
\partial_2 x = \begin{bmatrix}
3s_{0,1} + t_{0,1} & 3s_{0,2} + t_{0,2} & -3(t_{0,1} + t_{0,2}) \\
3s_{1,1} & 3s_{1,2} & 0 \\
3s_{2,1} & 0 & 0
\end{bmatrix}.
\]

The variables that are in play are those appearing in the matrix
\[
\begin{bmatrix}
0 & t_{0,1} & t_{0,2} \\
t_{1,0} & 0 & 3s_{1,2} \\
0 & 3s_{2,1} & 0
\end{bmatrix}.
\]

We want to identify a value \(a_1, a_2, a_3 \in \mathbb{Z}_9\) such that \(\partial_1 x \equiv a_1 m_1 + a_2 m_2 + a_3 m_3 \pmod{I}\). A formal expression for \(a_1 m_1 + a_2 m_2 + a_3 m_3\) is
\[
\begin{bmatrix}
1 & 0 & -3 \\
0 & 0 & 0 \\
0 & 0 & 0
\end{bmatrix} + a_2
\begin{bmatrix}
c_4 & 0 & 3(c_3 - c_4) \\
0 & 3 & 0 \\
-3c_4 & 0 & 0
\end{bmatrix} + a_3
\begin{bmatrix}
0 & 1 & 3c_6 \\
0 & 0 & 0 \\
3c_5 & 0 & 0
\end{bmatrix}.
\]
\[
\begin{bmatrix}
  a_1 + a_2 c_4 & a_3 & 3(-a_1 + a_2(c_3 - c_4) + a_3 c_6) \\
  0 & 3a_2 & 0 \\
  3(-a_2 c_4 + a_3 c_5) & 0 & 0
\end{bmatrix}.
\]

Comparing \((1, 0)\)-entries, we get \(0 \equiv 3s_{2,0}\) which gives no information.

Comparing \((0, 1)\)-entries, we get \(a_3 \equiv 3s_{1,1} \equiv 0\).

Comparing \((1, 1)\)-entries, we get \(a_2 \equiv s_{2,1}\).

Comparing \((0, 0)\)-entries, we get \(a_1 \equiv t_{1,0} - a_2 c_4\). Substituting \(a_2\) we obtain \(a_1 \equiv t_{1,0} - c_4 s_{2,1}\).

Comparing \((2, 0)\)-entries, we get \(-a_2 c_4 + a_3 c_5 \equiv -t_{1,0}\). Substituting \(a_2, a_3\) we obtain \(t_{1,0} \equiv c_4 s_{2,1}\).

Comparing \((0, 2)\)-entries, we get \(-a_1 + a_2(c_3 - c_4) + a_3 c_6 \equiv s_{1,2}\). Substituting \(a_1, a_2, a_3\) we obtain \(s_{1,2} \equiv -t_{1,0} + c_3 s_{2,1}\).

We see that \(\partial_1 x \in W_2\) if and only if

\(c_4 s_{2,1} \equiv t_{1,0} \quad (A1)\)

\(s_{1,2} \equiv -t_{1,0} + c_3 s_{2,1} \quad (A2)\).

We want to identify a value \(b_1, b_2, b_3 \in \mathbb{Z}_9\) such that \(\partial_2 x \equiv b_1 m_1 + b_2 m_2 + b_3 m_3\) (mod \(I\)). A formal expression for \(b_1 m_1 + b_2 m_2 + b_3 m_3\) is

\[
\begin{bmatrix}
  1 & 0 & -3 \\
  0 & 0 & 0
\end{bmatrix} + b_2
\begin{bmatrix}
  c_4 & 0 & 3(c_3 - c_4) \\
  0 & 3 & 0
\end{bmatrix} + b_3
\begin{bmatrix}
  0 & 1 & 3c_6 \\
  -3c_4 & 0 & 0 \\
  0 & 0 & 0 \\
  3c_5 & 0 & 0
\end{bmatrix}
\]
Comparing (1, 0)-entries, we get $0 \equiv 3s_{1,1}$ which gives no information.

Comparing (0, 1)-entries, we get $b_3 \equiv t_{0,2}$.

Comparing (1, 1)-entries, we get $b_2 \equiv s_{1,2}$.

Comparing (0, 0)-entries, we get $b_1 \equiv t_{0,1} - b_2c_4$. Substituting $b_2$ we obtain $b_1 \equiv t_{0,1} - c_4s_{1,2}$.

Comparing (2, 0)-entries, we get $-b_2c_4 + b_3c_5 \equiv s_{2,1}$. Substituting $b_2, b_3$ we obtain $s_{2,1} \equiv -c_4s_{1,2} + c_5t_{0,2}$.

Comparing (0, 2)-entries, we get $-b_1 + b_2(c_3 - c_4) + b_3c_6 \equiv -t_{0,1} - t_{0,2}$. Substituting $b_1, b_2, b_3$ we obtain $s_{1,2} \equiv (-c_3 - c_3c_6)t_{0,2}$.

We see that $\partial_2 x \in W_2$ if and only if

$$s_{2,1} \equiv -c_4s_{1,2} + c_5t_{0,2} \quad \text{(B1)}$$

$$s_{1,2} \equiv (-c_3 - c_3c_6)t_{0,2} \quad \text{(B2)}.$$

Substituting (B2) into (B1) we obtain $s_{2,1} \equiv (c_3c_4 + c_3c_4c_6 + c_5)t_{0,2}$. Substituting this into (A2) we obtain $s_{1,2} \equiv (-c_3c_4^2 - c_3c_4^2c_6 - c_4c_5 + c_4 + c_4c_6 + c_3c_5)t_{0,2}$.

For convenience let $s = -c_3c_4^2 - c_3c_4^2c_6 - c_4c_5 + c_4 + c_4c_6 + c_3c_5$. Thus $s_{1,2} \equiv st_{0,2}$.

Combining (A2) and (B2) we obtain $(s + c_3 + c_3c_6)t_{0,2} \equiv 0$. 

\[= \begin{bmatrix}
    b_1 + b_2c_4 & b_3 & 3(-b_1 + b_2(c_3 - c_4) + b_3c_6) \\
    0 & 3b_2 & 0 \\
    3(-b_2c_4 + b_3c_5) & 0 & 0
\end{bmatrix}.\]
Hence $x \in \partial^{-1}W_2$ if and only if

$$c_4s_{2,1} \equiv t_{1,0} \quad (A1)$$

$$s_{1,2} \equiv st_{0,2} \quad (A2)$$

$$s_{2,1} \equiv -c_4s_{1,2} + c_5t_{0,2} \quad (B1)$$

$$(s + c_3 + c_3c_6)t_{0,2} \equiv 0 \quad (B2).$$

If $s + c_3 + c_3c_6 \not\equiv 0$ then $t_{0,2} \equiv 0$. It follows that $s_{1,2} \equiv s_{2,1} \equiv 0$. Hence $\partial^{-1}W_2 = \partial^{-1}W_1$. Thus $\text{rank}(\partial^{-1}W_2/W_2) = \text{rank}(\partial^{-1}W_1/W_2)$ and $W_2$ is terminal and $W_2 \notin \hat{L}_2$.

If $s + c_3 + c_3c_6 \equiv 0$ then $t_{0,2}$ is a free variable. Taking $t_{0,2} \equiv 1$ and $t_{0,1} = 0$, the matrix $x$ becomes

$$v_2 = 
\begin{bmatrix}
0 & 0 & 1 \\
c_4(c_3c_4 + c_3c_4c_6 + c_5) & 0 & 3s \\
0 & 3(c_3c_4 + c_3c_4c_6 + c_5)
\end{bmatrix}.$$

We see that neither $v_2$ nor $3v_2$ is contained in $W_2$ and that $\partial^{-1}W_2 = \langle \partial^{-1}W_0, v_1, v_2 \rangle$. We know that $v_2 \in \partial^{-1}W_2$ and since $3v_2 \notin W_2$ then $v_2 + W_2$ is an element of the group $\partial^{-1}W_2/W_2$ whose order is larger than 3. Recall that $|\partial^{-1}W_2/W_2| = 3^3$ and that $\partial^{-1}W_1/W_2$ has rank 3. Since $\partial^{-1}W_2$ has an element of order larger than 3 then $\text{rank}(\partial^{-1}W_2/W_2)$ is not greater than 2 since $\partial^{-1}W_2/W_2 \cong \mathbb{Z}_9 \times \mathbb{Z}_3$ and therefore $W_2$ is terminal and $W_2 \notin \hat{L}_2$.  

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Case 6.2.1.2

Suppose $c_3 = 0$. Since $(c_3, c_4) \neq (0, 0)$ we get $c_4 \neq 0$. Thus $q_1 \equiv q_2 \equiv 0$ and $t_{0,1}$ and $s_{2,1}$ are free variables. Taking $s_{2,1} \equiv 1$ and $t_{0,1} = 0$, the matrix $x$ becomes

$$v_1 = \begin{bmatrix} 0 & 0 & 0 \\ c_4 & 0 & -3c_1 \\ 0 & 3 & 0 \end{bmatrix}.$$ 

Taking $t_{0,1} \equiv 1$ and $s_{2,1} = 0$, the matrix $x$ becomes

$$v_2 = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$ 

We see that this is the transpose of Case 4.2. Therefore we can conclude that $W_2$ is terminal and $W_2 \notin \hat{L}_2$.

9.2.2 Case 6.2.2

Suppose $c_1 \in \{1, 2\}$. Thus (B2) becomes $t_{0,1} \equiv c_1 q_2 s_{2,1}$.

Hence $x \in \partial^{-1}W_2$ if and only if

$$t_{1,0} \equiv c_4 s_{2,1} \quad (A1)$$

$$s_{1,2} \equiv (c_3 - c_4) s_{2,1} \quad (A2)$$

$$q_1 s_{2,1} \equiv 0 \quad (B1)$$

$$t_{0,2} \equiv c_1 q_2 s_{2,1} \quad (B2).$$

One can now argue that condition $q_1 \equiv 0$ holds if and only if $c_3 = 0$ and $c_4 \neq 0$. Thus $q_1 \equiv 0$ holds if and only if $(c_3, c_4)$ is $(0,1)$ or $(0,2)$. 

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Case 6.2.2.1
Suppose \((c_3, c_4)\) is neither \((0,1)\) nor \((0,2)\). Then \(q_1 \not\equiv 0\) and \(s_{2,1} \equiv 0\). Thus \(t_{1,0} \equiv s_{1,2} \equiv 0\). Hence \(\partial^{-1}W_2 = \partial^{-1}W_1\). Thus \(\text{rank}(\partial^{-1}W_2/W_2) = \text{rank}(\partial^{-1}W_1/W_2)\) and \(W_2\) is terminal and \(W_2 \notin \hat{\mathcal{L}}_2\).

Case 6.2.2.2
Suppose \((c_3, c_4)\) is either \((0,1)\) or \((0,2)\). Thus \(q_1 \equiv 0\) and so \((B1)\) is automatic so we may ignore it. Since \(c_3 = 0\) and \(c_4^2 \equiv 1\) then \(q_2 = -c_1\). We regard \(s_{2,1}\) as a free variable. Taking \(s_{2,1} \equiv 1\), the matrix \(x\) becomes

\[
v_1 = \begin{bmatrix}
0 & -1 & 0 \\
c_4 & 0 & -3c_4 \\
0 & 3 & 0
\end{bmatrix}.
\]

We see that \(\partial^{-1}W_1 = \langle \partial^{-1}W_0, v_1 \rangle\). Since \(v_1 \notin \partial^{-1}W_0\) and \(3v_1 \in \partial^{-1}W_0\), we know \(|\partial^{-1}W_1/\partial^{-1}W_0| = 3\). Since \(|\partial^{-1}W_0| = 3^7\) then \(|\partial^{-1}W_1| = 3^8\). Because \(|W_1| = 3^5\) then \(|\partial^{-1}W_1/W_1| = 3^3\). Also, \(v_1, y_2, y_3 \in \partial^{-1}W_1\) but are not contained in \(W_1\). Each of \(3v_1, 3y_2, 3y_3\) is contained in \(W_1\). So \(v_1 + W_1, y_2 + W_1, y_3 + W_1\) are elements of order 3 in the group \(\partial^{-1}W_1/W_1\). These three elements form a generating set for the group \(\partial^{-1}W_1/W_1\). Recall that \(|\partial^{-1}W_1/W_1| = 3^3\), we obtain \(\partial^{-1}W_1/W_1 \cong \mathbb{Z}_3 \times \mathbb{Z}_3 \times \mathbb{Z}_3\).

Since \(\text{rank}(\partial^{-1}W_1/W_1) = 3\) and \(\text{rank}(\partial^{-1}W_0/W_1) = 2\), then \(\text{rank}(\partial^{-1}W_0/W_1) < \text{rank}(\partial^{-1}W_1/W_1)\). Hence \(W_1\) is nonterminal and \(W_1 \in \hat{\mathcal{L}}_1\). A basis for \(\partial^{-1}W_1/W_1\) is \(v_1 + W_1, y_2 + W_1, y_3 + W_1\). A basis for \(\partial^{-1}W_0/W_1\) is \(y_2 + W_1, y_3 + W_1\).
We fix arbitrary values \( c_5, c_6 \in \{0, 1, 2\} \). There are \( 3^2 = 9 \) ways to choose these values. Let \( m_3 = c_5y_2 + c_6y_3 + v_1 \). Thus
\[
m_3 = \begin{bmatrix} 0 & -1 & 3c_6 \\ c_4 & 3c_5 & -3c_4 \\ 0 & 3 & 0 \end{bmatrix}.
\]

Let \( W_2 =< W_1, m_3 > \). There are 9 subgroups of \( W_2 \) of this type. Since \( m_3 \notin W_1 \) and \( 3m_3 \in W_1 \) then \( |W_2/W_1| = 3 \). Also, since \( |W_1| = 3^5 \) then \( |W_2| = 3^6 \).

Recall \( |\partial^{-1}W_1| = 3^8 \), then \( \partial^{-1}W_1/W_2 \cong \mathbb{Z}_3 \times \mathbb{Z}_3 \) and \( \text{rank}(\partial^{-1}W_1/W_2) = 2 \).

The subgroup \( W_2 \) is contained in the pattern subgroup
\[
\begin{bmatrix} 2 & 2 & 1 \\ 2 & 1 & 1 \\ 1 & 1 & 0 \end{bmatrix}.
\]

The pullback \( \partial^{-1}W_2 \) is contained in the pattern subgroup
\[
\begin{bmatrix} 2 & 2 & 2 \\ 2 & 2 & 1 \\ 2 & 1 & 1 \end{bmatrix} \cap \begin{bmatrix} 2 & 2 & 2 \\ 2 & 2 & 1 \\ 2 & 1 & 1 \end{bmatrix} = \begin{bmatrix} 2 & 2 & 2 \\ 2 & 2 & 1 \\ 2 & 1 & 1 \end{bmatrix}.
\]

Let
\[
x = \begin{bmatrix} 3s_{0,0} + t_{0,0} & 3s_{0,1} + t_{0,1} & 3s_{0,2} + t_{0,2} \\ 3s_{1,0} + t_{1,0} & 3s_{1,1} + t_{1,1} & 3s_{1,2} \\ 3s_{2,0} + t_{2,0} & 3s_{2,1} & 3s_{2,2} \end{bmatrix} \in \begin{bmatrix} 2 & 2 & 2 \\ 2 & 2 & 1 \\ 2 & 1 & 1 \end{bmatrix}.
\]
Thus

\[ \partial_1 x = \begin{bmatrix}
3s_{1,0} + t_{1,0} & 3s_{1,1} + t_{1,1} & 3s_{1,2} \\
3s_{2,0} + t_{2,0} & 3s_{2,1} & 3s_{2,2} \\
-3(t_{1,0} + t_{2,0}) & -3t_{1,1} & 0
\end{bmatrix} \]

and

\[ \partial_2 x = \begin{bmatrix}
3s_{0,1} + t_{0,1} & 3s_{0,2} + t_{0,2} & -3(t_{0,1} + t_{0,2}) \\
3s_{1,1} + t_{1,1} & 3s_{1,2} & -3t_{1,1} \\
3s_{2,1} & 3s_{2,2} & 0
\end{bmatrix}. \]

The variables that are in play are those appearing in the matrix

\[ \begin{bmatrix}
0 & t_{0,1} & t_{0,2} \\
t_{1,0} & t_{1,1} & 3s_{1,2} \\
t_{2,0} & 3s_{2,1} & 3s_{2,2}
\end{bmatrix}. \]

We want to identify a value \( a_1, a_2, a_3 \in \mathbb{Z}_9 \) such that \( \partial_1 x \equiv a_1 m_1 + a_2 m_2 + a_3 m_3 (\text{mod } I) \). A formal expression for \( a_1 m_1 + a_2 m_2 + a_3 m_3 \) is

\[ a_1 \begin{bmatrix} 1 & 0 & 3(c_1 - 1) \\ 0 & 0 & 0 \\ 3 & 0 & 0 \end{bmatrix} + a_2 \begin{bmatrix} c_4 & 0 & -3c_4 \\ 0 & 3 & 0 \\ -3c_4 & 0 & 0 \end{bmatrix} + a_3 \begin{bmatrix} 0 & -1 & 3c_6 \\ c_4 & 3c_5 & -3c_4 \\ 0 & 3 & 0 \end{bmatrix} \]

\[ = \begin{bmatrix}
a_1 + a_2 c_4 & -a_3 & 3(a_1(c_1 - 1) - a_2 c_4 + a_3 c_6) \\
a_3 c_4 & 3(a_2 + a_3 c_5) & -3a_3 c_4 \\
3(a_1 - a_2 c_4) & 3a_3 & 0
\end{bmatrix}. \]

Comparing (2,1)-entries, we get \( a_3 \equiv -t_{1,1} \).

Comparing (1,0)-entries, we get \( a_3 c_4 \equiv t_{2,0} \). Substituting \( a_3 \) we obtain \( t_{2,0} \equiv -c_4 t_{1,1} \).
Comparing $(1, 2)$-entries, we get $-a_3 c_4 \equiv s_{2, 2}$. Substituting $a_3$ we obtain $s_{2, 2} \equiv c_4 t_{1, 1}$.

Comparing $(0, 1)$-entries, we get $a_3 \equiv -t_{1, 1}$, which gives no new information.

Comparing $(1, 1)$-entries, we get $a_2 \equiv s_{2, 1} - a_3 c_5$. Substituting $a_3$ we obtain $a_2 \equiv s_{2, 1} + c_5 t_{1, 1}$.

Comparing $(0, 0)$-entries, we get $a_1 \equiv t_{1, 0} - a_2 c_4$. Substituting $a_2$ we obtain $a_1 \equiv t_{1, 0} - c_4 s_{2, 1} - c_5 c_3 t_{1, 1}$.

Comparing $(2, 0)$-entries, we get $a_1 - a_2 c_4 \equiv -t_{1, 0} - t_{2, 0}$. Substituting $a_2, a_3$ we obtain $t_{1, 0} \equiv c_4 s_{2, 1} + (-c_4 c_5 - c_4) t_{1, 1}$.

Comparing $(0, 2)$-entries, we get $a_1 (c_1 - 1) - a_2 c_4 + a_3 c_6 \equiv s_{1, 2}$. Substituting $a_1, a_2, a_3, t_{1, 0}$ we obtain $s_{1, 2} \equiv -c_4 s_{2, 1} + (c_1 c_4 c_5 - c_1 c_4 + c_4 c_5 + c_4 - c_6) t_{1, 1}$.

We see that $\partial_1 x \in W_2$ if and only if

$$t_{2, 0} \equiv -c_4 t_{1, 1} \quad (A1)$$

$$s_{2, 2} \equiv c_4 t_{1, 1} \quad (A2)$$

$$t_{1, 0} \equiv c_4 s_{2, 1} + (-c_4 c_5 - c_4) t_{1, 1} \quad (A3)$$

$$s_{1, 2} \equiv -c_4 s_{2, 1} + (c_1 c_4 c_5 - c_1 c_4 + c_4 c_5 + c_4 - c_6) t_{1, 1} \quad (A4).$$

We want to identify a value $b_1, b_2, b_3 \in \mathbb{Z}_9$ such that $\partial_2 x \equiv b_1 m_1 + b_2 m_2 + b_3 m_3 \pmod{I}$. A formal expression for $b_1 m_1 + b_2 m_2 + b_3 m_3$ is

$$b_1 \begin{bmatrix} 1 & 0 & 3(c_1 - 1) \\ 0 & 0 & 0 \\ 3 & 0 & 0 \end{bmatrix} + b_2 \begin{bmatrix} c_4 & 0 & -3c_4 \\ 0 & 3 & 0 \\ -3c_4 & 0 & 0 \end{bmatrix} + b_3 \begin{bmatrix} 0 & -1 & 3c_6 \\ c_4 & 0 & -3c_4 \\ 0 & 3 & 0 \end{bmatrix}$$

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\[
\begin{bmatrix}
    b_1 + b_2 c_4 & -b_3 & 3(b_1(c_1 - 1) - b_2 c_4 + b_3 c_6) \\
    b_3 c_4 & 3(b_2 + b_3 c_5) & -3b_3 c_4 \\
    3(b_1 - b_2 c_4) & 3b_3 & 0
\end{bmatrix}.
\]

Comparing (2,1)-entries, we get \( b_3 \equiv s_{2,2}. \)

Comparing (1,0)-entries, we get \( b_3 c_4 \equiv t_{1,1}. \) Substituting \( b_3 \) we obtain \( s_{2,2} \equiv c_4 t_{1,1}. \)

Comparing (1,2)-entries, we get \( b_3 c_4 \equiv t_{1,1} \) which gives us no new information.

Comparing (0,1)-entries, we get \( -b_3 \equiv t_{0,2}. \) Thus \( s_{2,2} \equiv -t_{0,2}. \)

Comparing (1,1)-entries, we get \( b_2 \equiv s_{1,2} - b_3 c_5. \) Thus we obtain \( b_2 \equiv s_{1,2} - c_4 c_5 t_{1,1}. \)

Comparing (0,0)-entries, we get \( b_1 \equiv t_{0,1} - b_2 c_4. \) Substituting \( b_2 \) we obtain \( b_1 \equiv t_{0,1} - c_4 s_{1,2} + c_5 t_{1,1}. \)

Comparing (0,2)-entries, we get \( b_1 c_1 - b_1 - b_2 c_4 + b_3 c_6 \equiv -t_{0,1} - t_{0,2}. \) Substituting \( b_1, b_2, b_3 \) we obtain \( t_{0,1} \equiv c_4 s_{1,2} + (-c_5 - c_1 c_4 c_6 + c_1 c_4) t_{1,1}. \)

Comparing (2,0)-entries, we get \( b_1 - b_2 c_4 \equiv s_{2,1}. \) Substituting \( b_1, b_2, t_{0,1} \) we obtain \( s_{1,2} \equiv -c_4 s_{2,1} + (c_1 c_6 - c_1) t_{1,1}. \)

We see that \( \partial_2 x \in W_2 \) if and only if

\begin{align*}
    s_{2,2} &\equiv c_4 t_{1,1} \quad \text{(B1)} \\
    s_{2,2} &\equiv -t_{0,2} \quad \text{(B2)} \\
    t_{0,1} &\equiv c_4 s_{1,2} + (-c_5 - c_1 c_4 c_6 + c_1 c_4) t_{1,1} \quad \text{(B3)} \\
    s_{1,2} &\equiv -c_4 s_{2,1} + (c_1 c_6 - c_1) t_{1,1} \quad \text{(B4)}.
\end{align*}

(B1) is redundant with (A2) so we may ignore it. Combining (A4) and (B4)
we obtain \((c_1 c_4 c_5 - c_1 c_4 + c_4 c_5 + c_4 - c_6 - c_1 c_6 + c_1) t_{1,1} \equiv 0\) which we will denote as (B4).

Substituting (A3) into (B4) we obtain \((c_1 c_4 c_5 - c_1 c_4 + c_4 c_5 + c_4 - c_6 - c_1 c_6 + c_1) t_{1,1} \equiv 0\).

For convenience let \(r = c_1 c_4 c_5 - c_1 c_4 + c_4 c_5 + c_4 - c_6 - c_1 c_6 + c_1\). Thus \(r t_{1,1} \equiv 0\).

Hence \(x \in \partial^{-1} W_2\) if and only if

\[
\begin{align*}
t_{2,0} &\equiv -c_4 t_{1,1} \quad \text{(A1)} \\
s_{2,2} &\equiv c_4 t_{1,1} \quad \text{(A2)} \\
t_{1,0} &\equiv c_4 s_{2,1} + (-c_4 c_5 - c_4) t_{1,1} \quad \text{(A3)} \\
s_{1,2} &\equiv -c_4 s_{2,1} + (c_1 c_4 c_5 - c_1 c_4 + c_4 c_5 + c_4 - c_6) t_{1,1} \quad \text{(A4)} \\
s_{2,2} &\equiv -t_{0,2} \quad \text{(B2)} \\
t_{0,1} &\equiv c_4 s_{1,2} + (-c_5 - c_1 c_4 c_6 + c_1 c_4) t_{1,1} \quad \text{(B3)} \\
r t_{1,1} &\equiv 0 \quad \text{(B4)}.
\end{align*}
\]

If \(r \neq 0\) then \(t_{1,1} \equiv 0\). This forces \(t_{2,0} \equiv s_{2,2} \equiv t_{0,2} \equiv 0\). (A4) (A5) and (B3) become \(t_{1,0} \equiv c_4 s_{2,1}, s_{1,2} \equiv -c_4 s_{2,1},\) and \(t_{0,1} \equiv -s_{2,1}\). These are the same congruences as Case 6.2.2. Hence \(\partial^{-1} W_2 = \partial^{-1} W_1\). Thus \(\text{rank}(\partial^{-1} W_2/W_2) = \text{rank}(\partial^{-1} W_1/W_2)\) and \(W_2\) is terminal and \(W_2 \notin \hat{L}_2\).

Suppose \(r \equiv 0\). Then \(t_{1,1}\) is a free variable. There are 15 possibilities for \((c_1, c_4, c_5, c_6)\). Taking \(t_{1,1} \equiv 1\) and \(s_{2,1} = 0\), the matrix \(x\) becomes

\[
v_2 = \begin{bmatrix} 0 & -c_5 - c_1 c_4 c_6 + c_1 c_4 & -c_4 \\ c_4 & 1 & 3(c_1 c_4 c_5 - c_1 c_4 + c_4 c_5 + c_4 - c_6) \\ -c_4 & 0 & 3 c_4 \end{bmatrix}.
\]

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We see that neither $v_2$ nor $3v_2$ is contained in $W_2$ and that
\[
\partial^{-1}W_2 = \langle \partial^{-1}W_0, v_1, v_2 \rangle.
\]
We know that $v_2 \in \partial^{-1}W_2$ and since $3v_2 \not\in W_2$ then
$v_2 + W_2$ is an element of the group $\partial^{-1}W_2/W_2$ whose order is larger than 3. Recall
that $|\partial^{-1}W_2/W_2| = 3^3$ and that $\partial^{-1}W_1/W_2$ has rank 3. Since $\partial^{-1}W_2$ has an element
of order larger than 3 then $\operatorname{rank}(\partial^{-1}W_2/W_2)$ is not greater than 2 since
$\partial^{-1}W_2/W_2 \cong \mathbb{Z}_9 \times \mathbb{Z}_3$ and therefore $W_2$ is terminal and $W_2 \not\in \hat{L}_2$.

In Case 6 we found 64 subgroups of $W_1 \in \mathcal{L}_1$, all of which satisfy $|W_1| = 3^5$.
Exactly 32 of these 64 members of $\mathcal{L}_1$ are nonterminal. We found that 30 of these
32 nonterminal members of $\mathcal{L}_1$ are each contained in 9 members of $\mathcal{L}_2$ and 2 of the 32
nonterminal members are each contained in 117 members of $\mathcal{L}_2$. Thus we found 504
subgroups $W_2 \in \mathcal{L}_2$, 342 of which satisfy $|W_2| = 3^6$ and 162 of which satisfy $|W_2| = 3^7$.
Every member of $\mathcal{L}_2$ is terminal. So in Case 6 we found a total of $64 + 504 = 568$
subgroups.
CHAPTER X

CASE 7

We fix arbitrary values $c_1, c_2, c_3 \in \{1, 2\}$. Let $m_1 = y_1 + c_1 y_4$, $m_2 = y_2 + c_2 y_4$, and $m_3 = y_3 + c_3 y_4$. Thus

$$m_1 = \begin{bmatrix} c_1 & 0 & -3c_1 \\ 0 & 0 & 0 \\ 3(1 - c_1) & 0 & 0 \end{bmatrix}, \quad m_2 = \begin{bmatrix} c_2 & 0 & -3c_2 \\ 0 & 3 & 0 \\ -3c_2 & 0 & 0 \end{bmatrix},$$

and $m_3 = \begin{bmatrix} c_3 & 0 & 3(1 - c_3) \\ 0 & 0 & 0 \\ -3c_3 & 0 & 0 \end{bmatrix}$.

Let $W_1 = < W_0, m_1, m_2, m_3 > \in \mathcal{L}_1$. The number of subgroups of this type is $2^3 = 8$. Note that $|W_1| = 3^6$ and $\partial^{-1}W_0/W_1 \cong \mathbb{Z}_3$ and rank $(\partial^{-1}W_0/W_1) = 1$.

We now calculate the pullback $\partial^{-1}W_1$. The subgroup $W_1$ is contained in the pattern subgroup

$$\begin{bmatrix} 2 & 1 & 1 \\ 1 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}.$$
Thus the pullback $\partial^{-1}W_1$ is contained in the pattern subgroup

$$\begin{bmatrix} 2 & 2 & 2 \\ 2 & 1 & 1 \\ 1 & 1 & 0 \end{bmatrix} \bigcap \begin{bmatrix} 2 & 2 & 1 \\ 2 & 1 & 1 \\ 1 & 1 & 0 \end{bmatrix} = \begin{bmatrix} 2 & 2 & 1 \\ 2 & 1 & 1 \\ 1 & 1 & 0 \end{bmatrix}.$$  

Let

$$x = \begin{bmatrix} 3s_{0,0} + t_{0,0} & 3s_{0,1} + t_{0,1} & 3s_{0,2} \\ 3s_{1,0} + t_{1,0} & 3s_{1,1} & 3s_{1,2} \\ 3s_{2,0} & 3s_{2,1} & 0 \end{bmatrix} \in \begin{bmatrix} 2 & 2 & 1 \\ 2 & 1 & 1 \\ 1 & 1 & 0 \end{bmatrix}.$$  

Thus

$$\partial_1 x = \begin{bmatrix} 3s_{1,0} + t_{1,0} & 3s_{1,1} & 3s_{1,2} \\ 3s_{2,0} & 3s_{2,1} & 0 \\ -3t_{1,0} & 0 & 0 \end{bmatrix} \quad \text{and} \quad \partial_2 x = \begin{bmatrix} 3s_{0,1} + t_{0,1} & 3s_{0,2} & -3t_{0,1} \\ 3s_{1,1} & 3s_{1,2} & 0 \\ 3s_{2,1} & 0 & 0 \end{bmatrix}.$$  

The variables that are in play are those appearing in the matrix

$$\begin{bmatrix} 0 & t_{0,1} & 0 \\ t_{1,0} & 0 & 3s_{1,2} \\ 0 & 3s_{2,1} & 0 \end{bmatrix}.$$
We want to identify a value \( a_1, a_2, a_3 \in \mathbb{Z}_9 \) such that \( \partial_1 x \equiv a_1 m_1 + a_2 m_2 + a_3 m_3 (\text{mod } I) \). A formal expression for \( a_1 m_1 + a_2 m_2 + a_3 m_3 \) is

\[
\begin{bmatrix}
  c_1 & 0 & -3c_1 \\
  0 & 0 & 0 \\
  3(1 - c_1) & 0 & 0
\end{bmatrix}
+ \begin{bmatrix}
  c_2 & 0 & -3c_2 \\
  0 & 3 & 0 \\
  -3c_2 & 0 & 0
\end{bmatrix}
+ \begin{bmatrix}
  c_3 & 0 & 3(1 - c_3) \\
  0 & 0 & 0 \\
  -3c_3 & 0 & 0
\end{bmatrix}
\]

\[
= \begin{bmatrix}
  a_1 c_1 + a_2 c_2 + a_3 c_3 & 0 & 3(-a_1 c_1 - a_2 c_2 + a_3 (1 - c_3)) \\
  0 & 3a_2 & 0 \\
  3(a_1 (1 - c_1) - a_2 c_2 - a_3 c_3) & 0 & 0
\end{bmatrix}.
\]

Comparing \((0,0)\)-entries, we get \( t_{1,0} \equiv c_1 a_1 + c_2 a_2 + c_3 a_3 \).

Comparing \((2,0)\)-entries, we get \( t_{1,0} \equiv -a_1 + c_1 a_1 + c_2 a_2 + c_3 a_3 \). Replacing \( c_1 a_1 + c_2 a_2 + c_3 a_3 \equiv t_{1,0} \), we obtain \( a_1 \equiv 0 \).

Comparing \((0,2)\)-entries, we get \( s_{1,2} \equiv -c_1 a_1 - c_2 a_2 - c_3 a_3 + a_3 \). Replacing \(-c_1 a_1 - c_2 a_2 - c_3 a_3 \) with \(-t_{1,0} \), we obtain \( s_{1,2} \equiv -t_{1,0} + a_3 \). Hence \( a_3 \equiv s_{1,2} + t_{1,0} \).

Comparing \((1,1)\)-entries, we get \( a_2 \equiv s_{2,1} \).

Comparing \((0,1)\)-entries, we get \( 3s_{1,1} \equiv 0 \) which gives no information.

Comparing \((1,0)\)-entries, we get \( 3s_{2,0} \equiv 0 \) which gives no information.

By substitution, \( t_{1,0} \) from the \((0,0)\) entry becomes \( t_{1,0} \equiv c_2 s_{2,1} + c_3 s_{1,2} + c_3 t_{1,0} \). This can be rewritten as \( s_{1,2} \equiv -c_2 c_3 s_{2,1} + c_3 (1 - c_3) t_{1,0} \).

We see that \( \partial_1 x \in W_1 \) if and only if

\[
s_{1,2} \equiv -c_2 c_3 s_{2,1} + c_3 (1 - c_3) t_{1,0} \quad (A1).
\]

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We want to identify a value $b_1, b_2, b_3 \in \mathbb{Z}_9$ such that $\partial^2 x \equiv b_1m_1 + b_2m_2 + b_3m_3 \pmod{I}$. A formal expression for $b_1m_1 + b_2m_2 + b_3m_3$ is

$$b_1 \begin{bmatrix} c_1 & 0 & -3c_1 \\ 0 & 0 & 0 \\ 3(1-c_1) & 0 & 0 \end{bmatrix} + b_2 \begin{bmatrix} c_2 & 0 & -3c_2 \\ 0 & 3 & 0 \\ -3c_2 & 0 & 0 \end{bmatrix} + b_3 \begin{bmatrix} c_3 & 0 & 3(1-c_3) \\ 0 & 0 & 0 \\ -3c_3 & 0 & 0 \end{bmatrix} = \begin{bmatrix} b_1c_1 + b_2c_2 + b_3c_3 & 0 & 3(b_1c_1 - b_2c_2 + b_3(1-c_3)) \\ 0 & 3b_2 & 0 \\ 3(b_1(1-c_1) - b_2c_2 - b_3c_3) & 0 & 0 \end{bmatrix}.$$ 

Comparing $(0,0)$-entries, we get $t_{0,1} \equiv c_1b_1 + c_2b_2 + c_3b_3$.

Comparing $(0,2)$-entries, we get $t_{0,1} \equiv b_1 - c_1b_1 - c_2b_2 - c_3b_3 - b_3$. Substituting $t_{0,1}$ for $c_1b_1 + c_2b_2 + c_3b_3$ we obtain $b_3 \equiv 0$.

Comparing $(2,0)$-entries, we get $s_{2,1} \equiv b_1 - c_1b_1 - c_2b_2 - c_3b_3$. Replacing $-c_1b_1 - c_2b_2 - c_3b_3$ with $-t_{0,1}$ we obtain $b_1 \equiv s_{2,1} + t_{0,1}$.

Comparing $(1,1)$-entries, we get $b_2 \equiv s_{1,2}$.

Comparing $(0,1)$-entries, we get $3s_{0,2} \equiv 0$ which gives no information.

Comparing $(1,0)$-entries, we get $3s_{1,1} \equiv 0$ which gives no information.

By substitution, $t_{0,1}$ from the $(0,0)$ entry becomes $t_{0,1} \equiv c_1s_{2,1} + c_1t_{0,1} + c_2s_{1,2}$. This can be rewritten as $s_{1,2} \equiv -c_1c_2s_{2,1} + c_2(1 - c_1)t_{0,1}$.

We see that $\partial^2 x \in W_1$ if and only if

$$s_{1,2} \equiv -c_1c_2s_{2,1} + c_2(1 - c_1)t_{0,1} \quad \text{(B1)}.$$
Hence \( x \in \partial^{-1}W_1 \) if and only if

\[ s_{1,2} \equiv -c_2c_3s_{2,1} + c_3(1 - c_3)t_{1,0} \quad \text{(A1)} \]
\[ s_{1,2} \equiv -c_1c_2s_{2,1} + c_2(1 - c_1)t_{0,1} \quad \text{(B1)}. \]

We now consider four separate cases: Case 7.1 \((c_1, c_3) = (1, 2)\), Case 7.2 \((c_1, c_3) = (2, 1)\), Case 7.3 \((c_1, c_3) = (2, 2)\), and Case 7.4 \((c_1, c_3) = (1, 1)\).

10.1 Case 7.1

Suppose \((c_1, c_3) = (1, 2)\). Then

\[
\begin{bmatrix}
1 & 0 & -3 \\
0 & 0 & 0 \\
0 & 0 & 0
\end{bmatrix}, \quad
\begin{bmatrix}
c_2 & 0 & -3c_2 \\
0 & 3 & 0 \\
-3c_2 & 0 & 0
\end{bmatrix}, \quad \text{and } m_3 = \begin{bmatrix} 0 & 0 & 0 \end{bmatrix}.
\]

Hence \( x \in \partial^{-1}W_1 \) if and only if

\[ s_{1,2} \equiv c_2s_{2,1} + t_{1,0} \quad \text{(A1)} \]
\[ s_{1,2} \equiv -c_2s_{2,1} \quad \text{(B1)}. \]

We can rewrite the (A1) as \(-c_2s_{2,1} \equiv c_2s_{2,1} + t_{1,0} \) which is equivalent to \( c_2s_{2,1} \equiv t_{1,0} \) which implies that \( s_{2,1} \equiv c_2t_{1,0} \). Now (B1) becomes \( s_{1,2} \equiv -t_{1,0} \).

Hence \( x \in \partial^{-1}W_1 \) if and only if

\[ s_{2,1} \equiv c_2t_{1,0} \quad \text{(A1)} \]
\[ s_{1,2} \equiv -t_{1,0} \quad \text{(B1)}. \]
We regard $t_{1,0}$ and $t_{0,1}$ as free variables. Taking $t_{1,0} \equiv 1$ and $t_{0,1} = 0$, the matrix $x$ becomes
\[
v_1 = \begin{bmatrix}
0 & 0 & 0 \\
1 & 0 & -3 \\
0 & 3c_2 & 0
\end{bmatrix}.
\]

Taking $t_{0,1} \equiv 1$ and $t_{1,0} = 0$, the matrix $x$ becomes
\[
v_2 = \begin{bmatrix}
0 & 1 & 0 \\
0 & 0 & 0 \\
0 & 0 & 0
\end{bmatrix}.
\]

We see that $\partial^{-1}W_1 = < \partial^{-1}W_0, v_1, v_2 >$. Since $v_1 \notin \partial^{-1}W_0$, $v_2 \notin < \partial^{-1}W_0, v_1 >$, $3v_1 \in \partial^{-1}W_0$, and $3v_2 \in < \partial^{-1}W_0, v_1 >$ we know $|\partial^{-1}W_1/\partial^{-1}W_0| = 3^2$. Since $|\partial^{-1}W_0| = 3^7$ then $|\partial^{-1}W_1| = 3^9$. Because $|W_1| = 3^6$ then $|\partial^{-1}W_1/W_1| = 3^3$. Also, $v_1, v_2, y_2 \in \partial^{-1}W_1$ but are not contained in $W_1$. Each of $3v_1, 3v_2, 3y_2$ is contained in $W_1$. So $v_1 + W_1, v_2 + W_1, y_2 + W_1$ are elements of order 3 in the group $\partial^{-1}W_1/W_1$. These three elements form a generating set for the group $\partial^{-1}W_1/W_1$. Recall that $|\partial^{-1}W_1/W_1| = 3^3$, we obtain $\partial^{-1}W_1/W_1 \cong \mathbb{Z}_3 \times \mathbb{Z}_3 \times \mathbb{Z}_3$.

Since rank($\partial^{-1}W_1/W_1$) = 3 and rank($\partial^{-1}W_0/W_1$) = 1, then rank($\partial^{-1}W_0/W_1$) < rank($\partial^{-1}W_1/W_1$). Hence $W_1$ is nonterminal and $W_1 \in \mathcal{L}_1$. A basis for $\partial^{-1}W_1/W_1$ is $v_1 + W_1, v_2 + W_1, y_2 + W_1$. A basis for $\partial^{-1}W_0/W_1$ is $v_1 + W_1, v_2 + W_1, y_2 + W_1$.

The subgroups $W_2$ belonging to $\mathcal{L}(W_1)$ correspond to the nontrivial proper subspaces $W_2/W_1$ of $\partial^{-1}W_1/W_1$ for which the intersection $W_2/W_1 \cap \partial^{-1}W_0/W_1$ is
trivial. Since $\partial^{-1}W_1/W_1$ has dimension 3 while its subspace $\partial^{-1}W_0/W_1$ has dimension 1, every such subspace $W_2/W_1$ has dimension either 1 or 2.

To help us define the subgroups $W_2$ belonging to $\mathcal{L}_2(W_1)$, it will be convenient to identify each element of the vector space $\partial^{-1}W_1/W_1$ with its coordinate vector with respect to the ordered basis $y_2 + W_1, v_1 + W_1, v_2 + W_1$. In this way, we identify $\partial^{-1}W_1/W_1$ with the vector space $\mathbb{Z}_3 \times \mathbb{Z}_3 \times \mathbb{Z}_3$ consisting of row vectors. Under this identification, the elements $y_2 + W_1, v_1 + W_1, v_2 + W_1$ are associated with the so-called standard basis vectors $[1,0,0], [0,1,0], [0,0,1]$, in $\mathbb{Z}_3 \times \mathbb{Z}_3 \times \mathbb{Z}_3$.

The subgroups $W_2$ belonging to $\mathcal{L}(W_1)$ are in one-to-one correspondence with the nontrivial proper subgroup $W_2/W_1$ of $\partial^{-1}W_1/W_1$ for which the intersection $W_2/W_1 \cap \partial^{-1}W_0/W_1$ is trivial. Note that $\partial^{-1}W_0/W_1$ is the 2-dimensional subspace generated by the element $y_2 + W_1$. Under our identification, each such subspace $W_2/W_1$ is associated with a subspace $S$ of $\mathbb{Z}_3 \times \mathbb{Z}_3 \times \mathbb{Z}_3$ that does not contain the standard basis vector $[1,0,0]$. Let $m$ denote the unique matrix in reduced row-echelon form that is row equivalent to the matrix whose rows are the members of an arbitrarily-chosen basis of such a subspace $S$.

In Case 7.1.1 we consider the 1-dimensional subspaces $W_2/W_1$. There are three possible forms for the matrix $m$. The first form is

$$m = [0,0,1]$$

(1 possibility), which is considered in Case 7.1.1.1. The second form is

$$m = [0,1,c_4] \quad \text{for } c_4 \in \{0,1,2\}$$
(3 possibilities), which is considered in Case 7.1.1.2. The third form is

\[ m = [1, c_4, c_5] \text{ for } c_4, c_5 \in \{0, 1, 2\}, (c_4, c_5) \neq (0, 0) \]

(8 possibilities), which is considered in Case 7.1.1.3.

In Case 7.1.2 we consider the 2-dimensional subspaces \( W_2/W_1 \). There are three possible forms for the matrix \( m \). The first form is

\[
m = \begin{bmatrix}
0 & 1 & 0 \\
0 & 0 & 1
\end{bmatrix}
\]

(1 possibility), which is considered in Case 7.1.2.1. The second form is

\[
m = \begin{bmatrix}
1 & c_4 & 0 \\
0 & 0 & 1
\end{bmatrix}
\text{ for } c_4 \in \{1, 2\}
\]

(2 possibilities), which is considered in Case 7.1.2.2. The third form is

\[
m = \begin{bmatrix}
1 & 0 & c_4 \\
0 & 1 & c_5
\end{bmatrix}
\text{ for } c_4 \in \{1, 2\} \text{ and } c_5 \in \{0, 1, 2\}
\]

(6 possibilities), which is considered in Case 7.1.2.3.

10.1.1 Case 7.1.1

We consider the 1-dimensional subspaces \( W_2/W_1 \). Let \( d_1, d_2, d_3 \) be unspecified variables. Let \( m_4 = d_1 y_2 + d_2 v_1 + d_3 v_2 \). A formal expression for \( m_4 \) is

\[
m_4 = d_1 \begin{bmatrix}
0 & 0 & 0 \\
0 & 3 & 0 \\
0 & 0 & 0
\end{bmatrix} + d_2 \begin{bmatrix}
0 & 0 & 0 \\
1 & 0 & -3 \\
0 & 3c_2 & 0
\end{bmatrix} + d_3 \begin{bmatrix}
0 & 1 & 0 \\
0 & 0 & 0 \\
0 & 0 & 0
\end{bmatrix} = \begin{bmatrix}
0 & d_3 & 0 \\
d_2 & 3d_1 & -3d_2 \\
0 & 3c_2d_2 & 0
\end{bmatrix}.
\]

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Let $W_2 = \langle W_1, m_4 \rangle \in \mathcal{L}_2$. We now calculate the pullback $\partial^{-1}W_2$. The subgroup $W_2$ is contained in the pattern subgroup

$$\begin{bmatrix}
2 & 2 & 1 \\
2 & 1 & 1 \\
1 & 1 & 0 \\
\end{bmatrix}.$$ 

Thus the pullback $\partial^{-1}W_2$ is contained in the pattern subgroup

$$\begin{bmatrix}
2 & 2 & 2 \\
2 & 2 & 1 \\
2 & 1 & 1 \\
\end{bmatrix} \cap \begin{bmatrix}
2 & 2 & 2 \\
2 & 2 & 1 \\
2 & 1 & 1 \\
\end{bmatrix} = \begin{bmatrix}
2 & 2 & 2 \\
2 & 2 & 1 \\
2 & 1 & 1 \\
\end{bmatrix}.$$ 

Let

$$x = \begin{bmatrix}
3s_{0,0} + t_{0,0} & 3s_{0,1} + t_{0,1} & 3s_{0,2} + t_{0,2} \\
3s_{1,0} + t_{1,0} & 3s_{1,1} + t_{1,1} & 3s_{1,2} \\
3s_{2,0} + t_{2,0} & 3s_{2,1} & 3s_{2,2} \\
\end{bmatrix} \in \begin{bmatrix}
2 & 2 & 2 \\
2 & 2 & 1 \\
2 & 1 & 1 \\
\end{bmatrix}.$$ 

Thus

$$\partial_1 x = \begin{bmatrix}
3s_{1,0} + t_{1,0} & 3s_{1,1} + t_{1,1} & 3s_{1,2} \\
3s_{2,0} + t_{2,0} & 3s_{2,1} & 3s_{2,2} \\
-3(t_{1,0} + t_{2,0}) & -3t_{1,1} & 0 \\
\end{bmatrix} \quad \text{and} \quad \partial_2 x = \begin{bmatrix}
3s_{0,1} + t_{0,1} & 3s_{0,2} + t_{0,2} & -3(t_{0,1} + t_{0,2}) \\
3s_{1,1} + t_{1,1} & 3s_{1,2} & -3t_{1,1} \\
3s_{2,1} & 3s_{2,2} & 0 \\
\end{bmatrix}.$$
The variables that are in play are those appearing in the matrix

\[
\begin{bmatrix}
0 & t_{0,1} & t_{0,2} \\
t_{1,0} & t_{1,1} & 3s_{1,2} \\
t_{2,0} & 3s_{2,1} & 3s_{2,2}
\end{bmatrix}.
\]

We wish to identify values \(a_1, a_2, a_3, a_4 \in \mathbb{Z}_9\) such that \(\partial_1 x \equiv a_1 m_1 + a_2 m_2 + a_3 m_3 + a_4 m_4 \pmod{I}\). A formal expression for \(a_1 m_1 + a_2 m_2 + a_3 m_3 + a_4 m_4\) is

\[
a_1 \begin{bmatrix} 1 & 0 & -3 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} + a_2 \begin{bmatrix} c_2 & 0 & -3c_2 \\ 0 & 3 & 0 \\ -3c_2 & 0 & 0 \end{bmatrix} + a_3 \begin{bmatrix} 2 & 0 & -3 \\ 0 & 0 & 0 \\ 3 & 0 & 0 \end{bmatrix} + a_4 \begin{bmatrix} 0 & d_3 & 0 \\ d_2 & 3d_1 & -3d_2 \\ 0 & 3c_2d_2 & 0 \end{bmatrix}
\]

\[
= \begin{bmatrix}
a_1 + a_2c_2 - a_3 & a_4d_3 & -3(a_1 + a_2c_2 + a_3) \\
a_4d_2 & 3(a_2 + a_4d_1) & -3a_4d_2 \\
3(-a_2c_2 + a_3) & 3a_4c_2d_2 & 0
\end{bmatrix}.
\]

Comparing \((0,1)\)-entries, we get \(t_{1,1} \equiv a_4d_3\).

Comparing \((2,1)\)-entries, we get \(-t_{1,1} \equiv a_4c_2d_2\).

Comparing \((1,0)\)-entries, we get \(t_{2,0} \equiv a_4d_2\).

Comparing \((1,2)\)-entries, we get \(s_{2,2} \equiv -a_4d_2\). Substituting \(a_4d_2 \equiv t_{2,0}\) we obtain \(s_{2,2} \equiv -t_{2,0}\).

Comparing \((1,1)\)-entries, we get \(a_2 \equiv s_{2,1} - a_4d_1\).

Comparing \((2,0)\)-entries, we get \(a_3 \equiv -t_{1,0} - t_{2,0} + a_2c_2\). Substituting \(a_2\) we obtain \(a_3 \equiv -t_{1,0} - t_{2,0} + c_2s_{2,1} - c_2a_4d_1\).
Comparing \((0,2)\)-entries, we get \(a_1 \equiv -s_{1,2} - c_2a_2 - a_3\). Substituting \(a_2\) we obtain

\[ a_1 \equiv c_2s_{2,1} - c_2a_4d_1 + t_{1,0} - t_{2,0}. \]

Comparing \((0,0)\)-entries, we get \(t_{1,0} \equiv a_1 + c_2a_2 - a_3\). Substituting \(a_1, a_2,\) and \(a_3\) we obtain \(s_{1,2} \equiv c_2s_{2,1} - c_2a_4d_1 + t_{1,0} - t_{2,0}\).

We see that \(\partial_1 x \in W_1\) if and only if

\[
\begin{align*}
t_{1,1} &\equiv a_4d_3 \quad (A1) \\
-t_{1,1} &\equiv a_4c_2d_2 \quad (A2) \\
t_{2,0} &\equiv a_4d_2 \quad (A3) \\
s_{2,2} &\equiv -t_{2,0} \quad (A4) \\
s_{1,2} &\equiv c_2s_{2,1} - c_2d_4a_4 + t_{1,0} - t_{2,0} \quad (A5).
\end{align*}
\]

We wish to identify values \(b_1, b_2, b_3, b_4 \in Z_9\) such that \(\partial_2 x \equiv b_1m_1 + b_2m_2 + b_3m_3 + b_4m_4 \pmod{I}\). A formal expression for \(b_1m_1 + b_2m_2 + b_3m_3 + b_4m_4\) is

\[
\begin{bmatrix}
1 & 0 & -3 \\
0 & 0 & 0 \\
0 & 0 & 0
\end{bmatrix}
+ b_2
\begin{bmatrix}
c_2 & 0 & -3c_2 \\
0 & 3 & 0 \\
-3c_2 & 0 & 0
\end{bmatrix}
+ b_3
\begin{bmatrix}
2 & 0 & -3 \\
0 & 0 & 0 \\
3 & 0 & 0
\end{bmatrix}
+ b_4
\begin{bmatrix}
0 & d_3 & 0 \\
d_2 & 3d_1 & -3d_2 \\
0 & 3c_2d_2 & 0
\end{bmatrix}
= 
\begin{bmatrix}
b_1 + b_2c_2 - b_3 & b_4d_3 & -3(b_1 + b_2c_2 + b_3) \\
b_4d_2 & 3(b_2 + b_4d_1) & -3b_4d_2 \\
3(-b_2c_2 + b_3) & 3b_4c_2d_2 & 0
\end{bmatrix}.
\]

Comparing \((0,1)\)-entries, we get \(t_{0,2} \equiv b_4d_3\).

Comparing \((2,1)\)-entries, we get \(s_{2,2} \equiv b_4c_2d_2\).
Comparing (1, 0)-entries, we get $t_{1,1} \equiv b_4d_2$.

Comparing (1, 2)-entries, we get $t_{1,1} \equiv b_4d_2$ which gives us no new information.

Comparing (1, 1)-entries, we get $b_2 \equiv s_{1,2} - b_4d_1$.

Comparing (2, 0)-entries, we get $b_3 \equiv s_{2,1} + b_2c_2$. Substituting $b_2$ we obtain $b_3 \equiv s_{2,1} + c_2s_{1,2} - c_2b_4d_1$.

Comparing (0, 2)-entries, we get $b_1 \equiv t_{0,1} + t_{0,2} - b_2c_2 - b_3$. Substituting $b_2$ and $b_3$ we obtain $b_1 \equiv t_{0,1} + t_{0,2} - c_2b_4d_1 - s_{2,1}$.

Comparing (0, 0)-entries, we get $t_{0,1} \equiv b_1 + b_2c_2 - b_3$. Substituting $b_1$, $b_2$, and $b_3$ we obtain $s_{1,2} \equiv -c_2t_{0,2} + b_4d_1 - c_2s_{2,1}$.

We see that $\partial_2x \in W_1$ if and only if

\begin{align*}
t_{0,2} & \equiv b_4d_3 \quad \text{(B1)} \\
s_{2,2} & \equiv b_4c_2d_2 \quad \text{(B2)} \\
t_{1,1} & \equiv b_4d_2 \quad \text{(B3)} \\
s_{1,2} & \equiv -c_2t_{0,2} + b_4d_1 - c_2s_{2,1} \quad \text{(B4)}. \\
\end{align*}

Case 7.1.1.1

Let $m_4 = v_2$. Thus

$$m_4 = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$ 

Let $W_2 = \langle W_1, m_4 \rangle \in \mathcal{L}_2$. The number of subgroups $W_2$ of this type is 1. Since $m_4 \notin W_1$ and $3m_4 \in W_1$ we have $|W_2/W_1| = 3$. Since $|W_1| = 3^6$ it follows
that $|W_2| = 3^7$. Note that $\vartheta^{-1}W_1/W_2 \cong \mathbb{Z}_3 \times \mathbb{Z}_3$ and $\text{rank}(\vartheta^{-1}W_1/W_2) = 2$. We now calculate the pullback of $\vartheta^{-1}W_2$. We observed in case 7.4.1 that $\vartheta^{-1}W_2$ is contained in the pattern subgroup

$$
\begin{bmatrix}
2 & 2 & 2 \\
2 & 2 & 1 \\
2 & 1 & 1
\end{bmatrix}.
$$

Let

$$
x = \begin{bmatrix}
3s_{0,0} + t_{0,0} & 3s_{0,1} + t_{0,1} & 3s_{0,2} + t_{0,2} \\
3s_{1,0} + t_{1,0} & 3s_{1,1} + t_{1,1} & 3s_{1,2} \\
3s_{2,0} + t_{2,0} & 3s_{2,1} & 3s_{2,2}
\end{bmatrix} \in \begin{bmatrix}
2 & 2 & 2 \\
2 & 2 & 1 \\
2 & 1 & 1
\end{bmatrix}.
$$

The variables that are in play are those appearing in the matrix

$$
\begin{bmatrix}
0 & t_{0,1} & t_{0,2} \\
t_{1,0} & t_{1,1} & 3s_{1,2} \\
t_{2,0} & 3s_{2,1} & 3s_{2,2}
\end{bmatrix}.
$$

In the notation of Case 7.4.1, we are taking $d_1 = 0$, $d_2 = 0$, and $d_3 = 1$. We wish to identify values $a_1, a_2, a_3, a_4 \in \mathbb{Z}_9$ such that $\vartheta_1x \equiv a_1m_1 + a_2m_2 + a_3m_3 + a_4m_4$ (mod $I$).
We see that $\partial_1 x \in W_2$ if and only if

\begin{align*}
t_{1,1} &\equiv a_4 \quad (A1) \\
-t_{1,1} &\equiv 0 \quad (A2) \\
t_{2,0} &\equiv 0 \quad (A3) \\
s_{2,2} &\equiv -t_{2,0} \quad (A4) \\
s_{1,2} &\equiv c_2 s_{2,1} + t_{1,0} \quad (A5).
\end{align*}

We wish to identify values $b_1, b_2, b_3, b_4 \in \mathbb{Z}_9$ such that $\partial_1 x \equiv b_1 m_1 + b_2 m_2 + b_3 m_3 + b_4 m_4 \ (\text{mod } I)$. We see that $\partial_2 x \in W_2$ if and only if

\begin{align*}
t_{0,2} &\equiv b_4 \quad (B1) \\
s_{2,2} &\equiv 0 \quad (B2) \\
t_{1,1} &\equiv 0 \quad (B3) \\
s_{1,2} &\equiv -c_2 t_{0,2} - c_2 s_{2,1} \quad (B4).
\end{align*}

Combining (A5) and (B4) we obtain $s_{2,1} \equiv c_2 t_{1,0} + t_{0,2}$ which we will denote as our new (B4). Substituting (B4) into (A5) we obtain $s_{1,2} \equiv c_2 t_{0,2} - t_{1,0}$.
Hence $x \in \partial^{-1}W_2$ if and only if

$$
\begin{align*}
t_{1,1} &\equiv 0 \\
t_{2,0} &\equiv 0 \\
s_{2,2} &\equiv 0 \\
s_{1,2} &\equiv c_2 t_{0,2} - t_{1,0} \\
s_{2,1} &\equiv c_2 t_{1,0} + t_{0,2}.
\end{align*}
$$

We regard $t_{1,0}$, $t_{0,1}$, and $t_{0,2}$ as free variables. Taking $t_{1,0} \equiv 1$, $t_{0,1} = 0$, and $t_{0,2} = 0$, the matrix $x$ becomes

$$
v_1 = \begin{bmatrix}
0 & 0 & 0 \\
1 & 0 & -3 \\
0 & 3c_2 & 0
\end{bmatrix}.
$$

Taking $t_{0,1} \equiv 1$, $t_{0,0} = 0$, and $t_{0,2} = 0$, the matrix $x$ becomes

$$
v_2 = \begin{bmatrix}
0 & 1 & 0 \\
0 & 0 & 0 \\
0 & 0 & 0
\end{bmatrix}.
$$

Taking $t_{0,2} \equiv 1$, $t_{0,1} = 0$, and $t_{1,0} = 0$, the matrix $x$ becomes

$$
v_3 = \begin{bmatrix}
0 & 0 & 1 \\
0 & 0 & 3c_2 \\
0 & 3 & 0
\end{bmatrix}.
$$

We see that neither $v_3$ nor $3v_3$ is contained in $W_2$ and that $\partial^{-1}W_2 =< \partial^{-1}W_0, v_1, v_2, v_3 >$. We know that $v_3 \in \partial^{-1}W_2$ and since $3v_3 \not\in W_2$ then $v_3 + W_2$
is an element of the group $\partial^{-1}W_2/W_2$ whose order is larger than 3. Recall that $|\partial^{-1}W_2/W_2| = 3^3$ and that $\partial^{-1}W_1/W_2$ has rank 2. We obtain $\partial^{-1}W_2/W_2 \cong \mathbb{Z}_9 \times \mathbb{Z}_3$.

Since $\text{rank}(\partial^{-1}W_2/W_2) = \text{rank}(\partial^{-1}W_1/W_2)$. Hence $W_2$ is terminal and $W_2 \not\in \hat{L}_2$.

Case 7.1.1.2

We fix arbitrary value $c_4 \in \{0, 1, 2\}$. There is 3 ways to choose the value $c_4$. Let $m_4 = v_1 + c_4 v_2$. Thus

$$m_4 = \begin{bmatrix} 0 & c_4 & 0 \\ 1 & 0 & -3 \\ 0 & 3c_2 & 0 \end{bmatrix}.$$ 

Let $W_2 = \langle W_1, m_4 \rangle \in \mathcal{L}_2$. The number of subgroups $W_2$ of this type is 3. Since $m_4 \notin W_1$ and $3m_4 \in W_1$ we have $|W_2/W_1| = 3$. Since $|W_1| = 3^6$ it follows that $|W_2| = 3^7$. Note that $\partial^{-1}W_1/W_2 \cong \mathbb{Z}_3 \times \mathbb{Z}_3$ and $\text{rank}(\partial^{-1}W_1/W_2) = 2$. We now calculate the pullback of $\partial^{-1}W_2$. We observed in case 7.4.1 that $\partial^{-1}W_2$ is contained in the pattern subgroup

$$\begin{bmatrix} 2 & 2 & 2 \\ 2 & 2 & 1 \\ 2 & 1 & 1 \end{bmatrix}.$$ 

Let

$$x = \begin{bmatrix} 3s_{0,0} + t_{0,0} & 3s_{0,1} + t_{0,1} & 3s_{0,2} + t_{0,2} \\ 3s_{1,0} + t_{1,0} & 3s_{1,1} + t_{1,1} & 3s_{1,2} \\ 3s_{2,0} + t_{2,0} & 3s_{2,1} & 3s_{2,2} \end{bmatrix} \in \begin{bmatrix} 2 & 2 & 2 \\ 2 & 2 & 1 \\ 2 & 1 & 1 \end{bmatrix}.$$
The variables that are in play are those appearing in the matrix

\[
\begin{bmatrix}
0 & t_{0,1} & t_{0,2} \\
t_{1,0} & t_{1,1} & 3s_{1,2} \\
t_{2,0} & 3s_{2,1} & 3s_{2,2}
\end{bmatrix}.
\]

In the notation of Case 7.4.1, we are taking \(d_1 = 0\), \(d_2 = 1\), and \(d_3 = c_4\). We wish to identify values \(a_1, a_2, a_3, a_4 \in \mathbb{Z}_9\) such that \(\partial_1 x \equiv a_1 m_1 + a_2 m_2 + a_3 m_3 + a_4 m_4 \pmod{I}\). We see that \(\partial_1 x \in W_2\) if and only if

\[
t_{1,1} \equiv a_4 c_4 \quad (A1)
\]

\[-t_{1,1} \equiv a_4 c_2 \quad (A2)
\]

\[t_{2,0} \equiv a_4 \quad (A3)
\]

\[s_{2,2} \equiv -t_{2,0} \quad (A4)
\]

\[s_{1,2} \equiv c_2 s_{2,1} + t_{1,0} - t_{2,0} \quad (A5).
\]

We wish to identify values \(b_1, b_2, b_3, b_4 \in \mathbb{Z}_9\) such that \(\partial_2 x \equiv b_1 m_1 + b_2 m_2 + b_3 m_3 + b_4 m_4 \pmod{I}\). We see that \(\partial_2 x \in W_2\) if and only if

\[
t_{0,2} \equiv b_4 c_4 \quad (B1)
\]

\[s_{2,2} \equiv b_4 c_2 \quad (B2)
\]

\[t_{1,1} \equiv b_4 \quad (B3)
\]

\[s_{1,2} \equiv -c_2 t_{0,2} - c_2 s_{2,1} \quad (B4).
\]

Substituting \(a_4\) into (A1) and (A2) we obtain \(t_{1,1} \equiv c_4 t_{2,0}\) and \(-t_{1,1} \equiv c_2 t_{2,0}\). Combining theses congruences we get \((c_4 + c_3)t_{2,0} \equiv 0\) which will we denote as our...
new (A2). Substituting $b_4$ into (B1) and (B2) we obtain $t_{0,2} \equiv c_4t_{1,1}$ and $s_{2,2} \equiv c_2t_{1,1}$.

Substituting $t_{1,1} \equiv c_4t_{2,0}$ into (B1) we obtain $t_{0,2} \equiv c_4^2t_{2,0}$. Combining (A5) and (B4) we obtain $s_{2,1} \equiv c_2t_{1,0} - (c_4^2 - c_2)t_{2,0}$ which will we denote as our new (B4).

Substituting (B4) into (A5) we obtain $s_{1,2} \equiv -t_{1,0} + (c_2 + 1)t_{2,0}$.

Hence $x \in \partial^{-1}W_2$ if and only if

$$t_{1,1} \equiv c_4t_{2,0} \quad (A1)$$

$$(c_4 + c_2)t_{2,0} \equiv 0 \quad (A2)$$

$$s_{2,2} \equiv -t_{2,0} \quad (A4)$$

$$s_{1,2} \equiv -t_{1,0} + (c_2 + 1)t_{2,0} \quad (A5)$$

$$t_{0,2} \equiv c_4^2t_{2,0} \quad (B1)$$

$$s_{2,2} \equiv c_2t_{1,1} \quad (B2)$$

$$s_{2,1} \equiv c_2t_{1,0} - (c_4^2 - c_2)t_{2,0} \quad (B4).$$

If $c_4 + c_2 \neq 0$ then $t_{2,0} \equiv 0$. It then follows that $s_{2,2} \equiv t_{1,1} \equiv t_{0,2} \equiv t_{2,0} \equiv 0$. Then (A5) becomes $s_{1,2} \equiv c_2s_{2,1} + t_{1,0}$ and (B4) becomes $s_{2,1} \equiv c_2t_{1,0}$. These are the same congruences as those in Case 7.1 therefore $\partial^{-1}W_2 = \partial^{-1}W_1$. Thus $\text{rank}(\partial^{-1}W_2/W_2) = \text{rank}(\partial^{-1}W_1/W_2)$, $W_2$ is terminal, and $W_2 \not\in \mathcal{L}_2$.

Therefore we will assume that $c_4 + c_2 \equiv 0$. Then $t_{2,0}$ is a free variable and $c_4 \equiv -c_2$. Hence $c_4^2 \equiv 1$.  

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Hence \( x \in \partial^{-1}W_2 \) if and only if

\[
\begin{align*}
t_{1,1} & \equiv -c_2 t_{2,0} \\
s_{2,2} & \equiv -t_{2,0} \\
s_{1,2} & \equiv -t_{1,0} + (c_2 + 1)t_{2,0} \\
t_{0,2} & \equiv t_{2,0} \\
s_{2,1} & \equiv c_2 t_{1,0} + (1 - c_2)t_{2,0}.
\end{align*}
\]

We regard \( t_{1,0}, t_{0,1}, \) and \( t_{2,0} \) as free variables. Taking \( t_{1,0} \equiv 1, t_{0,1} = 0, \) and \( t_{2,0} = 0, \) the matrix \( x \) becomes

\[
v_1 = \begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & -3 \\ 0 & 3c_2 & 0 \end{bmatrix}.
\]

Taking \( t_{0,1} \equiv 1, t_{1,0} = 0, \) and \( t_{2,0} = 0, \) the matrix \( x \) becomes

\[
v_2 = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}.
\]

Taking \( t_{2,0} \equiv 1, t_{0,1} = 0, \) and \( t_{1,0} = 0, \) the matrix \( x \) becomes

\[
v_3 = \begin{bmatrix} 0 & 0 & 1 \\ 0 & -c_2 & 3(c_2 + 1) \\ 1 & 3(1 - c_2) & -3 \end{bmatrix}.
\]

We see that neither \( v_3 \) nor \( 3v_3 \) is contained in \( W_2 \) and that \( \partial^{-1}W_2 = \partial^{-1}W_0, v_1, v_2, v_3 \). We know that \( v_3 \in \partial^{-1}W_2 \) and since \( 3v_3 \notin W_2 \) then \( v_3 + W_2 \)
is an element of the group $\partial^{-1}W_2/W_2$ whose order is larger than 3. Recall that $|\partial^{-1}W_2/W_2| = 3^3$ and that $\partial^{-1}W_1/W_2$ has rank 2. We obtain $\partial^{-1}W_2/W_2 \cong \mathbb{Z}_9 \times \mathbb{Z}_3$.

Since $\text{rank}(\partial^{-1}W_2/W_2) = \text{rank}(\partial^{-1}W_1/W_2)$. Hence $W_2$ is terminal and $W_2 \not\in \hat{\mathcal{L}}_2$.

Case 7.1.1.3

We fix arbitrary values $c_4, c_5 \in \{0, 1, 2\}$ such that $(c_4, c_5) \neq (0, 0)$. There is 8 ways to choose the values $c_4, c_5$. Let $m_4 = y_2 + c_4v_1 + c_5v_2$. Thus

$$m_4 = \begin{bmatrix} 0 & c_5 & 0 \\ c_4 & 3 & -3c_4 \\ 0 & 3c_2c_4 & 0 \end{bmatrix}.$$ 

Let $W_2 = \langle W_1, m_4 \rangle \in \mathcal{L}_2$. The number of subgroups $W_2$ of this type is 8. Since $m_4 \notin W_1$ and $3m_4 \in W_1$ we have $|W_2/W_1| = 3$. Since $|W_1| = 3^6$ it follows that $|W_2| = 3^7$. Note that $\partial^{-1}W_1/W_2 \cong \mathbb{Z}_3 \times \mathbb{Z}_3$ and $\text{rank}(\partial^{-1}W_1/W_2) = 2$. We now calculate the pullback of $\partial^{-1}W_2$. We observed in case 7.4.1 that $\partial^{-1}W_2$ is contained in the pattern subgroup

$$\begin{bmatrix} 2 & 2 & 2 \\ 2 & 2 & 1 \\ 2 & 1 & 1 \end{bmatrix}.$$

Let

$$x = \begin{bmatrix} 3s_{0,0} + t_{0,0} & 3s_{0,1} + t_{0,1} & 3s_{0,2} + t_{0,2} \\ 3s_{1,0} + t_{1,0} & 3s_{1,1} + t_{1,1} & 3s_{1,2} \\ 3s_{2,0} + t_{2,0} & 3s_{2,1} & 3s_{2,2} \end{bmatrix} \in \begin{bmatrix} 2 & 2 & 2 \\ 2 & 2 & 1 \\ 2 & 1 & 1 \end{bmatrix}.$$
The variables that are in play are those appearing in the matrix
\[
\begin{bmatrix}
0 & t_{0,1} & t_{0,2} \\
t_{1,0} & t_{1,1} & 3s_{1,2} \\
t_{2,0} & 3s_{2,1} & 3s_{2,2}
\end{bmatrix}.
\]

In the notation of Case 7.4.1, we are taking \(d_1 = 1, d_2 = c_4, \) and \(d_3 = c_5.\) We wish to identify values \(a_1, a_2, a_3, a_4 \in \mathbb{Z}_9\) such that \(\partial_1 x \equiv a_1 m_1 + a_2 m_2 + a_3 m_3 + a_4 m_4 \pmod{I}.\) We see that \(\partial_1 x \in W_2\) if and only if
\[
t_{1,1} \equiv a_4 c_5 \quad \text{(A1)}
\]
\[
-t_{1,1} \equiv a_4 c_2 c_4 \quad \text{(A2)}
\]
\[
t_{2,0} \equiv a_4 c_4 \quad \text{(A3)}
\]
\[
s_{2,2} \equiv -t_{2,0} \quad \text{(A4)}
\]
\[
s_{1,2} \equiv c_2 s_{2,1} - c_2 a_4 + t_{1,0} - t_{2,0} \quad \text{(A5)}.
\]

We wish to identify values \(b_1, b_2, b_3, b_4 \in \mathbb{Z}_9\) such that \(\partial_2 x \equiv b_1 m_1 + b_2 m_2 + b_3 m_3 + b_4 m_4 \pmod{I}.\) We see that \(\partial_2 x \in W_2\) if and only if
\[
t_{0,2} \equiv b_4 c_5 \quad \text{(B1)}
\]
\[
s_{2,2} \equiv b_4 c_2 c_4 \quad \text{(B2)}
\]
\[
t_{1,1} \equiv b_4 c_4 \quad \text{(B3)}
\]
\[
s_{1,2} \equiv -c_2 t_{0,2} - c_2 s_{2,1} + b_4 \quad \text{(B4)}.
\]

Combining (A5) and (B4) we obtain
\[-c_2 a_4 + t_{1,0} - t_{2,0} \equiv -c_2 t_{0,2} + c_2 s_{2,1} + b_4
\]
which will we denote as our new (B4).
Hence $x \in \partial^{-1}W_2$ if and only if

- $t_{1,1} \equiv a_4 c_5$ (A1)
- $-t_{1,1} \equiv a_4 c_2 c_4$ (A2)
- $t_{2,0} \equiv a_4 c_4$ (A3)
- $s_{2,2} \equiv -t_{2,0}$ (A4)
- $s_{1,2} \equiv c_2 s_{2,1} - c_2 a_4 + t_{1,0} - t_{2,0}$ (A5)
- $t_{0,2} \equiv b_4 c_5$ (B1)
- $s_{2,2} \equiv b_4 c_2 c_4$ (B2)
- $t_{1,1} \equiv b_4 c_4$ (B3)

- $-c_2 a_4 + t_{1,0} - t_{2,0} \equiv -c_2 t_{0,2} + c_2 s_{2,1} + b_4$ (B4).

It is convenient to consider the cases $c_4 = 0$ and $c_4 \neq 0$ separately.

**Case 7.1.1.3.1** First we consider the case $c_4 = 0$. Then $c_5 \neq 0$ and $c_5^2 \equiv 1$. From congruence (A2) we obtain $t_{1,1} \equiv 0$. It follows from (A1) that $a_4 \equiv 0$. From congruences (A3) and (A4) we obtain $t_{2,0} \equiv s_{2,2} \equiv 0$. The congruence (B1) gives us $b_4 \equiv c_5 t_{0,2}$.

Substituting $a_4$ and $t_{2,0}$ into (A5) we obtain $s_{1,2} \equiv c_2 s_{2,1} + t_{1,0}$. Substituting $a_4, t_{2,0}$, and $b_4$ into (B4) we obtain $s_{2,1} \equiv c_2 t_{1,0} + (1 - c_2 c_5) t_{0,2}$. Substituting (B4) into (A5) we obtain $s_{1,2} \equiv -t_{1,0} + c_2 (1 - c_2 c_5) t_{0,2}$.

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Hence \( x \in \partial^{-1}W_2 \) if and only if

\[
\begin{align*}
t_{1,1} &\equiv 0 \\
t_{2,0} &\equiv 0 \\
s_{2,2} &\equiv 0 \\
s_{1,2} &\equiv -t_{1,0} + c_2(1 - c_2c_5)t_{0,2} \\
s_{2,1} &\equiv c_2t_{1,0} + (1 - c_2c_5)t_{0,2}.
\end{align*}
\]

We regard \( t_{1,0}, t_{0,1}, \) and \( t_{0,2} \) as free variables. Taking \( t_{1,0} \equiv 1, t_{0,1} = 0, \) and \( t_{0,2} = 0, \) the matrix \( x \) becomes

\[
v_1 = \begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & -3 \\ 0 & 3c_2 & 0 \end{bmatrix}.
\]

Taking \( t_{0,1} \equiv 1, t_{1,0} = 0, \) and \( t_{0,2} = 0, \) the matrix \( x \) becomes

\[
v_2 = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}.
\]

Taking \( t_{0,2} \equiv 1, t_{0,1} = 0, \) and \( t_{1,0} = 0, \) the matrix \( x \) becomes

\[
v_3 = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 3c_2(1 - c_2c_5) \\ 1 & 3(1 - c_2c_5) & 0 \end{bmatrix}.
\]

We see that neither \( v_3 \) nor \( 3v_3 \) is contained in \( W_2 \) and that

\[
\partial^{-1}W_2 = \langle \partial^{-1}W_0, v_1, v_2, v_3 \rangle.
\]

We know that \( v_3 \in \partial^{-1}W_2 \) and since \( 3v_3 \notin W_2 \) then
$v_3 + W_2$ is an element of the group $\partial^{-1}W_2/W_2$ whose order is larger than 3. Recall that $|\partial^{-1}W_2/W_2| = 3^3$ and that $\partial^{-1}W_1/W_2$ has rank 2. We obtain $\partial^{-1}W_2/W_2 \cong \mathbb{Z}_9 \times \mathbb{Z}_3$. Since $\text{rank}(\partial^{-1}W_2/W_2) = \text{rank}(\partial^{-1}W_1/W_2)$. Hence $W_2$ is terminal and $W_2 \not\in \hat{\mathcal{L}}_2$.

**Case 7.1.1.3.2** Now we consider the case $c_4 \neq 0$. Then $c_4^2 \equiv 1$. From (A2) we obtain $a_4 \equiv -c_2c_4t_{1,1}$ and from (A3) we obtain $a_4 \equiv c_4t_{2,0}$. Combining these we obtain $0 \equiv c_4t_{0,2} + c_2c_4t_{1,1}$ which we will denote as (A2). Substituting $a_4 \equiv -c_2c_4t_{1,1}$ into (A1) we obtain $0 \equiv (1 + c_2c_4c_5)t_{1,1}$. From (B2) and (B3) we obtain $b_4 \equiv c_2c_4s_{2,2}$ and $b_4 \equiv c_4t_{1,1}$.

Combining these we get $0 \equiv c_2c_4s_{2,2} - c_4t_{1,1}$ which we will denote as (B2). Substituting $b_4 \equiv c_4t_{1,1}$ into (B1) we obtain $t_{0,2} \equiv c_4c_5t_{1,1}$. Substituting $a_4$ and $b_4$ into (B4) we obtain $s_{2,1} \equiv c_2t_{1,0} - c_2t_{2,0} + t_{0,2}$. (A5) becomes $s_{1,2} \equiv -t_{1,0} + t_{2,0} + (c_2 - c_2c_4)t_{0,2}$.

Hence $x \in \partial^{-1}W_2$ if and only if

\[
0 \equiv (1 + c_2c_4c_5)t_{1,1} \quad (A1)
\]
\[
0 \equiv c_4t_{2,0} + c_2c_4t_{1,1} \quad (A2)
\]
\[
s_{2,2} \equiv -t_{2,0} \quad (A4)
\]
\[
s_{1,2} \equiv -t_{1,0} + t_{2,0} + (c_2 - c_2c_4)t_{0,2} \quad (A5)
\]
\[
t_{0,2} \equiv c_4c_5t_{1,1} \quad (B1)
\]
\[
0 \equiv c_2c_4s_{2,2} - c_4t_{1,1} \quad (B2)
\]
\[
s_{2,1} \equiv c_2t_{1,0} - c_2t_{2,0} + t_{0,2} \quad (B4).
\]

If $1 + c_2c_4c_5 \neq 0$ then $t_{1,1} \equiv 0$. It follows that $t_{0,2} \equiv 0$. From (A2) we obtain $0 \equiv c_4t_{2,0}$. $c_4 \neq 0$ therefore $t_{2,0} \equiv 0$. It follows that $s_{2,2} \equiv 0$. The congruences
(A5) and (B4) become \( s_{1,2} \equiv -t_{0,1} \) and \( s_{2,1} \equiv c_2 t_{1,0} \). These are the same congruences as those in Case 7.1 therefore \( \partial^{-1}W_2 = \partial^{-1}W_1 \). Thus \( \text{rank}(\partial^{-1}W_2/W_2) = \text{rank}(\partial^{-1}W_1/W_2) \), \( W_2 \) is terminal, and \( W_2 \not\in \hat{\mathcal{L}}_2 \).

Therefore we assume that \( 1 + c_2 c_4 c_5 \equiv 0 \). Then \( c_5 \equiv -c_2 c_4 \) and \( t_{1,1} \) is a free variable. Then (B1) becomes \( t_{0,2} \equiv -c_2 t_{1,1} \). (B2) can we written as \( s_{2,2} \equiv c_2 t_{1,1} \).

Substituting (B2) into (A4) we obtain \( t_{2,0} \equiv -c_2 t_{1,1} \). The congruences (A5) and (B4) become \( s_{1,2} \equiv -t_{1,0} + (c_4 - c_2 - 1)t_{1,1} \) and \( s_{2,1} \equiv c_2 t_{1,0} + (1 - c_2) t_{1,1} \). Hence \( x \in \partial^{-1}W_2 \) if and only if

\[
\begin{align*}
t_{2,0} &\equiv -c_2 t_{1,1} \quad \text{(A4)} \\
s_{1,2} &\equiv -t_{1,0} + (c_4 - c_2 - 1)t_{1,1} \quad \text{(A5)} \\
t_{0,2} &\equiv -c_2 t_{1,1} \quad \text{(B1)} \\
s_{2,2} &\equiv c_2 t_{1,1} \quad \text{(B2)} \\
s_{2,1} &\equiv c_2 t_{1,0} + (1 - c_2) t_{1,1} \quad \text{(B4)}.
\end{align*}
\]

We regard \( t_{1,0}, t_{0,1}, \) and \( t_{1,1} \) as free variables. Taking \( t_{1,0} \equiv 1, t_{0,1} = 0, \) and \( t_{1,1} = 0, \) the matrix \( x \) becomes

\[
v_1 = \begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & -3 \\ 0 & 3c_2 & 0 \end{bmatrix}.
\]
Taking $t_{0,1} \equiv 1$, $t_{1,0} = 0$, and $t_{1,1} = 0$, the matrix $x$ becomes

$$v_2 = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$ 

Taking $t_{1,1} \equiv 1$, $t_{0,1} = 0$, and $t_{1,0} = 0$, the matrix $x$ becomes

$$v_3 = \begin{bmatrix} 0 & 0 & -c_2 \\ 0 & 1 & 3(c_4 - c_2 - 1) \\ -c_2 & 3(1 - c_2) & 3c_2 \end{bmatrix}.$$ 

We see that neither $v_3$ nor $3v_3$ is contained in $W_2$ and that $\partial^{-1}W_2 = \langle \partial^{-1}W_0, v_1, v_2, v_3 \rangle$. We know that $v_3 \in \partial^{-1}W_2$ and since $3v_3 \not\in W_2$ then $v_3 + W_2$ is an element of the group $\partial^{-1}W_2/W_2$ whose order is larger than 3. Recall that $|\partial^{-1}W_2/W_2| = 3^3$ and that $\partial^{-1}W_1/W_2$ has rank 2. We obtain $\partial^{-1}W_2/W_2 \cong \mathbb{Z}_9 \times \mathbb{Z}_3$. Since $\text{rank}(\partial^{-1}W_2/W_2) = \text{rank}(\partial^{-1}W_1/W_2)$. Hence $W_2$ is terminal and $W_2 \not\in \hat{L}_2$.

10.1.2 Case 7.1.2

We consider the 2-dimensional subspaces $W_2/W_1$. Let $d_1, d_2, d_3, e_1, e_2, e_3$ be unspec-ified variables. Let $m_4 = d_1y_2 + d_2v_1 + d_3v_2$ and $m_5 = e_1y_2 + e_2v_1 + e_3v_2$. In all the cases we consider the value of $e_1 = 0$ therefore we may exclude it from our expression of $m_5$. A formal expression for $m_4$ is

$$m_4 = d_1 \begin{bmatrix} 0 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 0 \end{bmatrix} + d_2 \begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & -3 \\ 0 & 0 & 0 \end{bmatrix} + d_3 \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & d_3 & 0 \\ d_2 & 3d_1 & -3d_2 \\ 0 & 3c_2d_2 & 0 \end{bmatrix}.$$
A formal expression for $m_5$ is

$$m_5 = e_2 \begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & -3 \\ 0 & 3c_2 & 0 \end{bmatrix} + e_3 \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & e_3 & 0 \\ 0 & 0 & -3e_2 \\ 0 & 3c_2e_2 & 0 \end{bmatrix}.$$ 

Let $W_2 = < W_1, m_4, m_5 > \in L_2$. We now calculate the pullback $\partial^{-1}W_2$. The subgroup $W_2$ is contained in the pattern subgroup

$$\begin{bmatrix} 2 & 2 & 1 \\ 2 & 1 & 1 \\ 1 & 1 & 0 \end{bmatrix}.$$ 

Thus the pullback $\partial^{-1}W_2$ is contained in the pattern subgroup

$$\begin{bmatrix} 2 & 2 & 2 \\ 2 & 2 & 1 \\ 2 & 1 & 1 \end{bmatrix} \cap \begin{bmatrix} 2 & 2 & 1 \\ 2 & 1 & 1 \end{bmatrix} = \begin{bmatrix} 2 & 2 & 1 \\ 2 & 1 & 1 \end{bmatrix}.$$ 

Let

$$x = \begin{bmatrix} 3s_{0,0} + t_{0,0} & 3s_{0,1} + t_{0,1} & 3s_{0,2} + t_{0,2} \\ 3s_{1,0} + t_{1,0} & 3s_{1,1} + t_{1,1} & 3s_{1,2} \\ 3s_{2,0} + t_{2,0} & 3s_{2,1} & 3s_{2,2} \end{bmatrix} \in \begin{bmatrix} 2 & 2 & 2 \\ 2 & 2 & 1 \\ 2 & 1 & 1 \end{bmatrix}.$$
Thus
\[ \partial_1 x = \begin{bmatrix}
3s_{1,0} + t_{1,0} & 3s_{1,1} + t_{1,1} & 3s_{1,2} \\
3s_{2,0} + t_{2,0} & 3s_{2,1} & 3s_{2,2} \\
-3(t_{1,0} + t_{2,0}) & -3t_{1,1} & 0
\end{bmatrix} \quad \text{and} \]
\[ \partial_2 x = \begin{bmatrix}
3s_{0,1} + t_{0,1} & 3s_{0,2} + t_{0,2} & -3(t_{0,1} + t_{0,2}) \\
3s_{1,1} + t_{1,1} & 3s_{1,2} & -3t_{1,1} \\
3s_{2,1} & 3s_{2,2} & 0
\end{bmatrix}. \]

The variables that are in play are those appearing in the matrix
\[
\begin{bmatrix}
0 & t_{0,1} & t_{0,2} \\
t_{1,0} & t_{1,1} & 3s_{1,2} \\
t_{2,0} & 3s_{2,1} & 3s_{2,2}
\end{bmatrix}.
\]

We wish to identify values \( a_1, a_2, a_3, a_4, a_5 \in \mathbb{Z}_9 \) such that \( \partial_1 x \equiv a_1 m_1 + a_2 m_2 + a_3 m_3 + a_4 m_4 + a_5 m_5 \) (mod I). A formal expression for \( a_1 m_1 + a_2 m_2 + a_3 m_3 + a_4 m_4 + a_5 m_5 \) is

\[
\begin{bmatrix}
1 & 0 & -3 \\
0 & 0 & 0 \\
0 & 0 & 0
\end{bmatrix} + a_2 \begin{bmatrix}
c_2 & 0 & -3c_2 \\
0 & 3 & 0 \\
-3c_2 & 0 & 0
\end{bmatrix} + a_3 \begin{bmatrix}
2 & 0 & -3 \\
0 & 0 & 0 \\
3 & 0 & 0
\end{bmatrix} + a_4 \begin{bmatrix}
0 & d_3 & 0 \\
d_2 & 3d_1 & -3d_2 \\
0 & 3c_2 d_2 & 0
\end{bmatrix}
\]

\[+ a_5 \begin{bmatrix}
0 & e_3 & 0 \\
e_2 & 0 & -3e_2 \\
0 & 3c_2 e_2 & 0
\end{bmatrix} = \begin{bmatrix}
a_1 + a_2 c_2 - a_3 & a_4 d_3 + a_5 e_3 & -3(a_1 + a_2 c_2 + a_3) \\
a_4 d_2 + a_5 e_2 & 3(a_2 + a_4 d_1) & -3(a_4 d_2 + a_5 e_2) \\
3(-c_2 a_2 + a_3) & 3c_2 (a_4 d_2 + a_5 e_2) & 0
\end{bmatrix}.\]
Comparing (0, 1)-entries, we get $t_{1,1} \equiv a_4d_3 + a_5e_3$.

Comparing (2, 1)-entries, we get $-c_2t_{1,1} \equiv a_4d_2 + a_5e_2$.

Comparing (1, 0)-entries, we get $t_{2,0} \equiv a_4d_2 + a_5e_2$. Substituting $a_4d_2 + a_5e_2 \equiv -c_2t_{1,1}$ we obtain $t_{2,0} \equiv -c_2t_{1,1}$.

Comparing (1, 2)-entries, we get $s_{2,2} \equiv -(a_4d_2 + a_5e_2)$. Substituting $a_4d_2 + a_5e_2 \equiv -c_2t_{1,1}$ we obtain $s_{2,2} \equiv c_2t_{1,1}$.

Comparing (1, 1)-entries, we get $a_2 \equiv s_{2,1} - a_4d_1$.

Comparing (2, 0)-entries, we get $a_3 \equiv -t_{0,1} - t_{2,0} + c_2a_2$. Substituting $a_2$ we obtain $a_3 \equiv -t_{1,0} - t_{2,0} + c_2s_{2,1} - c_2a_4d_1$.

Comparing (0, 2)-entries, we get $a_1 \equiv -a_2c_2 - a_3 - s_{1,2}$. Substituting $a_2$ and $a_3$ we obtain $a_1 \equiv c_2s_{2,1} - c_2a_4d_1 + t_{1,0} + t_{2,0} - s_{1,2}$.

Comparing (0, 0)-entries, we get $a_1 + a_2c_2 - a_3 \equiv t_{1,0}$. Substituting $a_1, a_2, a_3$, we obtain $s_{1,2} \equiv c_2s_{2,1} - c_2a_4d_1 + t_{1,0} - t_{2,0}$.

We see that $\partial_1 x \in W_2$ if and only if

\begin{align*}
  t_{1,1} & \equiv a_4d_3 + a_5e_3 \quad (A1) \\
  -c_2t_{1,1} & \equiv a_4d_2 + a_5e_2 \quad (A2) \\
  t_{2,0} & \equiv -c_2t_{1,1} \quad (A3) \\
  s_{2,2} & \equiv c_2t_{1,1} \quad (A4) \\
  s_{1,2} & \equiv c_2s_{2,1} - c_2a_4d_1 + t_{1,0} - t_{2,0} \quad (A5).
\end{align*}

We wish to identify values $b_1, b_2, b_3, b_4, b_5 \in \mathbb{Z}_9$ such that $\partial_2 x \equiv b_1m_1 + b_2m_2 + b_3m_3 + b_4m_4 + b_5m_5 (\text{mod } I)$. A formal expression for $b_1m_1 + b_2m_2 + b_3m_3 = b_4m_4 + b_5m_5$ is

321
\[
\begin{bmatrix}
1 & 0 & -3 \\
0 & 0 & 0
\end{bmatrix}
+ b_2
\begin{bmatrix}
0 & 3 & 0 \\
-3c_2 & 0 & 0
\end{bmatrix}
+ b_3
\begin{bmatrix}
2 & 0 & -3 \\
0 & 0 & 0
\end{bmatrix}
+ b_4
\begin{bmatrix}
d_2 & 3d_1 & -3d_2 \\
0 & 3c_2d_2 & 0
\end{bmatrix}
\]

\[
+ b_5
\begin{bmatrix}
0 & e_3 & 0 \\
e_2 & 0 & -3e_2 \\
0 & 3c_2e_2 & 0
\end{bmatrix}
= \begin{bmatrix}
b_1 + b_2c_2 - b_3 & b_4d_3 + b_5e_3 & -3(b_1 + b_2c_2 + b_3) \\
b_4d_2 + b_5e_2 & 3(b_2 + b_4d_1) & -3(b_4d_2 + b_5e_2)
\end{bmatrix}
\]

Comparing (0,1)-entries, we get \( t_{0,2} \equiv b_4d_3 + b_5e_3 \).

Comparing (2,1)-entries, we get \( c_2s_{2,2} \equiv b_4d_2 + b_5e_2 \).

Comparing (1,0)-entries, we get \( t_{1,1} \equiv b_4d_2 + b_5e_2 \). Substituting \( b_4d_2 + b_5e_2 \equiv c_2s_{2,2} \) we obtain \( t_{1,1} \equiv c_2s_{2,2} \).

Comparing (1,2)-entries, we get \( t_{1,1} \equiv c_2(b_4d_2 + b_5e_2) \) which gives us no new information.

Comparing (1,1)-entries, we get \( b_2 \equiv s_{1,2} - b_4d_1 \).

Comparing (2,0)-entries, we get \( b_3 \equiv s_{2,1} + c_2b_2 \). Substituting \( b_2 \) we obtain \( b_3 \equiv s_{2,1} + c_2s_{1,2} - c_2b_4d_1 \).

Comparing (0,2)-entries, we get \( b_1 \equiv t_{0,1} + t_{0,2} - b_2c_2 - b_3 \). Substituting \( b_2 \) and \( b_3 \) we obtain \( b_1 \equiv t_{0,1} + t_{0,2} + c_2s_{1,2} - c_2b_4d_1 - s_{2,1} \).

Comparing (0,0)-entries, we get \( b_1 + b_2c_2 - b_3 \equiv t_{0,1} \). Substituting \( b_1, b_2, b_3 \), we obtain \( s_{1,2} \equiv -c_2t_{0,2} + b_4d_1 - c_2s_{2,1} \).
We see that \( \partial_2 x \in W_2 \) if and only if

\[
t_{0,2} \equiv b_4d_3 + b_5e_3 \quad \text{(B1)}
\]

\[
c_{2}s_{2,2} \equiv b_4d_2 + b_5e_2 \quad \text{(B2)}
\]

\[
t_{1,1} \equiv c_2 s_{2,2} \quad \text{(B3)}
\]

\[
s_{1,2} \equiv -c_2 t_{0,2} + b_4d_1 - c_2 s_{2,1} \quad \text{(B4)}
\]

Case 7.1.2.1

Let \( m_4 = v_1 \) and \( m_5 = v_2 \). Thus

\[
m_4 = \begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & -3 \\ 0 & 3c_2 & 0 \end{bmatrix} \quad \text{and} \quad m_5 = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}.
\]

Let \( W_2 = \langle W_1, m_4, m_5 \rangle \in \mathcal{L}_2 \). The number of subgroups \( W_2 \) of this type is 1. Since \( m_4, m_5 \notin W_1 \) and \( 3m_4, 3m_5 \in W_1 \) we have \( |W_2/W_1| = 3^2 \). Since \( |W_1| = 3^6 \) it follows that \( |W_2| = 3^8 \). Note that \( \partial^{-1}W_1/W_2 \cong \mathbb{Z}_3 \) and \( \text{rank}(\partial^{-1}W_1/W_2) = 1 \).

We now calculate the pullback of \( \partial^{-1}W_2 \). We observed in case 7.2.2 that \( \partial^{-1}W_2 \) is contained in the pattern subgroup

\[
\begin{bmatrix} 2 & 2 & 2 \\ 2 & 2 & 1 \\ 2 & 1 & 1 \end{bmatrix}
\]
Let
\[ x = \begin{bmatrix}
3s_{0,0} + t_{0,0} & 3s_{0,1} + t_{0,1} & 3s_{0,2} + t_{0,2} \\
3s_{1,0} + t_{1,0} & 3s_{1,1} + t_{1,1} & 3s_{1,2} \\
3s_{2,0} + t_{2,0} & 3s_{2,1} & 3s_{2,2}
\end{bmatrix} \in \begin{bmatrix}
2 & 2 & 2 \\
2 & 2 & 1 \\
2 & 1 & 1
\end{bmatrix}. \]

The variables that are in play are those appearing in the matrix
\[
\begin{bmatrix}
0 & t_{0,1} & t_{0,2} \\
t_{1,0} & t_{1,1} & 3s_{1,2} \\
t_{2,0} & 3s_{2,1} & 3s_{2,2}
\end{bmatrix}.
\]

In the notation of Case 7.2.2, we are taking \( d_1 = 0, \ d_2 = 1, \ d_3 = 0, \ e_2 = 0, \) and \( e_3 = 1. \) We wish to identify values \( a_1, a_2, a_3, a_4, a_5 \in \mathbb{Z}_9 \) such that \( \partial_1 x \equiv a_1m_1 + a_2m_2 + a_3m_3 + a_4m_4 + a_5m_5 \pmod{I}. \)

Hence \( x \in \partial^{-1}W_2 \) if and only if

\[
\begin{align*}
t_{1,1} & \equiv a_5 \quad \text{(A1)} \\
-c_2t_{1,1} & \equiv a_4 \quad \text{(A2)} \\
t_{2,0} & \equiv -c_2t_{1,1} \quad \text{(A3)} \\
s_{2,2} & \equiv c_2t_{1,1} \quad \text{(A4)} \\
s_{1,2} & \equiv c_2s_{2,1} + t_{1,0} - t_{2,0} \quad \text{(A5)}.
\end{align*}
\]
We wish to identify values $b_1, b_2, b_3, b_4, b_5 \in \mathbb{Z}_9$ such that $\partial_1 x \equiv b_1 m_1 + b_2 m_2 + b_3 m_3 + b_4 m_4 + b_5 m_5 \pmod{I}$. We see that $\partial_2 x \in W_2$ if and only if

$$t_{0,2} \equiv b_5 \quad \text{(B1)}$$
$$c_2 s_{2,2} \equiv b_4 \quad \text{(B2)}$$
$$t_{1,1} \equiv c_2 s_{2,2} \quad \text{(B3)}$$
$$s_{1,2} \equiv -c_2 t_{0,2} - c_2 s_{2,1} \quad \text{(B4)}.$$

Multiplying (B3) by $c_2$ we obtain $s_{2,2} \equiv c_2 t_{1,1}$ which is redundant with (A4). Rewriting (A3) we obtain $t_{1,1} \equiv -c_2 t_{2,0}$. Substituting this into (A4) we obtain $s_{2,2} \equiv -t_{2,0}$. Combining (A5) and (B5) we obtain $s_{2,1} \equiv c_2 t_{1,0} - c_2 t_{2,0} + t_{0,2}$ which we will denote as our new (B4). Substituting (B4) into (A5) we obtain $s_{1,2} \equiv -t_{1,0} + t_{2,0} + c_2 t_{0,2}$. Hence $x \in \partial^{-1} W_2$ if and only if

$$t_{1,1} \equiv -c_2 t_{2,0} \quad \text{(A3)}$$
$$s_{2,2} \equiv -t_{2,0} \quad \text{(A4)}$$
$$s_{1,2} \equiv -t_{1,0} + t_{2,0} + c_2 t_{0,2} \quad \text{(A5)}$$
$$s_{2,1} \equiv c_2 t_{1,0} - c_2 t_{2,0} + t_{0,2} \quad \text{(B4)}.$$

We regard $t_{1,0}, t_{0,1}, t_{0,2}, t_{2,0}$ as the free variables. Taking $t_{1,0} \equiv 1$, $t_{0,1} = 0$, $t_{0,2} = 0$, and $t_{2,0} =$, the matrix $x$ becomes

$$v_1 = \begin{bmatrix}
0 & 0 & 0 \\
1 & 0 & -3 \\
0 & 3c_2 & 0
\end{bmatrix}.$$
Taking $t_{0,1} \equiv 1$, $t_{1,0} = 0$, $t_{0,2} = 0$, and $t_{2,0} = 0$, the matrix $x$ becomes
\[
v_2 = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}.
\]

Taking $t_{0,2} \equiv 1$, $t_{0,1} = 0$, $t_{1,0} = 0$, and $t_{2,0} = 0$, the matrix $x$ becomes
\[
v_3 = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 3c_2 \\ 0 & 3 & 0 \end{bmatrix}.
\]

Taking $t_{2,0} \equiv 1$, $t_{0,1} = 0$, $t_{1,0} = 0$, and $t_{0,2} = 0$, the matrix $x$ becomes
\[
v_4 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & -c_2 & 3 \\ 1 & -3c_2 & -3 \end{bmatrix}.
\]

We see that $\partial^{-1}W_2 = \langle \partial^{-1}W_1, v_3, v_4 \rangle$. Hence $|\partial^{-1}W_2/\partial^{-1}W_1| = 3^2$. Recall that $|\partial^{-1}W_1| = 3^9$ hence $|\partial^{-1}W_2| = 3^{11}$. Since $|W_2| = 3^8$, then $|\partial^{-1}W_2/W_2| = 3^3$.

We obtain $\partial^{-1}W_2/W_2 \cong \mathbb{Z}_9 \times \mathbb{Z}_3$ or $\mathbb{Z}_3 \times \mathbb{Z}_3 \times \mathbb{Z}_3$. We see that neither $v_3$ nor $3v_3$ is contained in $W_2$. We know that $v_3 \in \partial^{-1}W_2$ and since $3v_3 \not\in W_2$ then $v_3 + W_2$ is an element of the group $\partial^{-1}W_2/W_2$ whose order is larger than 3. Note
\[
m_1 - 3c_2 m_2 = \begin{bmatrix} 1 & 0 & -3 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} + \begin{bmatrix} -1 & 0 & 3 \\ 0 & -3c_2 & 0 \\ 3 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & -3c_2 & 0 \\ 3 & 0 & 0 \end{bmatrix} = 3v_4.
\]

Hence $3v_4 \in W_2$ and so $v_4 + W_2$ is an element of order 3. Hence $\partial^{-1}W_2/W_2 \cong \mathbb{Z}_9 \times \mathbb{Z}_3$ and rank$(\partial^{-1}W_2/W_2) = 2$. Recall that $\partial^{-1}W_1/W_2$ has rank 1. Since
rank(\(\partial^{-1}W_1/W_2\)) < rank(\(\partial^{-1}W_2/W_2\)) then \(W_2\) is nonterminal and \(W_2 \in \hat{\mathcal{L}}_2\). Note \(\Omega_1(\partial^{-1}W_2/W_2) \cong \mathbb{Z}_3 \times \mathbb{Z}_3\). Thus \(\Omega_1(\partial^{-1}W_2/W_2)\) has basis \(y_2 + W_1, v_4 + W_2\) while its subspace \(\partial^{-1}W_1/W_2\) has basis \(y_2 + W_2\).

Let \(c_4 \in \{0, 1, 2\}\). Let 
\[
m_6 = c_4 y_2 + v_4.
\]
Thus
\[
m_6 = \begin{bmatrix}
0 & 0 & 0 \\
0 & 3c_4 - c_2 & 3 \\
1 & -3c_2 & -3
\end{bmatrix}.
\]

Let \(W_3 =< W_2, m_6 >\). The number of subgroups of this type is 9. Since \(m_6 \notin W_2\) while \(3m_6 \in W_2\) we have \(|W_3/W_2| = 3\). Recalling \(|W_2| = 3^8\) we get \(|W_3| = 3^9\). Since \(|W_3/W_2| = 3\) and the antidiagonal of \(m_5\) has an entry not divisible by 3, by the Modified Terminal Lemma we conclude that \(\partial^{-1}W_3 = \partial^{-1}W_2\). Thus rank(\(\partial^{-1}W_3/W_3\)) = rank(\(\partial^{-1}W_2/W_3\)), \(W_3\) is terminal, and \(W_3 \notin \hat{\mathcal{L}}_3\).

Case 7.1.2.2

We fix arbitrary value \(c_4 \in \{1, 2\}\). There are 2 ways to choose the value of \(c_4\). Let 
\[
m_4 = y_2 + c_4 v_1 \quad \text{and} \quad m_5 = v_2.
\]
Thus
\[
m_4 = \begin{bmatrix}
0 & 0 & 0 \\
c_4 & 3 & -3c_4 \\
0 & 3c_2c_4 & 0
\end{bmatrix} \quad \text{and} \quad m_5 = \begin{bmatrix}
0 & 1 & 0 \\
0 & 0 & 0 \\
0 & 0 & 0
\end{bmatrix}.
\]

Let \(W_2 =< W_1, m_4, m_5 > \in \mathcal{L}_2\). The number of subgroups \(W_2\) of this type is 2. Since \(m_4, m_5 \notin W_1\) and \(3m_4, 3m_5 \in W_1\) we have \(|W_2/W_1| = 3^2\). Since \(|W_1| = 3^6\) it follows that \(|W_2| = 3^8\). Note that \(\partial^{-1}W_1/W_2 \cong \mathbb{Z}_3\) and rank(\(\partial^{-1}W_1/W_2\)) = 1.
We now calculate the pullback of $\partial^{-1}W_2$. We observed in case 7.2.2 that $\partial^{-1}W_2$ is contained in the pattern subgroup

$$
\begin{bmatrix}
2 & 2 & 2 \\
2 & 2 & 1 \\
2 & 1 & 1
\end{bmatrix}.
$$

Let

$$
x = \begin{bmatrix}
3s_{0,0} + t_{0,0} & 3s_{0,1} + t_{0,1} & 3s_{0,2} + t_{0,2} \\
3s_{1,0} + t_{1,0} & 3s_{1,1} + t_{1,1} & 3s_{1,2} \\
3s_{2,0} + t_{2,0} & 3s_{2,1} & 3s_{2,2}
\end{bmatrix} \in \begin{bmatrix}
2 & 2 & 2 \\
2 & 2 & 1 \\
2 & 1 & 1
\end{bmatrix}.
$$

The variables that are in play are those appearing in the matrix

$$
\begin{bmatrix}
0 & t_{0,1} & t_{0,2} \\
t_{1,0} & t_{1,1} & 3s_{1,2} \\
t_{2,0} & 3s_{2,1} & 3s_{2,2}
\end{bmatrix}.
$$

In the notation of Case 7.2.2, we are taking $d_1 = 1$, $d_2 = c_4$, $d_3 = 0$, $e_2 = 0$, and $e_3 = 1$. We wish to identify values $a_1, a_2, a_3, a_4, a_5 \in \mathbb{Z}_9$ such that $\partial_1 x \equiv a_1 m_1 + \cdots + a_5 m_5 \pmod{I}$. We see that $\partial_1 x \in W_2$ if and only if

- $t_{1,1} \equiv a_5$ (A1)
- $-c_2 t_{1,1} \equiv a_4 c_4$ (A2)
- $t_{2,0} \equiv -c_2 t_{1,1}$ (A3)
- $s_{2,2} \equiv c_2 t_{1,1}$ (A4)
- $s_{1,2} \equiv c_2 s_{2,1} - c_2 a_4 + t_{1,0} - t_{2,0}$ (A5).
We wish to identify values \( b_1, b_2, b_3, b_4, b_5 \in \mathbb{Z}_9 \) such that \( \partial_1 x \equiv b_1 m_1 + b_2 m_2 + b_3 m_3 + b_4 m_4 + b_5 m_5 \) (mod \( I \)). We see that \( \partial_2 x \in W_2 \) if and only if

\[
t_0,2 \equiv b_5 \quad \text{(B1)}
\]
\[
c_2 s_{2,2} \equiv b_4 c_4 \quad \text{(B2)}
\]
\[
t_{1,1} \equiv c_2 s_{2,2} \quad \text{(B3)}
\]
\[
s_{1,2} \equiv -c_2 t_{0,2} + b_4 - c_2 s_{2,1} \quad \text{(B4)}.
\]

Multiplying (B3) by \( c_2 \) we obtain \( s_{2,2} \equiv c_2 t_{1,1} \) which is redundant with (A4). Rewriting (A3) we obtain \( t_{1,1} \equiv -c_2 t_{2,0} \). Substituting this into (A4) we obtain \( s_{2,2} \equiv -t_{2,0} \). Since \( c_4 \neq 0 \), then we can rewrite (A2) and (B2) obtaining \( a_4 \equiv -c_2 c_4 t_{1,1} \) and \( b_4 \equiv c_2 c_4 s_{2,2} \). Combining (A5) and (B5) we obtain \(-c_2 a_4 + t_{1,0} - t_{2,0} \equiv -c_2 t_{0,2} + b_4 + c_2 s_{2,1} \). Substituting \( a_4, b_4, t_{1,1} \), and \( s_{2,2} \) we obtain \( s_{2,1} \equiv c_2 t_{1,0} - c_2 t_{2,0} + t_{0,2} \) which we will denote as our new (B4). Substituting (B4) and \( a_4 \) into (A5) we obtain \( s_{1,2} \equiv -t_{1,0} + (1 - c_2 c_4) t_{2,0} + c_2 t_{0,2} \).

Hence \( x \in \partial^{-1} W_2 \) if and only if

\[
t_{1,1} \equiv -c_2 t_{2,0} \quad \text{(A3)}
\]
\[
s_{2,2} \equiv -t_{2,0} \quad \text{(A4)}
\]
\[
s_{1,2} \equiv -t_{1,0} + (1 - c_2 c_4) t_{2,0} + c_2 t_{0,2} \quad \text{(A5)}
\]
\[
s_{2,1} \equiv c_2 t_{1,0} - c_2 t_{2,0} + t_{0,2} \quad \text{(B4)}.
\]
We regard $t_{1,0}$, $t_{0,1}$, $t_{0,2}$, $t_{2,0}$ as the free variables. Taking $t_{1,0} \equiv 1$, $t_{0,1} = 0$, $t_{0,2} = 0$, and $t_{2,0} = 0$, the matrix $x$ becomes

$$v_1 = \begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & -3 \\ 0 & 3c_2 & 0 \end{bmatrix}.$$ 

Taking $t_{0,1} \equiv 1$, $t_{1,0} = 0$, $t_{0,2} = 0$, and $t_{2,0} = 0$, the matrix $x$ becomes

$$v_2 = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$ 

Taking $t_{0,2} \equiv 1$, $t_{0,1} = 0$, $t_{1,0} = 0$, and $t_{2,0} = 0$, the matrix $x$ becomes

$$v_3 = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 3c_2 \\ 0 & 3 & 0 \end{bmatrix}.$$ 

Taking $t_{2,0} \equiv 1$, $t_{0,1} = 0$, $t_{1,0} = 0$, and $t_{0,2} = 0$, the matrix $x$ becomes

$$v_4 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & -c_2 & 3(1 - c_2c_4) \\ 1 & -3c_2 & -3 \end{bmatrix}.$$ 

We see that $\partial^{-1}W_2 =< \partial^{-1}W_1, v_3, v_4 >$. Hence $|\partial^{-1}W_2/\partial^{-1}W_1| = 3^2$. Recall that $|\partial^{-1}W_1| = 3^9$ hence $|\partial^{-1}W_2| = 3^{11}$. Since $|W_2| = 3^8$, then $|\partial^{-1}W_2/W_2| = 3^3$. We obtain $\partial^{-1}W_2/W_2 \cong \mathbb{Z}_9 \times \mathbb{Z}_3$ or $\mathbb{Z}_3 \times \mathbb{Z}_3 \times \mathbb{Z}_3$. We see that neither $v_3$ nor $3v_3$ is contained in $W_2$. We know that $v_3 \in \partial^{-1}W_2$ and since $3v_3 \not\in W_2$ then $v_3 + W_2$ is an
element of the group $\partial^{-1}W_2/W_2$ whose order is larger than 3. Note

$$m_1 - 3c_2m_2 = \begin{bmatrix} 1 & 0 & -3 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} + \begin{bmatrix} -1 & 0 & 3 \\ 0 & -3c_2 & 0 \\ 3 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & -3c_2 & 0 \\ 3 & 0 & 0 \end{bmatrix} = 3v_4.$$ 

Hence $3v_4 \in W_2$ and so $v_4 + W_2$ is an element of order 3. Hence $\partial^{-1}W_2/W_2 \cong \mathbb{Z}_9 \times \mathbb{Z}_3$ and $\text{rank}(\partial^{-1}W_2/W_2) = 2$. Recall that $\partial^{-1}W_1/W_2$ has rank 1. Since $\text{rank}(\partial^{-1}W_1/W_2) < \text{rank}(\partial^{-1}W_2/W_2)$ then $W_2$ is nonterminal and $W_2 \in \hat{\mathcal{L}}_2$. Note $\Omega_1(\partial^{-1}W_2/W_2) \cong \mathbb{Z}_3 \times \mathbb{Z}_3$. Thus $\Omega_1(\partial^{-1}W_2/W_2)$ has basis $y_2 + W_1, v_4 + W_2$ while its subspace $\partial^{-1}W_1/W_2$ has basis $y_2 + W_2$.

Let $c_5 \in \{0, 1, 2\}$. Let $m_6 = c_5y_2 + v_4$. Thus

$$m_6 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 3c_5 - c_2 & 3(1 - c_2c_4) \\ 1 & -3c_2 & -3 \end{bmatrix}.$$ 

Let $W_3 = \langle W_2, m_6 \rangle$. The number of subgroups of this type is 9. Since $m_6 \not\in W_2$ while $3m_6 \in W_2$ we have $|W_3/W_2| = 3$. Recalling $|W_2| = 3^8$ we get $|W_3| = 3^9$. Since $|W_3/W_2| = 3$ and the antidiagonal of $m_5$ has an entry not divisible by 3, by the Modified Terminal Lemma we conclude that $\partial^{-1}W_3 = \partial^{-1}W_2$. Thus $\text{rank}(\partial^{-1}W_3/W_3) = \text{rank}(\partial^{-1}W_2/W_3)$, $W_3$ is terminal, and $W_3 \not\in \hat{\mathcal{L}}_3$. 

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Case 7.1.2.3

We fix arbitrary values $c_4 \in \{1, 2\}$ and $c_5 \in \{0, 1, 2\}$. There are 6 ways to choose the values of $c_4, c_5$. Let $m_4 = y_2 + c_4 v_2$ and $m_5 = v_1 + c_5 v_2$. Thus

$$m_4 = \begin{bmatrix} 0 & c_4 & 0 \\ 0 & 3 & 3 \\ 0 & 0 & 0 \end{bmatrix} \quad \text{and} \quad m_5 = \begin{bmatrix} 0 & c_5 & 0 \\ 1 & 0 & -3 \\ 0 & 3c_2 & 0 \end{bmatrix}.$$  

Let $W_2 = < W_1, m_4, m_5 > \in \mathcal{L}_2$. The number of subgroups $W_2$ of this type is 6. Since $m_4, m_5 \notin W_1$ and $3m_4, 3m_5 \in W_1$ we have $|W_2/W_1| = 3^2$. Since $|W_1| = 3^6$ it follows that $|W_2| = 3^8$. Note that $\partial^{-1}W_1/W_2 \cong \mathbb{Z}_3$ and $\text{rank}(\partial^{-1}W_1/W_2) = 1$.

We now calculate the pullback of $\partial^{-1}W_2$. We observed in case 7.2.2 that $\partial^{-1}W_2$ is contained in the pattern subgroup

$$\begin{bmatrix} 2 & 2 & 2 \\ 2 & 2 & 1 \\ 2 & 1 & 1 \end{bmatrix}.$$  

Let

$$x = \begin{bmatrix} 3s_{0,0} + t_{0,0} & 3s_{0,1} + t_{0,1} & 3s_{0,2} + t_{0,2} \\ 3s_{1,0} + t_{1,0} & 3s_{1,1} + t_{1,1} & 3s_{1,2} \\ 3s_{2,0} + t_{2,0} & 3s_{2,1} & 3s_{2,2} \end{bmatrix} \in \begin{bmatrix} 2 & 2 & 2 \\ 2 & 2 & 1 \\ 2 & 1 & 1 \end{bmatrix}.$$  

The variables that are in play are those appearing in the matrix

$$\begin{bmatrix} 0 & t_{0,1} & t_{0,2} \\ t_{1,0} & t_{1,1} & 3s_{1,2} \\ t_{2,0} & 3s_{2,1} & 3s_{2,2} \end{bmatrix}.$$
In the notation of Case 7.2.2, we are taking $d_1 = 1$, $d_2 = 0$, $d_3 = c_4$, $e_2 = 1$, and $e_3 = c_5$. We wish to identify values $a_1, a_2, a_3, a_4, a_5 \in \mathbb{Z}_9$ such that $\partial_1 x \equiv a_1 m_1 + a_2 m_2 + a_3 m_3 + a_4 m_4 + a_5 m_5 \pmod{I}$. We see that $\partial_1 x \in W_2$ if and only if

$$t_{1,1} \equiv a_4 c_4 + a_5 c_5 \quad (A1)$$

$$-c_2 t_{1,1} \equiv a_5 \quad (A2)$$

$$t_{2,0} \equiv -c_2 t_{1,1} \quad (A3)$$

$$s_{2,2} \equiv c_2 t_{1,1} \quad (A4)$$

$$s_{1,2} \equiv c_2 s_{2,1} - c_2 a_4 + t_{1,0} - t_{2,0} \quad (A5).$$

We wish to identify values $b_1, b_2, b_3, b_4, b_5 \in \mathbb{Z}_9$ such that $\partial_1 x \equiv b_1 m_1 + b_2 m_2 + b_3 m_3 + b_4 m_4 + b_5 m_5 \pmod{I}$. We see that $\partial_2 x \in W_2$ if and only if

$$t_{0,2} \equiv b_4 c_4 + b_5 c_5 \quad (B1)$$

$$c_2 s_{2,2} \equiv b_5 \quad (B2)$$

$$t_{1,1} \equiv c_2 s_{2,2} \quad (B3)$$

$$s_{1,2} \equiv -c_2 t_{0,2} + b_4 - c_2 s_{2,1} \quad (B4).$$

Multiplying (B3) by $c_2$ we obtain $s_{2,2} \equiv c_2 t_{1,1}$ which is redundant with (A4). Rewriting (A3) we obtain $t_{1,1} \equiv -c_2 t_{2,0}$. Substituting this into (A4) we obtain $s_{2,2} \equiv -t_{2,0}$. Substituting $a_5$ and $b_5$ into (A1) and (B1) we obtain $a_4 \equiv (c_4 + c_2 c_4 c_5)t_{1,1}$ and $b_4 \equiv c_4 t_{0,2} - c_2 c_4 c_5 s_{2,2}$. Combining (A5) and (B5) we obtain $-c_2 a_4 + t_{1,0} - t_{2,0} \equiv -c_2 t_{0,2} + b_4 + c_2 s_{2,1}$. Substituting $a_4, b_4, t_{1,1}$, and $s_{2,2}$ we obtain $s_{2,1} \equiv c_2 t_{1,0} - c_2 t_{2,0} + c_2$. 

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\((c_2c_4c_5 - 1 - c_4c_5)t_{0,2}\) which we will denote as our new \((B4)\). Substituting \((B4)\) and \(a_4\) into \((A5)\) we obtain \(s_{1,2} \equiv -t_{1,0} + (-c_4 - c_2c_5 + c_2c_4c_5 - 1)t_{2,0} + (c_2 - c_4)t_{0,2}\).

Hence \(x \in \partial^{-1}W_2\) if and only if

\[
\begin{align*}
t_{1,1} &\equiv -c_2t_{2,0} \quad (A3) \\
s_{2,2} &\equiv -t_{2,0} \quad (A4) \\
s_{1,2} &\equiv -t_{1,0} + (-c_4 - c_2c_5 + c_2c_4c_5 - 1)t_{2,0} + (c_2 - c_4)t_{0,2} \quad (A5) \\
s_{2,1} &\equiv c_2t_{1,0} - c_2t_{2,0} + (c_2c_4c_5 - 1 - c_4c_5)t_{0,2} \quad (B4).
\end{align*}
\]

We regard \(t_{1,0}, t_{0,1}, t_{0,2}, t_{2,0}\) as the free variables. Taking \(t_{1,0} \equiv 1, t_{0,1} = 0, t_{0,2} = 0,\) and \(t_{2,0} = 0\), the matrix \(x\) becomes

\[
v_1 = \begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & -3 \\ 0 & 3c_2 & 0 \end{bmatrix}.
\]

Taking \(t_{0,1} \equiv 1, t_{1,0} = 0, t_{0,2} = 0,\) and \(t_{2,0} = 0\), the matrix \(x\) becomes

\[
v_2 = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}.
\]

Taking \(t_{0,2} \equiv 1, t_{0,1} = 0, t_{1,0} = 0,\) and \(t_{2,0} = 0\), the matrix \(x\) becomes

\[
v_3 = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 3(c_2 - c_4) \\ 0 & 3(1 - c_2c_4) & 0 \end{bmatrix}.
\]
Taking \( t_{2,0} \equiv 1, t_{0,1} = 0, t_{1,0} = 0, \) and \( t_{0,2} = 0, \) the matrix \( x \) becomes

\[
v_4 = \begin{bmatrix}
0 & 0 & 0 \\
0 & -c_2 & 3(-c_4 - c_2c_5 + c_2c_4c_5 - 1) \\
1 & -3(c_2c_4 - c_2) & -3
\end{bmatrix}.
\]

We see that \( \partial^{-1}W_2 = \langle \partial^{-1}W_1, v_3, v_4 \rangle. \) Hence \( |\partial^{-1}W_2/\partial^{-1}W_1| = 3^2. \) Recall that \( |\partial^{-1}W_1| = 3^9 \) hence \( |\partial^{-1}W_2| = 3^{11}. \) Since \( |W_2| = 3^8, \) then \( |\partial^{-1}W_2/W_2| = 3^3. \) We obtain \( \partial^{-1}W_2/W_2 \cong \mathbb{Z}_9 \times \mathbb{Z}_3 \) or \( \mathbb{Z}_3 \times \mathbb{Z}_3 \times \mathbb{Z}_3. \) We see that neither \( v_3 \) nor \( 3v_3 \) is contained in \( W_2. \) We know that \( v_3 \in \partial^{-1}W_2 \) and since \( 3v_3 \not\in W_2 \) then \( v_3 + W_2 \) is an element of the group \( \partial^{-1}W_2/W_2 \) whose order is larger than 3.

Note

\[
m_1 - 3c_2m_2 = \begin{bmatrix}
1 & 0 & -3 \\
0 & 0 & 0 \\
0 & 0 & 0
\end{bmatrix} + \begin{bmatrix}
-1 & 0 & 3 \\
0 & -3c_2 & 0 \\
3 & 0 & 0
\end{bmatrix} = \begin{bmatrix}
0 & 0 & 0 \\
0 & -3c_2 & 0 \\
3 & 0 & 0
\end{bmatrix} = 3v_4.
\]

Hence \( 3v_4 \in W_2 \) and so \( v_4 + W_2 \) is an element of order 3. Hence \( \partial^{-1}W_2/W_2 \cong \mathbb{Z}_9 \times \mathbb{Z}_3 \) and \( \text{rank}(\partial^{-1}W_2/W_2) = 2. \) Recall that \( \partial^{-1}W_1/W_2 \) has rank 1. Since \( \text{rank}(\partial^{-1}W_1/W_2) < \text{rank}(\partial^{-1}W_2/W_2) \) then \( W_2 \) is nonterminal and \( W_2 \in \mathcal{L}. \) Note \( \Omega_1(\partial^{-1}W_2/W_2) \cong \mathbb{Z}_3 \times \mathbb{Z}_3. \) Thus \( \Omega_1(\partial^{-1}W_2/W_2) \) has basis \( y_2 + W_1, v_4 + W_2 \) while its subspace \( \partial^{-1}W_1/W_2 \) has basis \( y_2 + W_2. \)

Let \( c_6 \in \{0, 1, 2\}. \) Let \( m_6 = c_6y_2 + v_4. \) Thus

\[
m_6 = \begin{bmatrix}
0 & 0 & 0 \\
0 & 3c_6 - c_2 & 3(-c_4 - c_2c_5 + c_2c_4c_5 - 1) \\
1 & -3(c_2c_4 - c_2) & -3
\end{bmatrix}.
\]
Let $W_3 = <W_2, m_6>$. The number of subgroups of this type is 9. Since $m_6 \not\in W_2$ while $3m_6 \in W_2$ we have $|W_3/W_2| = 3$. Recalling $|W_2| = 3^8$ we get $|W_3| = 3^9$. Since $|W_3/W_2| = 3$ and the antidiagonal of $m_5$ has an entry not divisible by 3, by the Modified Terminal Lemma we conclude that $\partial^{-1}W_3 = \partial^{-1}W_2$. Thus $\text{rank}(\partial^{-1}W_3/W_3) = \text{rank}(\partial^{-1}W_2/W_3)$, $W_3$ is terminal, and $W_3 \not\in \hat{L}_3$.

10.2 Case 7.2

Suppose $(c_1, c_3) = (2, 1)$. Then

$$
\begin{align*}
    m_1 &= \begin{bmatrix} 2 & 0 & 3 \\ 0 & 0 & 0 \\ -3 & 0 & 0 \end{bmatrix}, \\
    m_2 &= \begin{bmatrix} c_2 & 0 & -3c_2 \\ 0 & 3 & 0 \\ -3c_2 & 0 & 0 \end{bmatrix}, \quad \text{and} \\
    m_3 &= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ -3 & 0 & 0 \end{bmatrix}.
\end{align*}
$$

Hence $x \in \partial^{-1}W_2$ if and only if

$$
\begin{align*}
    s_{1,2} &\equiv -c_2s_{2,1} \quad \text{(A1)} \\
    s_{1,2} &\equiv c_2s_{2,1} - c_2t_{0,1} \quad \text{(B1)}.
\end{align*}
$$

We can rewrite (B1) as $-c_2s_{2,1} \equiv c_2s_{2,1} - c_2t_{0,1}$ which is equivalent to $s_{2,1} \equiv -t_{0,1}$. Now (A1) becomes $s_{1,2} \equiv c_2t_{0,1}$.

Hence $x \in \partial^{-1}W_2$ if and only if

$$
\begin{align*}
    s_{2,1} &\equiv -t_{0,1} \quad \text{(A1)} \\
    s_{1,2} &\equiv c_2t_{0,1} \quad \text{(B1)}.
\end{align*}
$$
We regard $t_{1,0}$ and $t_{0,1}$ as free variables. Taking $t_{1,0} \equiv 1$ and $t_{0,1} = 0$, the matrix $x$ becomes

$$v_1 = \begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$ 

Taking $t_{0,1} \equiv 1$ and $t_{1,0} = 0$, the matrix $x$ becomes

$$v_2 = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 3c_2 \\ 0 & -3 & 0 \end{bmatrix}.$$ 

We see that $\partial^{-1}W_1 = < \partial^{-1}W_0, v_1, v_2 >$. Since $v_1 \notin \partial^{-1}W_0$, $v_2 \notin < \partial^{-1}W_0, v_1 >$, $3v_1 \in \partial^{-1}W_0$, and $3v_2 \in < \partial^{-1}W_0, v_1 >$ we know $|\partial^{-1}W_1/\partial^{-1}W_0| = 3^2$. Since $|\partial^{-1}W_0| = 3^7$ then $|\partial^{-1}W_1| = 3^9$. Because $|W_1| = 3^6$ then $|\partial^{-1}W_1/W_1| = 3^3$. Also, $v_1, v_2, y_2 \in \partial^{-1}W_1$ but are not contained in $W_1$. Each of $3v_1, 3v_2, 3y_2$ is contained in $W_1$. So $v_1 + W_1, v_2 + W_1, y_2 + W_1$ are elements of order 3 in the group $\partial^{-1}W_1/W_1$. These three elements form a generating set for the group $\partial^{-1}W_1/W_1$. Recall that $|\partial^{-1}W_1/W_1| = 3^3$, we obtain $\partial^{-1}W_1/W_1 \cong \mathbb{Z}_3 \times \mathbb{Z}_3 \times \mathbb{Z}_3$.

Since $\text{rank}(\partial^{-1}W_1/W_1) = 3$ and $\text{rank}(\partial^{-1}W_0/W_1) = 1$, then $\text{rank}(\partial^{-1}W_0/W_1) < \text{rank}(\partial^{-1}W_1/W_1)$. Hence $W_1$ is nonterminal and $W_1 \in \hat{\mathcal{L}}_1$. A basis for $\partial^{-1}W_1/W_1$ is $v_1 + W_1, v_2 + W_1, y_2 + W_1$. A basis for $\partial^{-1}W_0/W_1$ is $y_2 + W_1$.

The subgroups $W_2$ belonging to $\mathcal{L}(W_1)$ correspond to the nontrivial proper subspaces $W_2/W_1$ of $\partial^{-1}W_1/W_1$ for which the intersection $W_2/W_1 \cap \partial^{-1}W_0/W_1$ is
trivial. Since $\partial^{-1}W_1/W_1$ has dimension 3 while its subspace $\partial^{-1}W_0/W_1$ has dimension 1, every such subspace $W_2/W_1$ has dimension either 1 or 2.

To help us define the subgroups $W_2$ belonging to $L_2(W_1)$, it will be convenient to identify each element of the vector space $\partial^{-1}W_1/W_1$ with its coordinate vector with respect to the ordered basis $y_2 + W_1, v_1 + W_1, v_2 + W_1$. In this way, we identify $\partial^{-1}W_1/W_1$ with the vector space $\mathbb{Z}_3 \times \mathbb{Z}_3 \times \mathbb{Z}_3$ consisting of row vectors. Under this identification, the elements $y_2 + W_1, v_1 + W_1, v_2 + W_1$ are associated with the so-called standard basis vectors $[1, 0, 0], [0, 1, 0], [0, 0, 1]$, in $\mathbb{Z}_3 \times \mathbb{Z}_3 \times \mathbb{Z}_3$.

The subgroups $W_2$ belonging to $L(W_1)$ are in one-to-one correspondence with the nontrivial proper subgroup $W_2/W_1$ of $\partial^{-1}W_1/W_1$ for which the intersection $W_2/W_1 \cap \partial^{-1}W_0/W_1$ is trivial. Note that $\partial^{-1}W_0/W_1$ is the 2-dimensional subspace generated by the element $y_2 + W_1$. Under our identification, each such subspace $W_2/W_1$ is associated with a subspace $S$ of $\mathbb{Z}_3 \times \mathbb{Z}_3 \times \mathbb{Z}_3$ that does not contain the standard basis vector $[1, 0, 0]$. Let $m$ denote the unique matrix in reduced row-echelon form that is row equivalent to the matrix whose rows are the members of an arbitrarily-chosen basis of such a subspace $S$.

In Case 7.2.1 we consider the 1-dimensional subspaces $W_2/W_1$. There are three possible forms for the matrix $m$. The first form is

$$m = [0, 0, 1]$$

(1 possibility), which is considered in Case 7.2.1.1. The second form is

$$m = [0, 1, c_4] \text{ for } c_4 \in \{0, 1, 2\}$$

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(3 possibilities), which is considered in Case 7.2.1.2. The third form is

\[ m = [1, c_4, c_5] \quad \text{for } c_4, c_5 \in \{0, 1, 2\}, \ (c_4, c_5) \neq (0, 0) \]

(8 possibilities), which is considered in Case 7.2.1.3.

In Case 7.2.2 we consider the 2-dimensional subspaces \( W_2/W_1 \). There are three possible forms for the matrix \( m \). The first form is

\[
m = \begin{bmatrix}
0 & 1 & 0 \\
0 & 0 & 1
\end{bmatrix}
\]

(1 possibility), which is considered in Case 7.2.2.1. The second form is

\[
m = \begin{bmatrix}
1 & c_4 & 0 \\
0 & 0 & 1
\end{bmatrix} \quad \text{for } c_4 \in \{1, 2\}
\]

(2 possibilities), which is considered in Case 7.2.2.2. The third form is

\[
m = \begin{bmatrix}
1 & 0 & c_4 \\
0 & 1 & c_5
\end{bmatrix} \quad \text{for } c_4 \in \{1, 2\} \quad \text{and } c_5 \in \{0, 1, 2\}
\]

(6 possibilities), which is considered in Case 7.2.2.3.

10.2.1 Case 7.2.1

We consider the 1-dimensional subspaces \( W_2/W_1 \). Let \( d_1, d_2, d_3 \) be unspecified variables. Let \( m_4 = d_1 y_2 + d_2 v_1 + d_3 v_2 \). A formal expression for \( m_4 \) is

\[
m_4 = d_1 \begin{bmatrix}
0 & 0 & 0 \\
0 & 3 & 0 \\
0 & 0 & 0
\end{bmatrix} + d_2 \begin{bmatrix}
0 & 0 & 0 \\
1 & 0 & 0 \\
0 & 0 & 0
\end{bmatrix} + d_3 \begin{bmatrix}
0 & 1 & 0 \\
0 & 0 & 3c_2 \\
0 & -3 & 0
\end{bmatrix} = \begin{bmatrix}
0 & d_3 & 0 \\
d_2 & 3d_1 & 3c_2d_3 \\
0 & -3d_3 & 0
\end{bmatrix}.
\]
Let $W_2 = \langle W_1, m_4 \rangle \in \mathcal{L}_2$. We now calculate the pullback $\partial^{-1} W_2$. The subgroup $W_2$ is contained in the pattern subgroup

$$
\begin{bmatrix}
2 & 2 & 1 \\
2 & 1 & 1 \\
1 & 1 & 0
\end{bmatrix}.
$$

Thus the pullback $\partial^{-1} W_2$ is contained in the pattern subgroup

$$
\begin{bmatrix}
2 & 2 & 2 \\
2 & 2 & 1 \\
2 & 1 & 1
\end{bmatrix} \cap
\begin{bmatrix}
2 & 2 & 2 \\
2 & 2 & 1 \\
2 & 1 & 1
\end{bmatrix} =
\begin{bmatrix}
2 & 2 & 2 \\
2 & 2 & 1 \\
2 & 1 & 1
\end{bmatrix}.
$$

Let

$$
x = \begin{bmatrix}
3s_{0,0} + t_{0,0} & 3s_{0,1} + t_{0,1} & 3s_{0,2} + t_{0,2} \\
3s_{1,0} + t_{1,0} & 3s_{1,1} + t_{1,1} & 3s_{1,2} \\
3s_{2,0} + t_{2,0} & 3s_{2,1} & 3s_{2,2}
\end{bmatrix} \in
\begin{bmatrix}
2 & 2 & 2 \\
2 & 2 & 1 \\
2 & 1 & 1
\end{bmatrix}.
$$

Thus

$$
\partial_1 x = \begin{bmatrix}
3s_{1,0} + t_{1,0} & 3s_{1,1} + t_{1,1} & 3s_{1,2} \\
3s_{2,0} + t_{2,0} & 3s_{2,1} & 3s_{2,2} \\
-3(t_{1,0} + t_{2,0}) & -3t_{1,1} & 0
\end{bmatrix}
$$

and

$$
\partial_2 x = \begin{bmatrix}
3s_{0,1} + t_{0,1} & 3s_{0,2} + t_{0,2} & -3(t_{0,1} + t_{0,2}) \\
3s_{1,1} + t_{1,1} & 3s_{1,2} & -3t_{1,1} \\
3s_{2,1} & 3s_{2,2} & 0
\end{bmatrix}.
$$
The variables that are in play are those appearing in the matrix

\[
\begin{pmatrix}
0 & t_{0,1} & t_{0,2} \\
t_{1,0} & t_{1,1} & 3s_{1,2} \\
t_{2,0} & 3s_{2,1} & 3s_{2,2}
\end{pmatrix}.
\]

We wish to identify values \(a_1, a_2, a_3, a_4 \in \mathbb{Z}_9\) such that \(\partial_1 x \equiv a_1 m_1 + a_2 m_2 + a_3 m_3 + a_4 m_4 \pmod{I}\). A formal expression for \(a_1 m_1 + a_2 m_2 + a_3 m_3 + a_4 m_4\) is

\[
\begin{pmatrix}
2 & 0 & 3 \\
0 & 0 & 0 \\
-3 & 0 & 0
\end{pmatrix} + a_2 \begin{pmatrix}
c_2 & 0 & -3c_2 \\
0 & 3 & 0 \\
-3c_2 & 0 & 0
\end{pmatrix} + a_3 \begin{pmatrix}
1 & 0 & 0 \\
0 & 0 & 0 \\
-3 & 0 & 0
\end{pmatrix} + a_4 \begin{pmatrix}
0 & d_3 & 0 \\
d_2 & 3d_1 & 3c_2d_3 \\
0 & -3d_3 & 0
\end{pmatrix} = \begin{pmatrix}
-a_1 + a_2c_2 + a_3 & a_4d_3 & 3(a_1 - a_2c_2) \\
a_4d_2 & 3(a_2 + a_4d_1) & -3a_4c_2d_3 \\
-3(a_1 + a_2c_2 + a_3) & -3a_4d_3 & 0
\end{pmatrix}.
\]

Comparing \((0,1)\)-entries, we get \(t_{1,1} \equiv a_4d_3\).

Comparing \((2,1)\)-entries, we get \(t_{1,1} \equiv a_4d_3\) which gives no new information.

Comparing \((1,0)\)-entries, we get \(t_{2,0} \equiv a_4d_2\).

Comparing \((1,2)\)-entries, we get \(s_{2,2} \equiv a_4c_2d_3\). Substituting \(a_4d_3 \equiv t_{1,1}\) we obtain \(s_{2,2} \equiv c_2t_{1,1}\).

Comparing \((1,1)\)-entries, we get \(a_2 \equiv s_{2,1} - a_4d_1\).

Comparing \((0,2)\)-entries, we get \(a_1 \equiv s_{1,2} + a_2c_2\). Substituting \(a_2\) we obtain \(a_1 \equiv s_{1,2} + c_2s_{2,1} - c_2a_4d_1\).
Comparing \((2,0)\)-entries, we get \(a_3 \equiv t_{1,0} + t_{2,0} - a_1 - c_2a_2\). Substituting \(a_1, a_2\) we
obtain \(a_3 \equiv t_{1,0} + t_{2,0} - s_{1,2} + c_2s_{2,1} - c_2a_4d_1\).

Comparing \((0,0)\)-entries, we get \(t_{1,0} \equiv -a_1 + c_2a_2 + a_3\). Substituting \(a_1, a_2,\) and \(a_3\)
we obtain \(s_{1,2} \equiv -c_2s_{2,1} + c_2a_4d_1 - t_{2,0}\).

We see that \(\partial_1 x \in W_2\) if and only if
\[ t_{1,1} \equiv a_4d_3 \quad \text{(A1)} \]
\[ t_{2,0} \equiv a_4d_2 \quad \text{(A2)} \]
\[ s_{2,2} \equiv c_2t_{1,1} \quad \text{(A3)} \]
\[ s_{1,2} \equiv -c_2s_{2,1} + c_2d_1a_4 - t_{2,0} \quad \text{(A4)}. \]

We wish to identify values \(b_1, b_2, b_3, b_4 \in \mathbb{Z}_9\) such that \(\partial_2 x \equiv b_1m_1 + b_2m_2 + b_3m_3 + b_4m_4\) (mod \(I\)). A formal expression for \(b_1m_1 + b_2m_2 + b_3m_3 + b_4m_4\) is
\[
\begin{bmatrix}
2 & 0 & 3 \\
0 & 0 & 0 \\
-3 & 0 & 0
\end{bmatrix}
+ b_2
\begin{bmatrix}
c_2 & 0 & -3c_2 \\
0 & 3 & 0 \\
-3c_2 & 0 & 0
\end{bmatrix}
+ b_3
\begin{bmatrix}
1 & 0 & 0 \\
0 & 0 & 0 \\
-3 & 0 & 0
\end{bmatrix}
+ b_4
\begin{bmatrix}
0 & d_3 & 0 \\
d_2 & 3d_1 & 3c_2d_3 \\
-3(b_1 + b_2c_2 + b_3) & -3b_4d_3 & 0
\end{bmatrix}
\]
\[
= \begin{bmatrix}
-b_1 + b_2c_2 + b_3 & b_4d_3 & 3(b_1 - b_2c_2) \\
& b_4d_2 & 3(b_2 + b_4d_1) & 3b_4c_2d_3 \\
& -3(b_1 + b_2c_2 + b_3) & -3b_4d_3 & 0
\end{bmatrix}.
\]

Comparing \((0,1)\)-entries, we get \(t_{0,2} \equiv b_4d_3\).

Comparing \((2,1)\)-entries, we get \(s_{2,2} \equiv -b_4d_3\). This implies \(t_{0,2} \equiv -s_{2,2}\).

Comparing \((1,0)\)-entries, we get \(t_{1,1} \equiv b_4d_2\).
Comparing (1,2)-entries, we get $-t_{1,1} \equiv b_4c_2d_3$. Substituting $b_4d_3$ we obtain $t_{1,1} \equiv c_2s_{2,2}$. Substituting $s_{2,2} \equiv -t_{0,2}$ we get $t_{1,1} \equiv -c_2t_{0,2}$.

Comparing (1,1)-entries, we get $b_2 \equiv s_{1,2} - b_4d_1$.

Comparing (0,2)-entries, we get $b_1 \equiv -t_{0,1} - t_{0,2} + b_2c_2$. Substituting $b_2$ we obtain $b_1 \equiv -t_{0,1} - t_{0,2} - c_2b_4d_1 + c_2s_{1,2}$.

Comparing (2,0)-entries, we get $b_3 \equiv -b_1 - c_2b_2 - s_{2,1}$. Substituting $b_1, b_2$ we obtain $b_3 \equiv t_{0,1} + t_{0,2} + c_2s_{1,2} - c_2b_4d_1 - s_{2,1}$.

Comparing (0,0)-entries, we get $t_{0,1} \equiv -b_1 + b_2c_2 + b_3$. Substituting $b_1, b_2, \text{ and } b_3$ we obtain $s_{1,2} \equiv c_2t_{0,2} - c_2t_{0,1} + b_4d_1 + c_2s_{2,1}$.

We see that $\partial_2 x \in W_2$ if and only if

\[ t_{0,2} \equiv b_4d_3 \quad \text{(B1)} \]

\[ s_{2,2} \equiv -t_{0,2} \quad \text{(B2)} \]

\[ t_{1,1} \equiv b_4d_2 \quad \text{(B3)} \]

\[ t_{1,1} \equiv -c_2t_{0,2} \quad \text{(B4)} \]

\[ s_{1,2} \equiv c_2t_{0,2} - c_2t_{0,1} + b_4d_1 + c_2s_{2,1} \quad \text{(B5).} \]

Case 7.2.1.1

Let $m_4 = v_2$. Thus

\[
\begin{bmatrix}
0 & 1 & 0 \\
0 & 0 & 3c_2 \\
0 & -3 & 0
\end{bmatrix}.
\]
Let $W_2 = < W_1, m_4 > \in \mathcal{L}_2$. The number of subgroups $W_2$ of this type is 1. Since $m_4 \notin W$ and $3m_4 \in W_1$ we have $|W_2/W_1| = 3$. Since $|W_1| = 3^6$ it follows that $|W_2| = 3^7$. Note that $\partial^{-1}W_1/W_2 \cong \mathbb{Z}_3 \times \mathbb{Z}_3$ and rank$(\partial^{-1}W_1/W_2) = 2$. We now calculate the pullback of $\partial^{-1}W_2$. We observed in case 7.2.1 that $\partial^{-1}W_2$ is contained in the pattern subgroup

$$
\begin{bmatrix}
2 & 2 & 2 \\
2 & 2 & 1 \\
2 & 1 & 1
\end{bmatrix}.
$$

Let

$$
x = \begin{bmatrix}
3s_{0,0} + t_{0,0} & 3s_{0,1} + t_{0,1} & 3s_{0,2} + t_{0,2} \\
3s_{1,0} + t_{1,0} & 3s_{1,1} + t_{1,1} & 3s_{1,2} \\
3s_{2,0} + t_{2,0} & 3s_{2,1} & 3s_{2,2}
\end{bmatrix} \in \begin{bmatrix}
2 & 2 & 2 \\
2 & 2 & 1 \\
2 & 1 & 1
\end{bmatrix}.
$$

The variables that are in play are those appearing in the matrix

$$
\begin{bmatrix}
0 & t_{0,1} & t_{0,2} \\
t_{1,0} & t_{1,1} & 3s_{1,2} \\
t_{2,0} & 3s_{2,1} & 3s_{2,2}
\end{bmatrix}.
$$

In the notation of Case 7.2.1, we are taking $d_1 = 0$, $d_2 = 0$, and $d_3 = 1$. We wish to identify values $a_1, a_2, a_3, a_4 \in \mathbb{Z}_9$ such that $\partial_1 x \equiv a_1 m_1 + a_2 m_2 + a_3 m_3 + a_4 m_4$ (mod $I$).
Hence \( x \in \partial^{-1}W_2 \) if and only if

\[
\begin{align*}
    t_{1,1} &\equiv a_4 \quad \text{(A1)} \\
    t_{2,0} &\equiv 0 \quad \text{(A2)} \\
    s_{2,2} &\equiv c_2 t_{1,1} \quad \text{(A3)} \\
    s_{1,2} &\equiv -c_2 s_{2,1} \quad \text{(A4)}.
\end{align*}
\]

We wish to identify values \( b_1, b_2, b_3, b_4 \in \mathbb{Z}_9 \) such that \( \partial_1 x \equiv b_1 m_1 + b_2 m_2 + b_3 m_3 + b_4 m_4 \pmod{I} \). We see that \( \partial_2 x \in W_2 \) if and only if

\[
\begin{align*}
    t_{0,2} &\equiv b_4 \quad \text{(B1)} \\
    s_{2,2} &\equiv -t_{0,2} \quad \text{(B2)} \\
    t_{1,1} &\equiv 0 \quad \text{(B3)} \\
    0 &\equiv -c_2 t_{0,2} \quad \text{(B4)} \\
    s_{1,2} &\equiv c_2 t_{0,2} - c_2 t_{0,1} + c_2 s_{2,1} \quad \text{(B5)}.
\end{align*}
\]

Since \( c_2 \neq 0 \) then congruence (B4) becomes \( t_{0,2} \equiv 0 \) and (B5) becomes

\[
    s_{1,2} \equiv -c_2 t_{0,1} + c_2 s_{2,1}.
\]
Hence $x \in \partial^{-1}W_2$ if and only if

\begin{align*}
t_{1,1} & \equiv 0 \\
t_{2,0} & \equiv 0 \\
s_{2,2} & \equiv 0 \\
t_{0,2} & \equiv 0 \\
s_{1,2} & \equiv -c_2s_{2,1} \\
s_{2,1} & \equiv -c_2t_{0,1} + c_2s_{2,1}.
\end{align*}

These are the same congruences as those in Case 7.2 therefore $\partial^{-1}W_2 = \partial^{-1}W_1$. Thus $\text{rank}(\partial^{-1}W_2/W_2) = \text{rank}(\partial^{-1}W_1/W_2)$, $W_2$ is terminal, and $W_2 \notin \hat{L}_2$.

Case 7.2.1.2

We fix arbitrary value $c_4 \in \{0, 1, 2\}$. There are three ways to choose the value $c_4$. Let $m_4 = v_1 + c_4v_2$. Thus

\[
m_4 = \begin{bmatrix}
0 & c_4 & 0 \\
1 & 0 & 3c_2c_4 \\
0 & -3c_4 & 0
\end{bmatrix}.
\]

Let $W_2 = < W_1, m_4 > \in L_2$. The number of subgroups $W_2$ of this type is 3. Since $m_4 \notin W_1$ and $3m_4 \in W_1$ we have $|W_2/W_1| = 3$. Since $|W_1| = 3^6$ it follows that $|W_2| = 3^7$. Note that $\partial^{-1}W_1/W_2 \cong \mathbb{Z}_3 \times \mathbb{Z}_3$ and $\text{rank}(\partial^{-1}W_1/W_2) = 2$. We now calculate the pullback of $\partial^{-1}W_2$. We observed in case 7.2.1 that $\partial^{-1}W_2$ is contained
in the pattern subgroup

\[
\begin{bmatrix}
2 & 2 & 2 \\
2 & 2 & 1 \\
2 & 1 & 1
\end{bmatrix}.
\]

Let

\[
x = \begin{bmatrix}
3s_{0,0} + t_{0,0} & 3s_{0,1} + t_{0,1} & 3s_{0,2} + t_{0,2} \\
3s_{1,0} + t_{1,0} & 3s_{1,1} + t_{1,1} & 3s_{1,2} \\
3s_{2,0} + t_{2,0} & 3s_{2,1} & 3s_{2,2}
\end{bmatrix} \in \begin{bmatrix}
2 & 2 & 2 \\
2 & 2 & 1 \\
2 & 1 & 1
\end{bmatrix}.
\]

The variables that are in play are those appearing in the matrix

\[
\begin{bmatrix}
0 & t_{0,1} & t_{0,2} \\
t_{1,0} & t_{1,1} & 3s_{1,2} \\
t_{2,0} & 3s_{2,1} & 3s_{2,2}
\end{bmatrix}.
\]

In the notation of Case 7.2.1, we are taking \( d_1 = 0, \ d_2 = 1, \) and \( d_3 = c_4. \) We wish to identify values \( a_1, a_2, a_3, a_4 \in \mathbb{Z}_9 \) such that \( \partial_1 x \equiv a_1 m_1 + a_2 m_2 + a_3 m_3 + a_4 m_4 \pmod{I}. \) We see that \( \partial_1 x \in W_2 \) if and only if

\[
t_{1,1} \equiv a_4 c_4 \quad (A1)
\]

\[
t_{2,0} \equiv a_4 \quad (A2)
\]

\[
s_{2,2} \equiv c_2 t_{1,1} \quad (A3)
\]

\[
s_{1,2} \equiv -c_2 s_{2,1} - t_{2,0} \quad (A4).
\]

We wish to identify values \( b_1, b_2, b_3, b_4 \in \mathbb{Z}_9 \) such that \( \partial_1 x \equiv b_1 m_1 + b_2 m_2 + b_3 m_3 + b_4 m_4 \pmod{I} \).
We see that $\partial_2 x \in W_2$ if and only if

$$t_{0,2} \equiv b_4 c_4 \quad \text{(B1)}$$

$$s_{2,2} \equiv -t_{0,2} \quad \text{(B2)}$$

$$t_{1,1} \equiv b_4 \quad \text{(B3)}$$

$$t_{1,1} \equiv -c_2 t_{0,2} \quad \text{(B4)}$$

$$s_{1,2} \equiv c_2 t_{0,2} - c_2 t_{0,1} + c_2 s_{2,1} \quad \text{(B5)}.$$

Substituting $a_4$ and $b_4$ into (A1) and (B1) we obtain $t_{1,1} \equiv c_4 t_{2,0}$ and $t_{0,2} \equiv c_4 t_{1,1}$. Combining (B1) and (B4) we obtain $(1 + c_2 c_4) t_{0,2} \equiv 0$ which we will denote as our new (B1). Combining (A4) and (B5) we obtain $s_{2,1} \equiv t_{0,2} - t_{0,1} + c_2 t_{2,0}$ which we will denote as our new (B5). Substituting (B5) into (A4) we obtain $s_{1,2} \equiv -c_2 t_{0,2} + c_2 t_{0,1} + t_{2,0}$.

Hence $x \in \partial^{-1} W_2$ if and only if

$$t_{1,1} \equiv c_4 t_{2,0} \quad \text{(A1)}$$

$$s_{2,2} \equiv c_2 t_{1,1} \quad \text{(A3)}$$

$$s_{1,2} \equiv -c_2 t_{0,2} + c_2 t_{0,1} + t_{2,0} \quad \text{(A4)}$$

$$(1 + c_2 c_4) t_{0,2} \equiv 0 \quad \text{(B1)}$$

$$s_{2,2} \equiv -t_{0,2} \quad \text{(B2)}$$

$$t_{1,1} \equiv -c_2 t_{0,2} \quad \text{(B4)}$$

$$s_{2,1} \equiv t_{0,2} - t_{0,1} + c_2 t_{2,0} \quad \text{(B5)}.$$

It is convenient to consider the cases $1 + c_2 c_4 \neq 0$ and $1 + c_2 c_4 = 0$ separately.
Case 7.2.1.2.1 Suppose $1 + c_2c_4 \neq 0$. Then $t_{0,2} \equiv 0$. It follows that $s_{2,2} \equiv t_{1,1} \equiv 0$. The congruence (A1) becomes $c_4t_{2,0} \equiv 0$. If $c_4 \neq 0$ then $t_{2,0} \equiv 0$. The congruences (A4) and (B5) become $s_{1,2} \equiv c_2t_{0,2}$ and $s_{2,1} \equiv -t_{0,1}$. These are the same congruences as those in Case 7.2 therefore $\partial^{-1}W_2 = \partial^{-1}W_1$. Thus $\rank(\partial^{-1}W_2/W_2) = \rank(\partial^{-1}W_1/W_2)$, $W_2$ is terminal, and $W_2 \notin \hat{L}_2$.

Therefore suppose $c_4 = 0$. Then $t_{2,0}$ is a free variable. We regard $t_{1,0}$, $t_{0,1}$, and $t_{2,0}$ as free variables. Taking $t_{1,0} \equiv 1$, $t_{0,1} = 0$, and $t_{2,0} = 0$, the matrix $x$ becomes

$$v_1 = \begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$  

Taking $t_{0,1} \equiv 1$, $t_{1,0} = 0$, and $t_{2,0} = 0$, the matrix $x$ becomes

$$v_2 = \begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & -3 \\ 0 & 3c_2 & 0 \end{bmatrix}.$$  

Taking $t_{0,2} \equiv 1$, $t_{0,1} = 0$, and $t_{1,0} = 0$, the matrix $x$ becomes

$$v_3 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 3 \\ 1 & 3c_2 & 0 \end{bmatrix}.$$  

We see that neither $v_3$ nor $3v_3$ is contained in $W_2$ and that $\partial^{-1}W_2 = < \partial^{-1}W_0, v_1, v_2, v_3 >$. We know that $v_3 \in \partial^{-1}W_2$ and since $3v_3 \notin W_2$ then $v_3 + W_2$ is an element of the group $\partial^{-1}W_2/W_2$ whose order is larger than 3. Recall that
$|\partial^{-1}W_2/W_2| = 3^3$ and that $\partial^{-1}W_1/W_2$ has rank 2. We obtain $\partial^{-1}W_2/W_2 \cong \mathbb{Z}_9 \times \mathbb{Z}_3$. Since rank($\partial^{-1}W_2/W_2$) = rank($\partial^{-1}W_1/W_2$). Hence $W_2$ is terminal and $W_2 \notin \hat{L}_2$.

**Case 7.2.1.2.2** Suppose $1 + c_2 c_4 \equiv 0$. Then $c_4 \equiv -c_2$ and $t_{0,2}$ is a free variable. Hence $x \in \partial^{-1}W_2$ if and only if

$$t_{1,1} \equiv -c_2 t_{2,0} \quad (A1)$$

$$s_{2,2} \equiv c_2 t_{1,1} \quad (A3)$$

$$s_{1,2} \equiv -c_2 t_{0,2} + c_2 t_{0,1} + t_{2,0} \quad (A4)$$

$$s_{2,2} \equiv -t_{0,2} \quad (B2)$$

$$t_{1,1} \equiv -c_2 t_{0,2} \quad (B4)$$

$$s_{2,1} \equiv t_{0,2} - t_{0,1} + c_2 t_{2,0} \quad (B5).$$

Combining (A1) and (B4) we obtain $t_{2,0} \equiv t_{0,2}$ which we will denote as our new (B4). Substituting (B4) into (A1) we obtain $t_{1,1} \equiv -c_2 t_{0,2}$.

Hence $x \in \partial^{-1}W_2$ if and only if

$$t_{2,0} \equiv t_{0,2}$$

$$t_{1,1} \equiv -c_2 t_{0,2}$$

$$s_{2,2} \equiv -t_{0,2}$$

$$s_{1,2} \equiv (1 - c_2) t_{0,2} + c_2 t_{0,2}$$

$$s_{2,1} \equiv (1 + c_2) t_{0,2} - t_{0,1}.$$
We regard \( t_{1,0}, t_{0,1}, \) and \( t_{0,2} \) as free variables. Taking \( t_{1,0} \equiv 1, t_{0,1} = 0, \) and \( t_{0,2} = 0, \) the matrix \( x \) becomes

\[
v_1 = \begin{bmatrix}
0 & 0 & 0 \\
1 & 0 & 0 \\
0 & 0 & 0
\end{bmatrix}.
\]

Taking \( t_{0,1} \equiv 1, t_{1,0} = 0, \) and \( t_{0,2} = 0, \) the matrix \( x \) becomes

\[
v_2 = \begin{bmatrix}
0 & 0 & 0 \\
1 & 0 & -3 \\
0 & 3c_2 & 0
\end{bmatrix}.
\]

Taking \( t_{0,2} \equiv 1, t_{0,1} = 0, \) and \( t_{1,0} = 0, \) the matrix \( x \) becomes

\[
v_3 = \begin{bmatrix}
0 & 0 & 1 \\
0 & -c_2 & 3(1 - c_2) \\
1 & 3(1 + c_2) & -3
\end{bmatrix}.
\]

We see that neither \( v_3 \) nor \( 3v_3 \) is contained in \( W_2 \) and that \( \partial^{-1}W_2 =< \partial^{-1}W_0, v_1, v_2, v_3 >. \) We know that \( v_3 \in \partial^{-1}W_2 \) and since \( 3v_3 \not\in W_2 \) then \( v_3 + W_2 \) is an element of the group \( \partial^{-1}W_2/W_2 \) whose order is larger than 3. Recall that \( |\partial^{-1}W_2/W_2| = 3^3 \) and that \( \partial^{-1}W_1/W_2 \) has rank 2. We obtain \( \partial^{-1}W_2/W_2 \cong \mathbb{Z}_9 \times \mathbb{Z}_3. \) Since \( \text{rank}(\partial^{-1}W_2/W_2) = \text{rank}(\partial^{-1}W_1/W_2) \). Hence \( W_2 \) is terminal and \( W_2 \notin \hat{\mathcal{L}}_2. \)
Case 7.2.1.3

We fix arbitrary values \( c_4, c_5 \in \{0, 1, 2\} \) where \((c_4, c_5) \neq (0, 0)\). There are 8 ways to choose the values \( c_4, c_5 \). Let \( m_4 = y_2 + c_4 v_1 + c_5 v_2 \). Thus

\[
  m_4 = \begin{bmatrix}
  0 & c_5 & 0 \\
  c_4 & 3 & 3c_2c_5 \\
  0 & -3c_5 & 0
  \end{bmatrix}.
\]

Let \( W_2 = < W_1, m_4 > \in \mathcal{L}_2 \). The number of subgroups \( W_2 \) of this type is 8. Since \( m_4 \notin W_1 \) and \( 3m_4 \in W_1 \) we have \( |W_2/W_1| = 3 \). Since \( |W_1| = 3^6 \) it follows that \( |W_2| = 3^7 \). Note that \( \partial^{-1}W_1/W_2 \cong \mathbb{Z}_3 \times \mathbb{Z}_3 \) and \( \text{rank}(\partial^{-1}W_1/W_2) = 2 \). We now calculate the pullback of \( \partial^{-1}W_2 \). We observed in case 7.2.1 that \( \partial^{-1}W_2 \) is contained in the pattern subgroup

\[
  \begin{bmatrix}
  2 & 2 & 2 \\
  2 & 2 & 1 \\
  2 & 1 & 1
  \end{bmatrix}.
\]

Let

\[
  x = \begin{bmatrix}
  3s_{0,0} + t_{0,0} & 3s_{0,1} + t_{0,1} & 3s_{0,2} + t_{0,2} \\
  3s_{1,0} + t_{1,0} & 3s_{1,1} + t_{1,1} & 3s_{1,2} \\
  3s_{2,0} + t_{2,0} & 3s_{2,1} & 3s_{2,2}
  \end{bmatrix} \in \begin{bmatrix}
  2 & 2 & 2 \\
  2 & 2 & 1 \\
  2 & 1 & 1
  \end{bmatrix}.
\]

The variables that are in play are those appearing in the matrix

\[
  \begin{bmatrix}
  0 & t_{0,1} & t_{0,2} \\
  t_{1,0} & t_{1,1} & 3s_{1,2} \\
  t_{2,0} & 3s_{2,1} & 3s_{2,2}
  \end{bmatrix}.
\]
In the notation of Case 7.2.1, we are taking $d_1 = 1$, $d_2 = c_4$, and $d_3 = c_5$. We wish to identify values $a_1, a_2, a_3, a_4 \in \mathbb{Z}_9$ such that $\partial_1 x \equiv a_1 m_1 + a_2 m_2 + a_3 m_3 + a_4 m_4 \pmod{I}$. We see that $\partial_1 x \in W_2$ if and only if

\[ t_{1,1} \equiv a_4 c_5 \quad (A1) \]
\[ t_{2,0} \equiv a_4 c_4 \quad (A2) \]
\[ s_{2,2} \equiv c_2 t_{1,1} \quad (A3) \]
\[ s_{1,2} \equiv -c_2 s_{2,1} + c_2 a_4 - t_{2,0} \quad (A4). \]

We wish to identify values $b_1, b_2, b_3, b_4 \in \mathbb{Z}_9$ such that $\partial_2 x \equiv b_1 m_1 + b_2 m_2 + b_3 m_3 + b_4 m_4 \pmod{I}$. We see that $\partial_2 x \in W_2$ if and only if

\[ t_{0,2} \equiv b_4 c_5 \quad (B1) \]
\[ s_{2,2} \equiv -t_{0,2} \quad (B2) \]
\[ t_{1,1} \equiv b_4 c_4 \quad (B3) \]
\[ t_{1,1} \equiv -c_2 t_{0,2} \quad (B4) \]
\[ s_{1,2} \equiv c_2 t_{0,2} - c_2 t_{0,1} + b_4 + c_2 s_{2,1} \quad (B5). \]

It is convenient to consider the cases $c_4 = 0$ and $c_4 \neq 0$ separately.

**Case 7.2.1.3.1** Suppose $c_4 = 0$. Then $c_5 \neq 0$ and $c_5^2 \equiv 1$. (B3) becomes $t_{1,1} \equiv 0$. Hence $a_4 \equiv 0$. It follows that $t_{0,2} \equiv s_{2,2} \equiv t_{2,0} \equiv 0$. The congruences (A4) and (B5) become $s_{1,2} \equiv -c_2 s_{2,1}$ and $s_{1,2} \equiv -c_2 t_{0,1} + c_2 s_{2,1}$. 

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Hence $x \in \partial^{-1}W_2$ if and only if

$$
t_{1,1} \equiv 0
$$

$$
t_{0,2} \equiv 0
$$

$$
s_{2,2} \equiv 0
$$

$$
s_{2,0} \equiv 0
$$

$$
s_{1,2} \equiv -c_2s_{2,1}
$$

$$
s_{1,2} \equiv -c_2t_{0,1} + c_2s_{2,1}.
$$

These are the same congruences as those in Case 7.2 therefore $\partial^{-1}W_2 = \partial^{-1}W_1$. Thus $\text{rank}(\partial^{-1}W_2/W_2) = \text{rank}(\partial^{-1}W_1/W_2)$, $W_2$ is terminal, and $W_2 \notin \hat{L}_2$.

**Case 7.2.1.3.2** Suppose $c_4 \neq 0$. Then $c_4^2 \equiv 1$. From (A2) nad (B3) we obtain $a_4 \equiv c_2t_{2,0}$ and $b_4 \equiv c_4t_{1,1}$. Substituting $a_4$ and $b_4$ into (A1) and (B1) we obtain $t_{1,1} \equiv c_4c_5t_{2,0}$ and $t_{0,2} \equiv c_4c_5t_{1,1}$. Combining (A4) and (B5) we obtain $-c_2s_{2,1} + c_2a_4 - t_{2,0} \equiv c_2t_{0,2} - c_2t_{0,1} + b_4 + c_2s_{2,1}$. Substituting $a_4$, $b_4$, and $t_{1,1}$ we obtain $s_{2,1} \equiv -t_{0,1} + (c_2 - c_4)t_{2,0} + (1 - c_4)t_{0,2}$ which we will denote as our new (B5). Substituting $a_4$, $b_4$, and $s_{2,1}$ into (A4) we obtain $s_{1,2} \equiv c_2t_{0,1} + (1 - c_2c_4)t_{2,0} + (c_2c_4 - c_2)t_{0,2}$. Combining (A1) and (B4) we get $t_{0,2} \equiv -c_2c_4c_5t_{2,0}$. Substituting this into (A4) and (B5) we obtain $s_{1,2} \equiv c_2t_{0,1} + (1 - c_2c_4 - c_5 + c_4c_5)t_{2,0}$ and $s_{2,1} \equiv -t_{0,1} + (c_2 - c_4 - c_2c_4c_5 + c_2c_5)t_{2,0}$.
Hence $x \in \partial^{-1}W_2$ if and only if

\[
t_{0,2} \equiv -c_2 c_4 c_5 t_{2,0} \\
t_{1,1} \equiv c_4 c_5 t_{2,0} \\
s_{2,2} \equiv c_2 c_4 c_5 t_{2,0} \\
s_{1,2} \equiv c_2 t_{0,1} + (1 - c_2 c_4 - c_5 + c_4 c_5) t_{2,0} \\
s_{2,1} \equiv -t_{0,1} + (c_2 - c_4 - c_2 c_4 c_5 + c_2 c_5) t_{2,0}.
\]

We regard $t_{1,0}$, $t_{0,1}$, and $t_{2,0}$ as free variables. Taking $t_{1,0} \equiv 1$, $t_{0,1} = 0$, and $t_{2,0} = 0$, the matrix $x$ becomes

\[
v_1 = \begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}.
\]

Taking $t_{0,1} \equiv 1$, $t_{1,0} = 0$, and $t_{2,0} = 0$, the matrix $x$ becomes

\[
v_2 = \begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & -3 \\ 0 & 3c_2 & 0 \end{bmatrix}.
\]

Taking $t_{2,0} \equiv 1$, $t_{0,1} = 0$, and $t_{1,0} = 0$, the matrix $x$ becomes

\[
v_3 = \begin{bmatrix} 0 & 0 & -c_2 c_4 c_5 \\ 0 & c_4 c_5 & 3(1 - c_2 c_4 - c_5 + c_4 c_5) \\ 1 & 3(c_2 - c_4 - c_2 c_4 c_5 + c_2 c_5) & 3c_2 c_4 c_5 \end{bmatrix}.
\]
We see that neither $v_3$ nor $3v_3$ is contained in $W_2$ and that

$$\partial^{-1}W_2 = \langle \partial^{-1}W_0, v_1, v_2, v_3 \rangle.$$  We know that $v_3 \in \partial^{-1}W_2$ and since $3v_3 \not\in W_2$ then $v_3 + W_2$ is an element of the group $\partial^{-1}W_2/W_2$ whose order is larger than 3. Recall that $|\partial^{-1}W_2/W_2| = 3^3$ and that $\partial^{-1}W_1/W_2$ has rank 2. We obtain $\partial^{-1}W_2/W_2 \cong \mathbb{Z}_9 \times \mathbb{Z}_3$.

Since rank($\partial^{-1}W_2/W_2$) = rank($\partial^{-1}W_1/W_2$). Hence $W_2$ is terminal and $W_2 \not\in \hat{L}_2$.

10.2.2 Case 7.2.2

We consider the 2-dimensional subspaces $W_2/W_1$. Let $d_1, d_2, d_3, e_1, e_2, e_3$ be unspecified variables. Let $m_4 = d_1 y_2 + d_2 v_1 + d_3 v_2$ and $m_5 = e_1 y_2 + e_2 v_1 + e_3 v_2$. In all the cases we consider the value of $e_1 = 0$ therefore we may exclude it from our expression of $m_5$. A formal expression for $m_4$ is

$$m_4 = d_1 \begin{bmatrix} 0 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 0 \end{bmatrix} + d_2 \begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} + d_3 \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 3c_2 \\ 0 & -3 & 0 \end{bmatrix} = \begin{bmatrix} 0 & d_3 & 0 \\ d_2 & 3d_1 & 3c_2d_3 \\ 0 & -3d_3 & 0 \end{bmatrix}.$$

A formal expression for $m_5$ is

$$m_5 = e_2 \begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} + e_3 \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 3c_2 \\ 0 & -3 & 0 \end{bmatrix} = \begin{bmatrix} 0 & e_3 & 0 \\ e_2 & 0 & 3c_2e_3 \\ 0 & -3e_3 & 0 \end{bmatrix}.$$

Let $W_2 = \langle W_1, m_4, m_5 \rangle \in \mathcal{L}_2$. We now calculate the pullback $\partial^{-1}W_2$. The subgroup $W_2$ is contained in the pattern subgroup

$$\begin{bmatrix} 2 & 2 & 1 \\ 2 & 1 & 1 \\ 1 & 1 & 0 \end{bmatrix}.$$
Thus the pullback $\partial^{-1}W_2$ is contained in the pattern subgroup

$$\begin{bmatrix} 2 & 2 & 2 \\ 2 & 2 & 1 \\ 2 & 1 & 1 \end{bmatrix} \cap \begin{bmatrix} 2 & 2 & 2 \\ 2 & 2 & 1 \\ 2 & 1 & 1 \end{bmatrix} = \begin{bmatrix} 2 & 2 \\ 2 & 1 \\ 2 & 1 \end{bmatrix}.$$ 

Let

$$x = \begin{bmatrix} 3s_{0,0} + t_{0,0} & 3s_{0,1} + t_{0,1} & 3s_{0,2} + t_{0,2} \\ 3s_{1,0} + t_{1,0} & 3s_{1,1} + t_{1,1} & 3s_{1,2} \\ 3s_{2,0} + t_{2,0} & 3s_{2,1} & 3s_{2,2} \end{bmatrix} \in \begin{bmatrix} 2 & 2 & 2 \\ 2 & 1 & 1 \end{bmatrix}.$$ 

Thus

$$\partial_1 x = \begin{bmatrix} 3s_{1,0} + t_{1,0} & 3s_{1,1} + t_{1,1} & 3s_{1,2} \\ 3s_{2,0} + t_{2,0} & 3s_{2,1} & 3s_{2,2} \\ -3(t_{1,0} + t_{2,0}) & -3t_{1,1} & 0 \end{bmatrix} \quad \text{and} \quad \partial_2 x = \begin{bmatrix} 3s_{0,1} + t_{0,1} & 3s_{0,2} + t_{0,2} & -3(t_{0,1} + t_{0,2}) \\ 3s_{1,1} + t_{1,1} & 3s_{1,2} & -3t_{1,1} \\ 3s_{2,1} & 3s_{2,2} & 0 \end{bmatrix}.$$ 

The variables that are in play are those appearing in the matrix

$$\begin{bmatrix} 0 & t_{0,1} & t_{0,2} \\ t_{1,0} & t_{1,1} & 3s_{1,2} \\ t_{2,0} & 3s_{2,1} & 3s_{2,2} \end{bmatrix}.$$ 

We wish to identify values $a_1, a_2, a_3, a_4, a_5 \in \mathbb{Z}_9$ such that $\partial_1 x \equiv a_1m_1 + a_2m_2 + a_3m_3 + a_4m_4 + a_5m_5 \pmod{I}$. A formal expression for $a_1m_1 + a_2m_2 + a_3m_3 + a_4m_4 + a_5m_5$ (mod I).
\[ a_4 m_4 + a_5 m_5 \]
\[
\begin{bmatrix}
2 & 0 & 3 \\
0 & 0 & 0 \\
-3 & 0 & 0
\end{bmatrix}
+ a_2
\begin{bmatrix}
c_2 & 0 & -3c_2 \\
0 & 3 & 0 \\
-3c_2 & 0 & 0
\end{bmatrix}
+ a_3
\begin{bmatrix}
1 & 0 & 0 \\
0 & 0 & 0 \\
-3 & 0 & 0
\end{bmatrix}
+ a_4
\begin{bmatrix}
d_2 & 3d_1 & 3c_2 d_3 \\
0 & 0 & 0 \\
0 & -3d_3 & 0
\end{bmatrix}
+ a_5
\begin{bmatrix}
e_3 & 0 \\
e_2 & 0 & 3c_2 e_3 \\
0 & -3e_3 & 0
\end{bmatrix}
\]
\[
= \begin{bmatrix}
-a_1 + a_2 c_2 + a_3 & a_4 d_3 + a_5 e_3 & 3(a_1 - a_2 c_2) \\
a_4 d_2 + a_5 e_2 & 3(a_2 + a_4 d_1) & 3c_2(a_4 d_3 + a_5 e_3)
\end{bmatrix}.
\]

Comparing (0,1)-entries, we get \( t_{1,1} \equiv a_4 d_3 + a_5 e_3 \).

Comparing (2,1)-entries, we get \( t_{1,1} \equiv a_4 d_3 + a_5 e_3 \) which gives us no new information.

Comparing (1,0)-entries, we get \( t_{2,0} \equiv a_4 d_2 + a_5 e_2 \).

Comparing (1,2)-entries, we get \( s_{2,2} \equiv c_2(a_4 d_3 + a_5 e_3) \). Substituting \( a_4 d_3 + a_5 e_3 \equiv t_{1,1} \)
we obtain \( s_{2,2} \equiv c_2 t_{1,1} \).

Comparing (1,1)-entries, we get \( a_2 \equiv s_{2,1} - a_4 d_1 \).

Comparing (0,2)-entries, we get \( a_1 \equiv s_{1,2} + c_2 a_2 \). Substituting \( a_2 \) we obtain \( a_1 \equiv s_{1,2} + c_2 s_{2,1} - c_2 a_4 d_1 \).

Comparing (2,0)-entries, we get \( a_3 \equiv t_{1,0} + t_{2,0} - a_1 - c_2 a_2 \). Substituting \( a_1, a_2 \) we obtain \( a_3 \equiv t_{1,0} + t_{2,0} - s_{1,2} + c_2 s_{2,1} - c_2 a_4 d_1 \).

Comparing (0,0)-entries, we get \( -a_1 + a_2 c_2 + a_3 \equiv t_{1,0} \). Substituting \( a_1, a_2, a_3 \), we obtain \( s_{1,2} \equiv -c_2 s_{2,1} + c_2 a_4 d_1 - t_{2,0} \).
We see that $\partial_1 x \in W_2$ if and only if

$$t_{1,1} \equiv a_4 d_3 + a_5 e_3 \quad (A1)$$

$$t_{2,0} \equiv a_4 d_2 + a_4 e_2 \quad (A2)$$

$$s_{2,2} \equiv c_2 t_{1,1} \quad (A3)$$

$$s_{1,2} \equiv -c_2 s_{2,1} + c_2 a_4 d_1 - t_{2,0} \quad (A4).$$

We wish to identify values $b_1, b_2, b_3, b_4, b_5 \in \mathbb{Z}_9$ such that $\partial_2 x \equiv b_1 m_1 + b_2 m_2 + b_3 m_3 + b_4 m_4 + b_5 m_5 \pmod{I}$. A formal expression for $b_1 m_1 + b_2 m_2 + b_3 m_3 = b_4 m_4 + b_5 m_5$ is

\[
\begin{pmatrix}
2 & 0 & 3 \\
0 & 0 & 0 \\
-3 & 0 & 0
\end{pmatrix}
\begin{pmatrix}
b_1 \\
b_2 \\
b_3
\end{pmatrix}
+ \begin{pmatrix}
0 & e_3 & 0 \\
e_2 & 0 & 3c_2 e_3 \\
0 & -3e_3 & 0
\end{pmatrix}
= \begin{pmatrix}
c_2 & 0 & -3c_2 \\
0 & 0 & 0 \\
-3c_2 & 0 & 0
\end{pmatrix}
\begin{pmatrix}
1 & 0 & 0 \\
0 & d_3 & 0 \\
-3 & 0 & 0
\end{pmatrix}
\begin{pmatrix}
b_4 \\
b_5
\end{pmatrix}
= \begin{pmatrix}
t_0 \cdot b_4 & b_4 d_2 + b_5 e_2 & 3(b_2 + b_4 d_1) \\
b_1 + b_2 c_2 + b_3 & b_4 d_3 + b_5 e_3 & 3(b_1 - b_2 c_2) \\
0 & 0 & 0
\end{pmatrix}
\begin{pmatrix}
t_0 \cdot b_4 & b_4 d_3 + b_5 e_3 \\
-3(b_1 + c_2 b_2 + b_3) & -3(b_4 d_3 + b_5 e_3) & 0
\end{pmatrix}.
\]

Comparing (0,1)-entries, we get $t_{0,2} \equiv b_4 d_3 + b_5 e_3$.

Comparing (2,1)-entries, we get $s_{2,2} \equiv -(b_4 d_3 + b_5 e_3)$. Substituting $b_4 d - 3 + b_5 e_3 \equiv t_{0,2}$ we obtain $s_{2,2} \equiv -t_{0,2}$.

Comparing (1,0)-entries, we get $t_{1,1} \equiv b_4 d_2 + b_5 e_2$.

Comparing (1,2)-entries, we get $-t_{1,1} \equiv c_2 (b_4 d_3 + b_5 e_3)$. Substituting $b_4 d_3 + b_5 e_3 \equiv t_{0,2}$ we obtain $t_{1,1} \equiv -c_2 t_{0,2}$.

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Comparing $(1,1)$-entries, we get $b_2 \equiv s_{1,2} - b_4 d_1$.

Comparing $(0,2)$-entries, we get $b_1 \equiv -t_{0,1} - t_{0,2} + b_2 c_2$. Substituting $b_2$ we obtain $b_1 \equiv -t_{0,1} - t_{0,2} + c_2 s_{1,2} - c_2 b_4 d_1$.

Comparing $(2,0)$-entries, we get $b_3 \equiv -b_1 - b_2 c_2 - s_{2,1}$. Substituting $b_1, b_2$ we obtain $b_3 \equiv t_{0,1} + t_{0,2} + c_2 s_{1,2} - c_2 b_4 d_1 - s_{2,1}$.

Comparing $(0,0)$-entries, we get $-b_1 + b_2 c_2 + b_3 \equiv t_{0,1}$. Substituting $b_1, b_2, b_3$, we obtain $s_{1,2} \equiv c_2 t_{0,2} - c_2 t_{0,1} + b_4 d_1 + c_2 s_{2,1}$.

We see that $\partial_2 x \in W_2$ if and only if

$$t_{0,2} \equiv b_4 d_3 + b_5 e_3 \quad \text{(B1)}$$

$$s_{2,2} \equiv -t_{0,2} \quad \text{(B2)}$$

$$t_{1,1} \equiv b_4 d_2 + b_5 e_2 \quad \text{(B3)}$$

$$t_{1,1} \equiv -c_2 t_{0,2} \quad \text{(B4)}$$

$$s_{1,2} \equiv c_2 t_{0,2} - c_2 t_{0,1} + b_4 d_1 + c_2 s_{2,1} \quad \text{(B5)}.$$  

Case 7.2.2.1

Let $m_4 = v_1$ and $m_5 = v_2$. Thus

$$m_4 = \begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \quad \text{and} \quad m_5 = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 3c_2 \\ 0 & -3 & 0 \end{bmatrix}.$$

Let $W_2 = \langle W_1, m_4, m_5 \rangle \in \mathcal{L}_2$. The number of subgroups $W_2$ of this type is 1. Since $m_4, m_5 \notin W_1$ and $3m_4, 3m_5 \in W_1$ we have $|W_2/W_1| = 3^2$. Since $|W_1| = 3^6$ it follows that $|W_2| = 3^8$. Note that $\partial^{-1}W_1/W_2 \cong \mathbb{Z}_3$ and $\text{rank}(\partial^{-1}W_1/W_2) = 1$. 

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We now calculate the pullback of $\partial^{-1}W_2$. We observed in case 7.2.2 that $\partial^{-1}W_2$ is contained in the pattern subgroup

$$\begin{bmatrix} 2 & 2 & 2 \\ 2 & 2 & 1 \\ 2 & 1 & 1 \end{bmatrix}.$$ 

Let

$$x = \begin{bmatrix} 3s_{0,0} + t_{0,0} & 3s_{0,1} + t_{0,1} & 3s_{0,2} + t_{0,2} \\ 3s_{1,0} + t_{1,0} & 3s_{1,1} + t_{1,1} & 3s_{1,2} \\ 3s_{2,0} + t_{2,0} & 3s_{2,1} & 3s_{2,2} \end{bmatrix} \in \begin{bmatrix} 2 & 2 & 2 \\ 2 & 2 & 1 \\ 2 & 1 & 1 \end{bmatrix}.$$ 

The variables that are in play are those appearing in the matrix

$$\begin{bmatrix} 0 & t_{0,1} & t_{0,2} \\ t_{1,0} & t_{1,1} & 3s_{1,2} \\ t_{2,0} & 3s_{2,1} & 3s_{2,2} \end{bmatrix}.$$ 

In the notation of Case 7.2.2, we are taking $d_1 = 0$, $d_2 = 1$, $d_3 = 0$, $e_2 = 0$, and $e_3 = 1$. We wish to identify values $a_1, a_2, a_3, a_4, a_5 \in \mathbb{Z}_9$ such that $\partial_1 x \equiv a_1 m_1 + a_2 m_2 + a_3 m_3 + a_4 m_4 + a_5 m_5 \pmod{I}$. We see that $\partial_1 x \in W_2$ if and only if

$$t_{1,1} \equiv a_5 \quad (A1)$$

$$t_{2,0} \equiv a_4 \quad (A2)$$

$$s_{2,2} \equiv c_2 t_{1,1} \quad (A3)$$

$$s_{1,2} \equiv -c_2 s_{2,1} - t_{2,0} \quad (A4).$$
We wish to identify values $b_1, b_2, b_3, b_4, b_5 \in \mathbb{Z}_9$ such that $\partial_1 x \equiv b_1 m_1 + b_2 m_2 + b_3 m_3 + b_4 m_4 + b_5 m_5 \pmod{I}$. We see that $\partial_2 x \in W_2$ if and only if

$$t_{0,2} \equiv b_5 \quad \text{(B1)}$$
$$s_{2,2} \equiv -t_{0,2} \quad \text{(B2)}$$
$$t_{1,1} \equiv b_4 \quad \text{(B3)}$$
$$t_{1,1} \equiv -c_2 t_{0,2} \quad \text{(B4)}$$
$$s_{1,2} \equiv c_2 t_{0,2} - c_2 t_{0,1} + c_2 s_{2,1} \quad \text{(B5)}.$$

Combining (A4) and (B5) we obtain $s_{2,1} \equiv t_{0,2} - c_2 t_{0,1} + c_2 t_{2,0}$ which we will denote as our new (B5). Substituting (B5) into (A4) we obtain $s_{1,2} \equiv -c_2 t_{0,2} + t_{0,1} + t_{2,0}$. Hence $x \in \partial^{-1} W_2$ if and only if

$$s_{2,2} \equiv c_2 t_{1,1} \quad \text{(A3)}$$
$$s_{1,2} \equiv -c_2 t_{0,2} + t_{0,1} + t_{2,0} \quad \text{(A4)}$$
$$s_{2,2} \equiv -t_{0,2} \quad \text{(B2)}$$
$$t_{1,1} \equiv -c_2 t_{0,2} \quad \text{(B4)}$$
$$s_{2,1} \equiv t_{0,2} - c_2 t_{0,2} + c_2 t_{2,0} \quad \text{(B5)}.$$

We regard $t_{1,0}, t_{0,1}, t_{0,2}, t_{2,0}$ as the free variables. Taking $t_{1,0} \equiv 1, t_{0,1} = 0, t_{0,2} = 0,$ and $t_{2,0} = 0$, the matrix $x$ becomes

$$v_1 = \begin{bmatrix}
0 & 0 & 0 \\
1 & 0 & 0 \\
0 & 0 & 0
\end{bmatrix}.$$
Taking \( t_{0,1} \equiv 1, t_{1,0} = 0, t_{0,2} = 0, \) and \( t_{2,0} = 0, \) the matrix \( x \) becomes
\[
v_2 = \begin{bmatrix}
0 & 1 & 0 \\
0 & 0 & 3c_2 \\
0 & -3 & 0
\end{bmatrix}.
\]
Taking \( t_{0,2} \equiv 1, t_{0,1} = 0, t_{1,0} = 0, \) and \( t_{2,0} = 0, \) the matrix \( x \) becomes
\[
v_3 = \begin{bmatrix}
0 & 0 & 1 \\
0 & -c_2 & -3c_2 \\
0 & 3 & -3
\end{bmatrix}.
\]
Taking \( t_{2,0} \equiv 1, t_{0,1} = 0, t_{1,0} = 0, \) and \( t_{0,2} = 0, \) the matrix \( x \) becomes
\[
v_4 = \begin{bmatrix}
0 & 0 & 0 \\
0 & 0 & 3 \\
1 & 3c_2 & -3
\end{bmatrix}.
\]

We see that \( \partial^{-1}W_2 = \langle \partial^{-1}W_1, v_3, v_4 \rangle. \) Hence \( |\partial^{-1}W_2/\partial^{-1}W_1| = 3^2. \) Recall that \( |\partial^{-1}W_1| = 3^9 \) hence \( |\partial^{-1}W_2| = 3^{11}. \) Since \( |W_2| = 3^8, \) then \( |\partial^{-1}W_2/W_2| = 3^3. \)

We obtain \( \partial^{-1}W_2/W_2 \cong \mathbb{Z}_9 \times \mathbb{Z}_3 \) or \( \mathbb{Z}_3 \times \mathbb{Z}_3 \times \mathbb{Z}_3. \) We see that neither \( v_4 \) nor \( 3v_4 \) is contained in \( W_2. \) We know that \( v_4 \in \partial^{-1}W_2 \) and since \( 3v_4 \notin W_2 \) then \( v_4 + W_2 \) is an element of the group \( \partial^{-1}W_2/W_2 \) whose order is larger than 3. Note
\[
-c_2m_2 + m_3 = \begin{bmatrix}
-1 & 0 & - \\
0 & -3c_2 & 0 \\
3 & 0 & 0
\end{bmatrix} + \begin{bmatrix}
1 & 0 & 0 \\
0 & 0 & 0 \\
-3 & 0 & 0
\end{bmatrix} = \begin{bmatrix}
0 & 0 & 3 \\
0 & -3c_2 & 0 \\
0 & 0 & 0
\end{bmatrix} = 3v_3.
\]

Hence \( 3v_3 \in W_2 \) and so \( v_3 + W_2 \) is an element of order 3. Hence \( \partial^{-1}W_2/W_2 \cong \mathbb{Z}_9 \times \mathbb{Z}_3 \) and \( \text{rank}(\partial^{-1}W_2/W_2) = 2. \) Recall that \( \partial^{-1}W_1/W_2 \) has rank 1. Since
\[ \text{rank}(\partial^{-1}W_1/W_2) < \text{rank}(\partial^{-1}W_2/W_2) \] then \( W_2 \) is nonterminal and \( W_2 \in \hat{L}_2 \). Note \( \Omega_1(\partial^{-1}W_2/W_2) \cong \mathbb{Z}_3 \times \mathbb{Z}_3 \). Thus \( \Omega_1(\partial^{-1}W_2/W_2) \) has basis \( y_2 + W_1, v_3 + W_2 \) while its subspace \( \partial^{-1}W_1/W_2 \) has basis \( y_2 + W_2 \).

Let \( c_4 \in \{0, 1, 2\} \). Let \( m_6 = c_4y_2 + v_3 \). Thus

\[
\begin{bmatrix}
0 & 0 & 1 \\
0 & 3c_4 - c_2 & -3c_2 \\
0 & 3 & -3
\end{bmatrix}
\]

Let \( W_3 = \langle W_2, m_6 \rangle \). The number of subgroups of this type is 9. Since \( m_6 \not\in W_2 \) while \( 3m_6 \in W_2 \) we have \( |W_3/W_2| = 3 \). Recalling \( |W_2| = 3^8 \) we get \( |W_3| = 3^9 \). Since \( |W_3/W_2| = 3 \) and the antidiagonal of \( m_5 \) has an entry not divisible by 3, by the Modified Terminal Lemma we conclude that \( \partial^{-1}W_3 = \partial^{-1}W_2 \). Thus \( \text{rank}(\partial^{-1}W_3/W_3) = \text{rank}(\partial^{-1}W_2/W_3) \). \( W_3 \) is terminal, and \( W_3 \not\in \hat{L}_3 \).

Case 7.2.2.2

We fix arbitrary value \( c_4 \in \{1, 2\} \). There are 2 ways to choose the value \( c_4 \). Let \( m_4 = y_2 + c_4v_1 \) and \( m_5 = v_2 \). Thus

\[
\begin{bmatrix}
0 & 0 & 0 \\
c_4 & 3 & 0 \\
0 & 0 & 0
\end{bmatrix}
\text{ and }
\begin{bmatrix}
0 & 1 & 0 \\
0 & 0 & 3c_2 \\
0 & -3 & 0
\end{bmatrix}
\]

Let \( W_2 = \langle W_1, m_4, m_5 \rangle \in L_2 \). The number of subgroups \( W_2 \) of this type is 2. Since \( m_4, m_5 \not\in W_1 \) and \( 3m_4, 3m_5 \in W_1 \) we have \( |W_2/W_1| = 3^2 \). Since \( |W_1| = 3^6 \) it follows that \( |W_2| = 3^8 \). Note that \( \partial^{-1}W_1/W_2 \cong \mathbb{Z}_3 \) and \( \text{rank}(\partial^{-1}W_1/W_2) = 1 \).
We now calculate the pullback of $\partial^{-1}W_2$. We observed in case 7.2.2 that $\partial^{-1}W_2$ is contained in the pattern subgroup

$$
\begin{bmatrix}
2 & 2 & 2 \\
2 & 2 & 1 \\
2 & 1 & 1 \\
\end{bmatrix}.
$$

Let

$$
x = \begin{bmatrix}
3s_{0,0} + t_{0,0} & 3s_{0,1} + t_{0,1} & 3s_{0,2} + t_{0,2} \\
3s_{1,0} + t_{1,0} & 3s_{1,1} + t_{1,1} & 3s_{1,2} \\
3s_{2,0} + t_{2,0} & 3s_{2,1} & 3s_{2,2} \\
\end{bmatrix} \in \begin{bmatrix}
2 & 2 & 2 \\
2 & 2 & 1 \\
2 & 1 & 1 \\
\end{bmatrix}.
$$

The variables that are in play are those appearing in the matrix

$$
\begin{bmatrix}
0 & t_{0,1} & t_{0,2} \\
t_{1,0} & t_{1,1} & 3s_{1,2} \\
t_{2,0} & 3s_{2,1} & 3s_{2,2} \\
\end{bmatrix}.
$$

In the notation of Case 7.2.2, we are taking $d_1 = 1$, $d_2 = c_4$, $d_3 = 0$, $e_2 = 0$, and $e_3 = 1$. We wish to identify values $a_1, a_2, a_3, a_4, a_5 \in \mathbf{Z}_9$ such that $\partial_1 x \equiv a_1 m_1 + a_2 m_2 + a_3 m_3 + a_4 m_4 + a_5 m_5 \pmod{I}$. We see that $\partial_1 x \in W_2$ if and only if

$$
t_{1,1} \equiv a_5 \quad (A1)
$$

$$
t_{2,0} \equiv a_4 c_4 \quad (A2)
$$

$$
s_{2,2} \equiv c_2 t_{1,1} \quad (A3)
$$

$$
s_{1,2} \equiv -c_2 s_{2,1} + c_2 a_4 - t_{2,0} \quad (A4).
$$
We wish to identify values \( b_1, b_2, b_3, b_4, b_5 \in \mathbb{Z}_9 \) such that \( \partial_1 x \equiv b_1 m_1 + b_2 m_2 + b_3 m_3 + b_4 m_4 + b_5 m_5 \pmod{I} \).

Hence \( x \in \partial^{-1} W_2 \) if and only if

\[
\begin{align*}
t_{0,2} &\equiv b_5 \quad (B1) \\
s_{2,2} &\equiv -t_{0,2} \quad (B2) \\
t_{1,1} &\equiv b_4 c_4 \quad (B3) \\
t_{1,1} &\equiv -c_2 t_{0,2} \quad (B4) \\
s_{1,2} &\equiv c_2 t_{0,2} - c_2 t_{0,1} + b_4 + c_2 s_{2,1} \quad (B5).
\end{align*}
\]

(A2) becomes \( a_4 \equiv c_4 t_{2,0} \) and (B3) becomes \( b_4 \equiv c_4 t_{1,1} \). Combining (A4) and (B5) we obtain \( -c_2 s_{2,1} + c_2 a_4 - t_{2,0} \equiv c_2 t_{0,2} - c_2 t_{0,1} + b_4 + c_2 s_{2,1} \). Substituting \( a_4 \) and \( b_4 \) we obtain \( s_{2,1} \equiv (c_2 - c_4) t_{2,0} + (1 - c_4) t_{0,2} - t_{0,1} \) which we will denote as our new (B5).

Substituting (B5) into (A4) we obtain \( s_{1,2} \equiv (1 - c_2 c_4) t_{2,0} + (c_2 c_4 - c_2) t_{0,2} + c_2 t_{0,1} \).

Hence \( x \in \partial^{-1} W_2 \) if and only if

\[
\begin{align*}
s_{2,2} &\equiv c_2 t_{1,1} \quad (A3) \\
s_{1,2} &\equiv (1 - c_2 c_4) t_{2,0} + (c_2 c_4 - c_2) t_{0,2} + c_2 t_{0,1} \quad (A4) \\
s_{2,2} &\equiv -t_{0,2} \quad (B2) \\
t_{1,1} &\equiv -c_2 t_{0,2} \quad (B4) \\
s_{2,1} &\equiv (c_2 - c_4) t_{2,0} + (1 - c_4) t_{0,2} - t_{0,1} \quad (B5).
\end{align*}
\]
We regard $t_{1,0}$, $t_{0,1}$, $t_{0,2}$, $t_{2,0}$ as the free variables. Taking $t_{1,0} \equiv 1$, $t_{0,2} = 0$, and $t_{2,0} = 0$, the matrix $x$ becomes
\[
v_1 = \begin{bmatrix}
0 & 0 & 0 \\
1 & 0 & 0 \\
0 & 0 & 0
\end{bmatrix}.
\]

Taking $t_{0,1} \equiv 1$, $t_{1,0} = 0$, $t_{0,2} = 0$, and $t_{2,0} = 0$, the matrix $x$ becomes
\[
v_2 = \begin{bmatrix}
0 & 1 & 0 \\
0 & 0 & 3c_2 \\
0 & -3 & 0
\end{bmatrix}.
\]

Taking $t_{0,2} \equiv 1$, $t_{0,1} = 0$, $t_{1,0} = 0$, and $t_{2,0} = 0$, the matrix $x$ becomes
\[
v_3 = \begin{bmatrix}
0 & 0 & 1 \\
0 & -c_2 & 3(c_2c_4 - c_2) \\
0 & 3(1 - c_4) & -3
\end{bmatrix}.
\]

Taking $t_{2,0} \equiv 1$, $t_{0,1} = 0$, $t_{1,0} = 0$, and $t_{0,2} = 0$, the matrix $x$ becomes
\[
v_4 = \begin{bmatrix}
0 & 0 & 0 \\
0 & 0 & 3(1 - c_2c_4) \\
1 & 3(c_2 - c_4) & 0
\end{bmatrix}.
\]

We see that $\partial^{-1}W_2 = \langle \partial^{-1}W_1, v_3, v_4 \rangle$. Hence $|\partial^{-1}W_2/\partial^{-1}W_1| = 3^2$. Recall that $|\partial^{-1}W_1| = 3^9$ hence $|\partial^{-1}W_2| = 3^{11}$. Since $|W_2| = 3^8$, then $|\partial^{-1}W_2/W_2| = 3^3$. We obtain $\partial^{-1}W_2/W_2 \cong \mathbb{Z}_9 \times \mathbb{Z}_3$ or $\mathbb{Z}_3 \times \mathbb{Z}_3 \times \mathbb{Z}_3$. We see that neither $v_4$ nor $3v_4$ is contained in $W_2$. We know that $v_4 \in \partial^{-1}W_2$ and since $3v_4 \not\in W_2$ then $v_4 + W_2$ is an
element of the group $\partial^{-1}W_2/W_2$ whose order is larger than 3. Note

$$-c_2m_2 + m_3 = \begin{bmatrix} -1 & 0 & - \\ 0 & -3c_2 & 0 \\ 3 & 0 & 0 \end{bmatrix} + \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ -3 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 3 \\ 0 & -3c_2 & 0 \\ 0 & 0 & 0 \end{bmatrix} = 3v_3.$$  

Hence $3v_3 \in W_2$ and so $v_3 + W_2$ is an element of order 3. Hence $\partial^{-1}W_2/W_2 \cong \mathbb{Z}_9 \times \mathbb{Z}_3$ and $\text{rank}(\partial^{-1}W_2/W_2) = 2$. Recall that $\partial^{-1}W_1/W_2$ has rank 1. Since $\text{rank}(\partial^{-1}W_1/W_2) < \text{rank}(\partial^{-1}W_2/W_2)$ then $W_2$ is nonterminal and $W_2 \in \hat{\mathcal{L}}_2$. Note $\Omega_1(\partial^{-1}W_2/W_2) \cong \mathbb{Z}_3 \times \mathbb{Z}_3$. Thus $\Omega_1(\partial^{-1}W_2/W_2)$ has basis $y_2 + W_1, v_3 + W_2$ while its subspace $\partial^{-1}W_1/W_2$ has basis $y_2 + W_2$.

Let $c_5 \in \{0, 1, 2\}$. Let $m_6 = c_5 y_2 + v_3$. Thus

$$m_6 = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 3c_5 - c_2 & 3(c_2c_4 - c_2) \\ 0 & 3(1 - c_4) & -3 \end{bmatrix}.$$  

Let $W_3 = < W_2, m_6 >$. The number of subgroups of this type is 9. Since $m_6 \not\in W_2$ while $3m_6 \in W_2$ we have $|W_3/W_2| = 3$. Recalling $|W_2| = 3^8$ we get $|W_3| = 3^9$. Since $|W_3/W_2| = 3$ and the antidiagonal of $m_5$ has an entry not divisible by 3, by the Modified Terminal Lemma we conclude that $\partial^{-1}W_3 = \partial^{-1}W_2$. Thus $\text{rank}(\partial^{-1}W_3/W_3) = \text{rank}(\partial^{-1}W_2/W_3)$, $W_3$ is terminal, and $W_3 \not\in \hat{\mathcal{L}}_3$. 

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Case 7.2.2.3

We fix arbitrary values $c_4 \in \{1, 2\}$ and $c_5 \in \{0, 1, 2\}$. There are 6 ways to choose the values $c_4, c_5$. Let $m_4 = y_2 + c_4v_2$ and $m_5 = v_1 + c_5v_2$. Thus

$$m_4 = \begin{bmatrix} 0 & c_4 & 0 \\ 0 & 3 & 3c_2c_4 \\ 0 & -3c_4 & 0 \end{bmatrix} \quad \text{and} \quad m_5 = \begin{bmatrix} 0 & c_5 & 0 \\ 1 & 0 & 3c_2c_5 \\ 0 & -3c_5 & 0 \end{bmatrix}.$$ 

Let $W_2 = \langle W_1, m_4, m_5 \rangle \in \mathcal{L}_2$. The number of subgroups $W_2$ of this type is 6. Since $m_4, m_5 \notin W_1$ and $3m_4, 3m_5 \in W_1$ we have $|W_2/W_1| = 3^2$. Since $|W_1| = 3^6$ it follows that $|W_2| = 3^8$. Note that $\partial^{-1}W_1/W_2 \cong \mathbb{Z}_3$ and $\text{rank}(\partial^{-1}W_1/W_2) = 1$.

We now calculate the pullback of $\partial^{-1}W_2$. We observed in case 7.2.2 that $\partial^{-1}W_2$ is contained in the pattern subgroup

$$\begin{bmatrix} 2 & 2 & 2 \\ 2 & 2 & 1 \\ 2 & 1 & 1 \end{bmatrix}.$$ 

Let

$$x = \begin{bmatrix} 3s_{0,0} + t_{0,0} & 3s_{0,1} + t_{0,1} & 3s_{0,2} + t_{0,2} \\ 3s_{1,0} + t_{1,0} & 3s_{1,1} + t_{1,1} & 3s_{1,2} \\ 3s_{2,0} + t_{2,0} & 3s_{2,1} & 3s_{2,2} \end{bmatrix} \in \begin{bmatrix} 2 & 2 & 2 \\ 2 & 2 & 1 \\ 2 & 1 & 1 \end{bmatrix}.$$

The variables that are in play are those appearing in the matrix

$$\begin{bmatrix} 0 & t_{0,1} & t_{0,2} \\ t_{1,0} & t_{1,1} & 3s_{1,2} \\ t_{2,0} & 3s_{2,1} & 3s_{2,2} \end{bmatrix}.$$ 

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In the notation of Case 7.2.2, we are taking $d_1 = 1$, $d_2 = 0$, $d_3 = c_4$, $e_2 = 1$, and $e_3 = c_5$. We wish to identify values $a_1, a_2, a_3, a_4, a_5 \in \mathbb{Z}_9$ such that $\partial_1 x \equiv a_1 m_1 + a_2 m_2 + a_3 m_3 + a_4 m_4 + a_5 m_5 \pmod{I}$. We see that $\partial_1 x \in W_2$ if and only if

\[ t_{1,1} \equiv a_4 c_4 + a_5 c_5 \quad \text{(A1)} \]
\[ t_{2,0} \equiv a_5 \quad \text{(A2)} \]
\[ s_{2,2} \equiv c_2 t_{1,1} \quad \text{(A3)} \]
\[ s_{1,2} \equiv -c_2 s_{2,1} + c_2 a_4 - t_{2,0} \quad \text{(A4)} \]

We wish to identify values $b_1, b_2, b_3, b_4, b_5 \in \mathbb{Z}_9$ such that $\partial_2 x \equiv b_1 m_1 + b_2 m_2 + b_3 m_3 + b_4 m_4 + b_5 m_5 \pmod{I}$. We see that $\partial_2 x \in W_2$ if and only if

\[ t_{0,2} \equiv b_4 c_4 + b_5 c_5 \quad \text{(B1)} \]
\[ s_{2,2} \equiv -t_{0,2} \quad \text{(B2)} \]
\[ t_{1,1} \equiv b_5 \quad \text{(B3)} \]
\[ t_{1,1} \equiv -c_2 t_{0,2} \quad \text{(B4)} \]
\[ s_{1,2} \equiv c_2 t_{0,2} - c_2 t_{0,1} + b_4 + c_2 s_{2,1} \quad \text{(B5)} \]

Substituting (A2) and (B3) into (A1) and (B1) we obtain $a_4 \equiv c_4 t_{1,1} - c_4 c_5 t_{2,0}$ and $b_4 \equiv c_4 t_{0,2} - c_4 c_5 t_{1,1}$. Combining (A4) and (B5) and substituting $a_4$ and $b_4$ we obtain $s_{2,1} \equiv -t_{0,1} + (c_4 c_5 + c_2) t_{2,0} + (1 - c_2 c_4 + c_4 c_3) t_{0,2}$ which we will denote as our new (B5). Substituting (B5) into (A4) we obtain $s_{1,2} \equiv -c_2 t_{0,1} + (c_2 c_4 c_5) t_{2,0} + (-c_2 - c_2 c_4 c_5) t_{0,2}$. 

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Hence \( x \in \partial^{-1}W_2 \) if and only if

\[
s_{2,2} \equiv c_2 t_{1,1} \quad \text{(A3)}
\]

\[
s_{1,2} \equiv -c_2 t_{0,1} + (c_2 c_4 c_5) t_{2,0} + (-c_2 - c_2 c_4 c_5) t_{0,2} \quad \text{(A4)}
\]

\[
s_{2,2} \equiv -t_{0,2} \quad \text{(B2)}
\]

\[
t_{1,1} \equiv -c_2 t_{0,2} \quad \text{(B4)}
\]

\[
s_{2,1} \equiv -t_{0,1} + (c_4 c_5 + c_2) t_{2,0} + (1 - c_2 c_4 + c_4 c_5) t_{0,2} \quad \text{(B5)}.
\]

We regard \( t_{1,0}, t_{0,1}, t_{0,2}, t_{2,0} \) as the free variables. Taking \( t_{1,0} \equiv 1, t_{0,1} = 0, t_{0,2} = 0, \) and \( t_{2,0} = 0, \) the matrix \( x \) becomes

\[
v_1 = \begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}.
\]

Taking \( t_{0,1} \equiv 1, t_{1,0} = 0, t_{0,2} = 0, \) and \( t_{2,0} = 0, \) the matrix \( x \) becomes

\[
v_2 = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 3c_2 \\ 0 & -3 & 0 \end{bmatrix}.
\]

Taking \( t_{0,2} \equiv 1, t_{0,1} = 0, t_{1,0} = 0, \) and \( t_{2,0} = 0, \) the matrix \( x \) becomes

\[
v_3 = \begin{bmatrix} 0 & 0 & 1 \\ 0 & -c_2 & 3(-c_2 - c_2 c_4 c_5) \\ 0 & 3(1 - c_2 c_4 + c_4 c_5) & -3 \end{bmatrix}.
\]
Taking $t_{2,0} \equiv 1$, $t_{0,1} = 0$, $t_{1,0} = 0$, and $t_{0,2} = 0$, the matrix $x$ becomes

$$
v_4 = \begin{bmatrix}
0 & 0 & 0 \\
0 & 0 & 3(c_4 c_5 + c_2) \\
1 & 3(c_4 c_5 + c_2) & 0
\end{bmatrix}.
$$

We see that $\partial^{-1}W_2 = < \partial^{-1}W_1, v_3, v_4 >$. Hence $|\partial^{-1}W_2/\partial^{-1}W_1| = 3^2$. Recall that $|\partial^{-1}W_1| = 3^9$ hence $|\partial^{-1}W_2| = 3^{11}$. Since $|W_2| = 3^8$, then $|\partial^{-1}W_2/W_2| = 3^3$. We obtain $\partial^{-1}W_2/W_2 \cong \mathbb{Z}_9 \times \mathbb{Z}_3$ or $\mathbb{Z}_3 \times \mathbb{Z}_3 \times \mathbb{Z}_3$. We see that neither $v_4$ nor $3v_4$ is contained in $W_2$. We know that $v_4 \in \partial^{-1}W_2$ and since $3v_4 \not\in W_2$ then $v_4 + W_2$ is an element of the group $\partial^{-1}W_2/W_2$ whose order is larger than 3. Note

$$
-c_2m_2 + m_3 = \begin{bmatrix}
-1 & 0 & - \\
0 & -3c_2 & 0 \\
3 & 0 & 0
\end{bmatrix} + \begin{bmatrix}
1 & 0 & 0 \\
0 & 0 & 0 \\
-3 & 0 & 0
\end{bmatrix} = \begin{bmatrix}
0 & 0 & 3 \\
0 & -3c_2 & 0 \\
0 & 0 & 0
\end{bmatrix} = 3v_3.
$$

Hence $3v_3 \in W_2$ and so $v_3 + W_2$ is an element of order 3. Hence $\partial^{-1}W_2/W_2 \cong \mathbb{Z}_9 \times \mathbb{Z}_3$ and rank($\partial^{-1}W_2/W_2$) = 2. Recall that $\partial^{-1}W_1/W_2$ has rank 1. Since rank($\partial^{-1}W_1/W_2$) < rank($\partial^{-1}W_2/W_2$) then $W_2$ is nonterminal and $W_2 \in \mathcal{L}_2$. Note $\Omega_1(\partial^{-1}W_2/W_2) \cong \mathbb{Z}_3 \times \mathbb{Z}_3$. Thus $\Omega_1(\partial^{-1}W_2/W_2)$ has basis $y_2 + W_1, v_3 + W_2$ while its subspace $\partial^{-1}W_1/W_2$ has basis $y_2 + W_2$.

Let $c_6 \in \{0, 1, 2\}$. Let $m_6 = c_6y_2 + v_3$. Thus

$$
m_6 = \begin{bmatrix}
0 & 0 & 1 \\
0 & 3c_6 - c_2 & 3(-c_2 - c_2 c_4 c_5) \\
0 & 3(1 - c_2 c_4 + c_4 c_5) & -3
\end{bmatrix}.
$$
Let $W_3 = \langle W_2, m_6 \rangle$. The number of subgroups of this type is 9. Since $m_6 \not\in W_2$ while $3m_6 \in W_2$ we have $|W_3/W_2| = 3$. Recalling $|W_2| = 3^8$ we get $|W_3| = 3^9$. Since $|W_3/W_2| = 3$ and the antidiagonal of $m_5$ has an entry not divisible by 3, by the Modified Terminal Lemma we conclude that $\partial^{-1}W_3 = \partial^{-1}W_2$. Thus $\text{rank}(\partial^{-1}W_3/W_3) = \text{rank}(\partial^{-1}W_2/W_3)$, $W_3$ is terminal, and $W_3 \not\in \hat{L}_3$.

10.3 Case 7.3

Suppose $(c_1, c_3) = (2, 2)$. Then

$$m_1 = \begin{bmatrix} 2 & 0 & 3 \\ 0 & 0 & 0 \\ -3 & 0 & 0 \end{bmatrix}, \quad m_2 = \begin{bmatrix} c_2 & 0 & -3c_2 \\ 0 & 3 & 0 \\ -3c_2 & 0 & 0 \end{bmatrix}, \quad \text{and} \quad m_3 = \begin{bmatrix} 2 & 0 & -3 \\ 0 & 0 & 0 \\ 3 & 0 & 0 \end{bmatrix}.$$ 

Hence $x \in \partial^{-1}W_2$ if and only if

$s_{1,2} \equiv c_2 s_{2,1} + t_{1,0}$ \hspace{1cm} (A1)

$s_{1,2} \equiv c_2 s_{2,1} - c_2 t_{0,1}$ \hspace{1cm} (B1).

The preceding two congruences together imply $t_{1,0} \equiv -c_2 t_{0,1}$, or equivalently $t_{0,1} \equiv -c_2 t_{1,0}$.

Hence $x \in \partial^{-1}W_2$ if and only if

$$t_{0,1} \equiv -c_2 t_{1,0} \hspace{1cm} (A1)$$

$$s_{1,2} \equiv c_2 s_{2,1} + t_{1,0} \hspace{1cm} (B1).$$
We regard $t_{1,0}$ and $s_{2,1}$ as free variables. Taking $t_{1,0} \equiv 1$ and $s_{2,1} = 0$, the matrix $x$ becomes

$$v_1 = \begin{bmatrix} 0 & c_2 & 0 \\ 1 & 0 & 3 \\ 0 & 0 & 0 \end{bmatrix}. $$

Taking $s_{2,1} \equiv 1$ and $t_{1,0} = 0$, the matrix $x$ becomes

$$v_2 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 3c_2 \\ 0 & 3 & 0 \end{bmatrix}. $$

We see that $\partial^{-1}W_1 = <\partial^{-1}W_0, v_1, v_2>$. Since $v_1 \notin \partial^{-1}W_0$, $v_2 \notin <\partial^{-1}W_0, v_1>$, $3v_1 \in \partial^{-1}W_0$, and $3v_2 \in <\partial^{-1}W_0, v_1>$ we know $|\partial^{-1}W_1/\partial^{-1}W_0| = 3^2$. Since $|\partial^{-1}W_0| = 3^7$ then $|\partial^{-1}W_1| = 3^9$. Because $|W_1| = 3^6$ then $|\partial^{-1}W_1/W_1| = 3^3$. Also, $v_1, v_2, y_2 \in \partial^{-1}W_1$ but are not contained in $W_1$. Each of $3v_1, 3v_2, 3y_2$ is contained in $W_1$. So $v_1 + W_1, v_2 + W_1, y_2 + W_1$ are elements of order 3 in the group $\partial^{-1}W_1/W_1$. These three elements form a generating set for the group $\partial^{-1}W_1/W_1$. Recall that $|\partial^{-1}W_1/W_1| = 3^3$, we obtain $\partial^{-1}W_1/W_1 \cong \mathbb{Z}_3 \times \mathbb{Z}_3 \times \mathbb{Z}_3$.

Since rank($\partial^{-1}W_1/W_1$) = 3 and rank(\partial^{-1}W_0/W_1) = 1, then rank($\partial^{-1}W_0/W_1$) < rank($\partial^{-1}W_1/W_1$). Hence $W_1$ is nonterminal and $W_1 \in \hat{L}_1$. A basis for $\partial^{-1}W_1/W_1$ is $v_1 + W_1, v_2 + W_1, y_2 + W_1$. A basis for $\partial^{-1}W_0/W_1$ is $y_2 + W_1$.

The subgroups $W_2$ belonging to $\mathcal{L}(W_1)$ correspond to the nontrivial proper subspaces $W_2/W_1$ of $\partial^{-1}W_1/W_1$ for which the intersection $W_2/W_1 \cap \partial^{-1}W_0/W_1$ is
trivial. Since $\mathcal{L}_2(W_1)$ has dimension 3 while its subspace $\mathcal{L}_2(W_0/W_1)$ has dimension 1, every such subspace $W_2/W_1$ has dimension either 1 or 2.

To help us define the subgroups $W_2$ belonging to $\mathcal{L}_2(W_1)$, it will be convenient to identify each element of the vector space $\mathcal{L}_2(W_1)$ with its coordinate vector with respect to the ordered basis $y_2 + W_1, v_1 + W_1, v_2 + W_1$. In this way, we identify $\mathcal{L}_2(W_1)$ with the vector space $\mathbb{Z}_3 \times \mathbb{Z}_3 \times \mathbb{Z}_3$ consisting of row vectors. Under this identification, the elements $y_2 + W_1, v_1 + W_1, v_2 + W_1$ are associated with the so-called standard basis vectors $[1, 0, 0], [0, 1, 0], [0, 0, 1]$, in $\mathbb{Z}_3 \times \mathbb{Z}_3 \times \mathbb{Z}_3$.

The subgroups $W_2$ belonging to $\mathcal{L}(W_1)$ are in one-to-one correspondence with the nontrivial proper subgroup $W_2/W_1$ of $\mathcal{L}_2(W_1)$ for which the intersection $W_2/W_1 \cap \mathcal{L}_2(W_0/W_1)$ is trivial. Note that $\mathcal{L}_2(W_0/W_1)$ is the 2-dimensional subspace generated by the element $y_2 + W_1$. Under our identification, each such subspace $W_2/W_1$ is associated with a subspace $S$ of $\mathbb{Z}_3 \times \mathbb{Z}_3 \times \mathbb{Z}_3$ that does not contain the standard basis vector $[1, 0, 0]$. Let $m$ denote the unique matrix in reduced row-echelon form that is row equivalent to the matrix whose rows are the members of an arbitrarily-chosen basis of such a subspace $S$.

In Case 7.3.1 we consider the 1-dimensional subspaces $W_2/W_1$. There are three possible forms for the matrix $m$. The first form is

$$m = [0, 0, 1]$$

(1 possibility), which is considered in Case 7.3.1.1. The second form is

$$m = [0, 1, c_4] \text{ for } c_4 \in \{0, 1, 2\}$$
(3 possibilities), which is considered in Case 7.3.1.2. The third form is

\[ m = [1, c_4, c_5] \quad \text{for} \quad c_4, c_5 \in \{0, 1, 2\}, \ (c_4, c_5) \neq (0, 0) \]

(8 possibilities), which is considered in Case 7.3.1.3.

In Case 7.3.2 we consider the 2-dimensional subspaces \( W_2/W_1 \). There are three possible forms for the matrix \( m \). The first form is

\[
m = \begin{bmatrix}
0 & 1 & 0 \\
0 & 0 & 1
\end{bmatrix}
\]

(1 possibility), which is considered in Case 7.3.2.1. The second form is

\[
m = \begin{bmatrix}
1 & c_4 & 0 \\
0 & 0 & 1
\end{bmatrix}
\quad \text{for} \quad c_4 \in \{1, 2\}
\]

(2 possibilities), which is considered in Case 7.3.2.2. The third form is

\[
m = \begin{bmatrix}
1 & 0 & c_4 \\
0 & 1 & c_5
\end{bmatrix}
\quad \text{for} \quad c_4 \in \{1, 2\} \quad \text{and} \quad c_5 \in \{0, 1, 2\}
\]

(6 possibilities), which is considered in Case 7.3.2.3.

10.3.1 Case 7.3.1

We consider the 1-dimensional subspaces \( W_2/W_1 \). Let \( d_1, d_2, d_3 \) be unspecified variables. Let \( m_4 = d_1 y_2 + d_2 v_1 + d_3 v_2 \). A formal expression for \( m_4 \) is

\[
m_4 = d_1 \begin{bmatrix}
0 & 0 & 0 \\
0 & 0 & 0 \\
0 & 0 & 0
\end{bmatrix} + d_2 \begin{bmatrix}
0 & c_2 & 0 \\
0 & 0 & 0 \\
0 & 0 & 0
\end{bmatrix} + d_3 \begin{bmatrix}
0 & 0 & 3c_2 \\
0 & 0 & 0 \\
0 & 0 & 0
\end{bmatrix} = \begin{bmatrix}
0 & c_2d_2 & 0 \\
d_2 & 3d_1 & 3(d_2 + c_2d_3) \\
0 & 3d_3 & 0
\end{bmatrix}.
\]

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Let \( W_2 = \langle W_1, m_4 \rangle \in \mathcal{L}_2 \). We now calculate the pullback \( \partial^{-1}W_2 \). The subgroup \( W_2 \) is contained in the pattern subgroup

\[
\begin{bmatrix}
2 & 2 & 1 \\
2 & 1 & 1 \\
1 & 1 & 0
\end{bmatrix}.
\]

Thus the pullback \( \partial^{-1}W_2 \) is contained in the pattern subgroup

\[
\begin{bmatrix}
2 & 2 & 2 \\
2 & 2 & 1 \\
2 & 1 & 1
\end{bmatrix} \cap \begin{bmatrix}
2 & 2 & 2 \\
2 & 1 & 1 \\
2 & 1 & 1
\end{bmatrix} = \begin{bmatrix}
2 & 2 & 2 \\
2 & 2 & 1 \\
2 & 1 & 1
\end{bmatrix}.
\]

Let

\[
x = \begin{bmatrix}
3s_{0,0} + t_{0,0} & 3s_{0,1} + t_{0,1} & 3s_{0,2} + t_{0,2} \\
3s_{1,0} + t_{1,0} & 3s_{1,1} + t_{1,1} & 3s_{1,2} \\
3s_{2,0} + t_{2,0} & 3s_{2,1} & 3s_{2,2}
\end{bmatrix} \in \begin{bmatrix}
2 & 2 & 2 \\
2 & 2 & 1 \\
2 & 1 & 1
\end{bmatrix}.
\]

Thus

\[
\partial_1 x = \begin{bmatrix}
3s_{1,0} + t_{1,0} & 3s_{1,1} + t_{1,1} & 3s_{1,2} \\
3s_{2,0} + t_{2,0} & 3s_{2,1} & 3s_{2,2} \\
-3(t_{1,0} + t_{2,0}) & -3t_{1,1} & 0
\end{bmatrix}
\]

and

\[
\partial_2 x = \begin{bmatrix}
3s_{0,1} + t_{0,1} & 3s_{0,2} + t_{0,2} & -3(t_{0,1} + t_{0,2}) \\
3s_{1,1} + t_{1,1} & 3s_{1,2} & -3t_{1,1} \\
3s_{2,1} & 3s_{2,2} & 0
\end{bmatrix}.
\]
The variables that are in play are those appearing in the matrix

\[
\begin{bmatrix}
0 & t_{0,1} & t_{0,2} \\
t_{1,0} & t_{1,1} & 3s_{1,2} \\
t_{2,0} & 3s_{2,1} & 3s_{2,2}
\end{bmatrix}.
\]

We wish to identify values \(a_1, a_2, a_3, a_4 \in \mathbb{Z}_9\) such that \(\partial_1 x \equiv a_1 m_1 + a_2 m_2 + a_3 m_3 + a_4 m_4 \pmod{I}\). A formal expression for \(a_1 m_1 + a_2 m_2 + a_3 m_3 + a_4 m_4\) is

\[
\begin{bmatrix}
2 & 0 & 3 \\
0 & 0 & 0 \\
-3 & 0 & 0
\end{bmatrix}
\begin{bmatrix}
c_2 & 0 & -3c_2 \\
0 & 3c_2 & 0 \\
-3c_2 & 0 & 0
\end{bmatrix}
\begin{bmatrix}
2 & -3 \\
0 & 0 \\
3 & 0
\end{bmatrix}
\begin{bmatrix}
0 & c_{2d_2} & 0 \\
d_2 & 3d_1 & 3(d_2 + c_{2d_2}) \\
0 & 3d_3 & 0
\end{bmatrix}
\]

\[
= \begin{bmatrix}
-a_1 + a_2 c_2 - a_3 & a_4 c_{2d_2} & 3(a_1 - a_2 c_2 - a_3) \\
a_4 d_2 & 3(a_2 + a_4 d_1) & 3a_4(d_2 + c_{2d_2}) \\
-3(-a_1 - a_2 c_2 + a_3) & 3a_4 d_3 & 0
\end{bmatrix}.
\]

Comparing (0,1)-entries, we get \(t_{1,1} \equiv a_4 c_{2d_3}\).

Comparing (2,1)-entries, we get \(-t_{1,1} \equiv a_4 d_3\).

Comparing (1,0)-entries, we get \(t_{2,0} \equiv a_4 d_2\).

Comparing (1,2)-entries, we get \(s_{2,2} \equiv a_4(d_2 + c_{2d_2})\). Substituting \(a_4 d_2 \equiv t_{2,0}\) and \(-t_{1,1} \equiv a_4 d_3\) we obtain \(s_{2,2} \equiv t_{2,0} - c_2 t_{1,1}\).

Comparing (1,1)-entries, we get \(a_2 \equiv s_{2,1} - a_4 d_1\).

Comparing (0,0)-entries, we get \(a_1 \equiv c_2 s_{2,1} - c_2 a_4 d_1 - a_3 - t_{1,0}\).
Comparing \((2, 0)\)-entries, we get \(a_3 \equiv -t_{1,0} - t_{2,0} + a_1 + a_2c_2\). Substituting \(a_2\) we obtain \(a_3 \equiv -t_{1,0} + t_{2,0} + c_2s_{2,1} - c_2a_4d_1\).

Comparing \((0, 2)\)-entries, we get \(s_{1,2} \equiv a_1 - c_2a_2 - a_3\). Substituting \(a_1, a_2, a_3\) we obtain \(s_{1,2} \equiv t_{1,0} + t_{2,0} + c_2s_{2,1} - c_2a_4d_1\).

We see that \(\partial_1 x \in W_2\) if and only if

\[
t_{1,1} \equiv a_4c_2d_2 \quad (A1)
- t_{1,1} \equiv a_4d_3 \quad (A2)
\]
\[
t_{2,0} \equiv a_4d_2 \quad (A3)
\]
\[
s_{2,2} \equiv t_{2,0} - c_2t_{1,1} \quad (A4)
\]
\[
s_{1,2} \equiv t_{1,0} + t_{2,0} + c_2s_{2,1} - c_2a_4d_1 \quad (A4).
\]

We wish to identify values \(b_1, b_2, b_3, b_4 \in \mathbb{Z}_9\) such that \(\partial_2 x \equiv b_1m_1 + b_2m_2 + b_3m_3 + b_4m_4 \pmod{I}\). A formal expression for \(b_1m_1 + b_2m_2 + b_3m_3 + b_4m_4\) is

\[
\begin{bmatrix}
2 & 0 & 3 \\
0 & 0 & 0 \\
-3 & 0 & 0
\end{bmatrix}
+ b_2
\begin{bmatrix}
c_2 & 0 & -3c_2 \\
0 & 3 & 0 \\
-3c_2 & 0 & 0
\end{bmatrix}
+ b_3
\begin{bmatrix}
2 & 0 & -3 \\
0 & 0 & 0 \\
3 & 0 & 0
\end{bmatrix}
+ b_4
\begin{bmatrix}
0 & c_2d_2 & 0 \\
d_2 & 3d_1 & 3(d_2 + c_2d_2) \\
0 & 3d_3 & 0
\end{bmatrix}
= \begin{bmatrix}
-b_1 + b_2c_2 - b_3 & b_4c_2d_2 & 3(b_1 - b_2c_2 - b_3) \\
b_4d_2 & 3(b_2 + b_4d_1) & 3b_4(d_2 + c_2d_2) \\
-3(-b_1 - b_2c_2 + b_3) & 3b_4d_3 & 0
\end{bmatrix}.
\]

Comparing \((0, 1)\)-entries, we get \(t_{0,2} \equiv b_4c_2d_2\).

Comparing \((2, 1)\)-entries, we get \(s_{2,2} \equiv b_4d_3\).
Comparing (1, 0)-entries, we get \( t_{1,1} \equiv b_4 d_2 \).

Comparing (1, 2)-entries, we get \(-t_{1,1} \equiv b_4 d_2 + b_4 c_2 d_3\). Substituting \( b_4 d_3 \) and \( b_4 d_2 \) we obtain \( t_{1,1} \equiv c_2 s_{2,2} \).

Comparing (1, 1)-entries, we get \( b_2 \equiv s_{1,2} - b_4 d_1 \).

Comparing (0, 0)-entries, we get \( b_1 \equiv c_2 s_{1,2} - c_2 b_4 d_1 - b_3 - t_{0,1} \).

Comparing (2, 0)-entries, we get \( b_3 \equiv s_{2,1} + b_1 + b_2 c_2 \). Substituting \( b_2 \) we obtain \( b_3 \equiv -s_{2,1} + c_2 s_{1,2} - c_2 b_4 d_1 + t_{0,1} \).

Comparing 90, 2)-entries, we get \(-t_{0,1} - t_{0,2} \equiv b_1 - c_2 b_2 - b_3\). Substituting \( b_1, b_2, b_3 \) we obtain \( s_{1,2} \equiv -c_2 t_{0,2} + c_2 s_{2,1} + b_4 d_1 - c_2 t_{0,1} \).

We see that \( \partial_2 x \in W_2 \) if and only if

\[
\begin{align*}
t_{0,2} &\equiv b_4 c_2 d_2 \quad \text{(B1)} \\
s_{2,2} &\equiv b_4 d_3 \quad \text{(B2)} \\
t_{1,1} &\equiv b_4 d_2 \quad \text{(B3)} \\
t_{1,1} &\equiv -c_2 s_{2,2} \quad \text{(B4)} \\
s_{1,2} &\equiv -c_2 t_{0,2} + c_2 s_{2,1} + b_4 d_1 - c_2 t_{0,1} \quad \text{(B5).}
\end{align*}
\]

Case 7.3.1.1

Let \( m_4 = v_2 \). Thus

\[
m_4 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 3c_2 \\ 0 & 3 & 0 \end{bmatrix}.
\]
Let $W_2 = \langle W_1, m_4 \rangle \in \mathcal{L}_2$. The number of subgroups $W_2$ of this type is 1. Since $m_4 \notin W_1$ and $3m_4 \in W_1$ we have $|W_2/W_1| = 3$. Since $|W_1| = 3^6$ it follows that $|W_2| = 3^7$. Note that $\partial^{-1}W_1/W_2 \cong \mathbb{Z}_3 \times \mathbb{Z}_3$ and rank$(\partial^{-1}W_1/W_2) = 2$. We now calculate the pullback of $\partial^{-1}W_2$. We observed in case 7.3.1 that $\partial^{-1}W_2$ is contained in the pattern subgroup

$$\begin{bmatrix}
2 & 2 & 2 \\
2 & 2 & 1 \\
2 & 1 & 1
\end{bmatrix}.$$

Let

$$x = \begin{bmatrix}
3s_{0,0} + t_{0,0} & 3s_{0,1} + t_{0,1} & 3s_{0,2} + t_{0,2} \\
3s_{1,0} + t_{1,0} & 3s_{1,1} + t_{1,1} & 3s_{1,2} \\
3s_{2,0} + t_{2,0} & 3s_{2,1} & 3s_{2,2}
\end{bmatrix} \in \begin{bmatrix}
2 & 2 & 2 \\
2 & 2 & 1 \\
2 & 1 & 1
\end{bmatrix}.$$

The variables that are in play are those appearing in the matrix

$$\begin{bmatrix}
0 & t_{0,1} & t_{0,2} \\
t_{1,0} & t_{1,1} & 3s_{1,2} \\
t_{2,0} & 3s_{2,1} & 3s_{2,2}
\end{bmatrix}.$$

In the notation of Case 7.3.1, we are taking $d_1 = 0$, $d_2 = 0$, and $d_3 = 1$. We wish to identify values $a_1, a_2, a_3, a_4 \in \mathbb{Z}_9$ such that $\partial_1 x \equiv a_1m_1 + a_2m_2 + a_3m_3 + a_4m_4$ (mod I).
We see that $\partial_1 x \in W_2$ if and only if

\begin{align*}
t_{1,1} &\equiv 0 \quad (A1) \\
-t_{1,1} &\equiv a_4 \quad (A2) \\
t_{2,0} &\equiv 0 \quad (A3) \\
s_{2,2} &\equiv 0 \quad (A4) \\
s_{1,2} &\equiv t_{10} + c_2 s_{2,1} \quad (A5).
\end{align*}

We wish to identify values $b_1, b_2, b_3, b_4 \in \mathbb{Z}_9$ such that $\partial_1 x \equiv b_1 m_1 + b_2 m_2 + b_3 m_3 + b_4 m_4 \pmod{I}$. We see that $\partial_2 x \in W_2$ if and only if

\begin{align*}
t_{0,2} &\equiv 0 \quad (B1) \\
s_{2,2} &\equiv b_4 \quad (B2) \\
t_{1,1} &\equiv 0 \quad (B3) \\
s_{1,2} &\equiv c_2 s_{2,1} - c_2 t_{0,1} \quad (B5).
\end{align*}

Hence $x \in \partial^{-1} W_2$ if and only if

\begin{align*}
t_{1,1} &\equiv 0 \\
t_{2,0} &\equiv 0 \\
s_{2,2} &\equiv 0 \\
t_{0,2} &\equiv 0 \\
s_{1,2} &\equiv t_{1,0} + c_2 s_{2,1} \\
s_{1,2} &\equiv c_2 s_{2,1} - c_2 t_{0,1}.
\end{align*}
These are the same congruences as those in Case 7.3 therefore \( \partial^{-1}W_2 = \partial^{-1}W_1 \). Thus \( \text{rank}(\partial^{-1}W_2/W_2) = \text{rank}(\partial^{-1}W_1/W_2) \), \( W_2 \) is terminal, and \( W_2 \not\in \hat{\mathcal{L}}_2 \).

Case 7.3.1.2

We fix arbitrary value \( c_4 \in \{0, 1, 2\} \). There is 3 ways to choose the value \( c_4 \). Let \( m_4 = v_1 + c_4 v_2 \). Thus

\[
m_4 = \begin{bmatrix}
0 & c_2 & 0 \\
1 & 0 & 3(1 + c_2 c_4) \\
0 & 3 c_4 & 0
\end{bmatrix}.
\]

Let \( W_2 = \langle W_1, m_4 \rangle \in \mathcal{L}_2 \). The number of subgroups \( W_2 \) of this type is 3. Since \( m_4 \not\in W_1 \) and \( 3 m_4 \in W_1 \) we have \( |W_2/W_1| = 3 \). Since \( |W_1| = 3^6 \) it follows that \( |W_2| = 3^7 \). Note that \( \partial^{-1}W_1/W_2 \cong \mathbb{Z}_3 \times \mathbb{Z}_3 \) and \( \text{rank}(\partial^{-1}W_1/W_2) = 2 \). We now calculate the pullback of \( \partial^{-1}W_2 \). We observed in case 7.3.1 that \( \partial^{-1}W_2 \) is contained in the pattern subgroup

\[
\begin{bmatrix}
2 & 2 & 2 \\
2 & 2 & 1 \\
2 & 1 & 1
\end{bmatrix}
\]

Let

\[
x = \begin{bmatrix}
3 s_{0,0} + t_{0,0} & 3 s_{0,1} + t_{0,1} & 3 s_{0,2} + t_{0,2} \\
3 s_{1,0} + t_{1,0} & 3 s_{1,1} + t_{1,1} & 3 s_{1,2} \\
3 s_{2,0} + t_{2,0} & 3 s_{2,1} & 3 s_{2,2}
\end{bmatrix} \in \begin{bmatrix}
2 & 2 & 2 \\
2 & 2 & 1 \\
2 & 1 & 1
\end{bmatrix}.
\]
The variables that are in play are those appearing in the matrix

\[
\begin{bmatrix}
0 & t_{0,1} & t_{0,2} \\
t_{1,0} & t_{1,1} & 3s_{1,2} \\
t_{2,0} & 3s_{2,1} & 3s_{2,2}
\end{bmatrix}
\]

In the notation of Case 7.3.1, we are taking \(d_1 = 0, \ d_2 = 1, \) and \(d_3 = c_4.\) We wish to identify values \(a_1, a_2, a_3, a_4 \in \mathbb{Z}_9\) such that \(\partial_1 x \equiv a_1 m_1 + a_2 m_2 + a_3 m_3 + a_4 m_4 \pmod{I}\). We see that \(\partial_1 x \in W_2\) if and only if

\[
t_{1,1} \equiv a_4 c_2 \quad \text{(A1)}
\]

\[-t_{1,1} \equiv a_4 c_4 \quad \text{(A2)}
\]

\[t_{2,0} \equiv a_4 \quad \text{(A3)}
\]

\[
s_{2,2} \equiv t_{2,0} - c_2 t_{1,1} \quad \text{(A4)}
\]

\[s_{1,2} \equiv t_{10} + t_{2,0} + c_2 s_{2,1} \quad \text{(A5)}.
\]

We wish to identify values \(b_1, b_2, b_3, b_4 \in \mathbb{Z}_9\) such that \(\partial_2 x \equiv b_1 m_1 + b_2 m_2 + b_3 m_3 + b_4 m_4 \pmod{I}\). We see that \(\partial_2 x \in W_2\) if and only if

\[
t_{0,2} \equiv b_4 c_2 \quad \text{(B1)}
\]

\[
s_{2,2} \equiv b_4 c_4 \quad \text{(B2)}
\]

\[t_{1,1} \equiv b_4 \quad \text{(B3)}
\]

\[t_{1,1} \equiv c_2 s_{2,2} \quad \text{(B4)}
\]

\[s_{1,2} \equiv -c_2 t_{0,2} + c_2 s_{2,1} - c_2 t_{0,1} \quad \text{(B5)}.
\]
Substituting $a_4$ and $b_4$ into (A1), (A2), (B1), and (B2) we obtain $t_{1,1} \equiv c_2 t_{2,0}, -t_{1,1} \equiv c_4 t_{2,0}, t_{0,2} \equiv c_2 t_{1,1}$ and $s_{2,2}c_4 t_{1,1}$. Combining (A1) and (A2) we get $(c_2 + c_4)t_{2,0} \equiv 0$.

It is convenient to consider the cases $c_2 + c_4 \neq 0$ and $c_2 + c_4 = 0$ separately.

**Case 7.3.1.2.1** Suppose $c_2 + c_4 \neq 0$. Then $t_{2,0} \equiv 0$. It follows that $t_{1,1} \equiv t_{0,2} \equiv s_{2,2} \equiv 0$. (A5) and (B5) become $s_{1,2} \equiv t_{1,0} + c_2 s_{2,1}$ and $s_{1,2} \equiv c_2 s_{2,1} - c_2 t_{0,1}$. These are the same congruences as those in Case 7.3 therefore $\partial^{-1}W_2 = \partial^{-1}W_1$. Thus $\text{rank}(\partial^{-1}W_2/W_2) = \text{rank}(\partial^{-1}W_1/W_2)$, $W_2$ is terminal, and $W_2 \notin \hat{L}_2$.

**Case 7.3.1.2.2** Suppose $c_2 + c_4 \equiv 0$. Then $c_4 \equiv -c_2$ and $t_{2,0}$ is a free variable. Substituting (A1) into (B1) and (B2) we obtain $t_{0,2} \equiv t_{2,0}$ and $s_{2,2} \equiv -t_{2,0}$.

Hence $x \in \partial^{-1}W_2$ if and only if

\[
s_{2,2} \equiv -t_{2,0} \\
t_{0,2} \equiv t_{2,0} \\
t_{1,1} \equiv c_2 t_{2,0} \\
s_{1,2} \equiv t_{1,0} + t_{2,0} + c_2 s_{2,1} \\
t_{0,1} \equiv (-1 - c_2)t_{2,0} - c_2 t_{1,0}.
\]
We regard $t_{1,0}$, $s_{2,1}$, and $t_{2,0}$ as free variables. Taking $t_{1,0} \equiv 1$, $s_{2,1} = 0$, and $t_{2,0} = 0$, the matrix $x$ becomes

$$v_1 = \begin{bmatrix} 0 & c_2 & 0 \\ 1 & 0 & 3 \\ 0 & 0 & 0 \end{bmatrix}. $$

Taking $s_{2,1} \equiv 1$, $t_{1,0} = 0$, and $t_{2,0} = 0$, the matrix $x$ becomes

$$v_2 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 3c_2 \\ 0 & 3 & 0 \end{bmatrix}. $$

Taking $t_{2,0} \equiv 1$, $t_{1,0} = 0$, and $s_{2,1} = 0$, the matrix $x$ becomes

$$v_3 = \begin{bmatrix} 0 & -1 - c_2 & 1 \\ 0 & c_2 & 3 \\ 1 & 0 & -3 \end{bmatrix}. $$

We see that neither $v_3$ nor $3v_3$ is contained in $W_2$ and that $\partial^{-1}W_2 = < \partial^{-1}W_0, v_1, v_2, v_3 >$. We know that $v_3 \in \partial^{-1}W_2$ and since $3v_3 \not\in W_2$ then $v_3 + W_2$ is an element of the group $\partial^{-1}W_2/W_2$ whose order is larger than 3. Recall that $|\partial^{-1}W_2/W_2| = 3^3$ and that $\partial^{-1}W_1/W_2$ has rank 2. We obtain $\partial^{-1}W_2/W_2 \cong \mathbb{Z}_9 \times \mathbb{Z}_3$. Since $\text{rank}(\partial^{-1}W_2/W_2) = \text{rank}(\partial^{-1}W_1/W_2)$. Hence $W_2$ is terminal and $W_2 \not\in \hat{\mathcal{L}}_2$. 

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Case 7.3.1.3

We fix arbitrary values $c_4 \in \{1, 2\}$, $c_5 \in \{0, 1, 2\}$. There is 8 ways to choose the values $c_4, c_5$. Let $m_4 = y_2 + c_4 v_1 + c_4 v_2$. Thus

\[
m_4 = \begin{bmatrix}
0 & c_2 c_4 & 0 \\
c_4 & 3 & 3(c_4 + c_2 c_5) \\
0 & 3 c_5 & 0
\end{bmatrix}.
\]

Let $W_2 =< W_1, m_4 > \in \mathcal{L}_2$. The number of subgroups $W_2$ of this type is 8. Since $m_4 \notin W_1$ and $3m_4 \in W_1$ we have $|W_2/W_1| = 3$. Since $|W_1| = 3^6$ it follows that $|W_2| = 3^7$. Note that $\partial^{-1}W_1/W_2 \cong \mathbb{Z}_3 \times \mathbb{Z}_3$ and $\text{rank}(\partial^{-1}W_1/W_2) = 2$. We now calculate the pullback of $\partial^{-1}W_2$. We observed in case 7.3.1 that $\partial^{-1}W_2$ is contained in the pattern subgroup

\[
\begin{bmatrix}
2 & 2 & 2 \\
2 & 2 & 1 \\
2 & 1 & 1
\end{bmatrix}.
\]

Let

\[
x = \begin{bmatrix}
3s_{0,0} + t_{0,0} & 3s_{0,1} + t_{0,1} & 3s_{0,2} + t_{0,2} \\
3s_{1,0} + t_{1,0} & 3s_{1,1} + t_{1,1} & 3s_{1,2} \\
3s_{2,0} + t_{2,0} & 3s_{2,1} & 3s_{2,2}
\end{bmatrix} \in \begin{bmatrix}
2 & 2 & 2 \\
2 & 2 & 1 \\
2 & 1 & 1
\end{bmatrix}.
\]

The variables that are in play are those appearing in the matrix

\[
\begin{bmatrix}
0 & t_{0,1} & t_{0,2} \\
t_{1,0} & t_{1,1} & 3s_{1,2} \\
t_{2,0} & 3s_{2,1} & 3s_{2,2}
\end{bmatrix}.
\]
In the notation of Case 7.3.1, we are taking $d_1 = 1$, $d_2 = c_4$, and $d_3 = c_5$. We wish to identify values $a_1, a_2, a_3, a_4 \in \mathbb{Z}_9$ such that $\partial_1 x \equiv a_1 m_1 + a_2 m_2 + a_3 m_3 + a_4 m_4 \pmod{I}$. We see that $\partial_1 x \in W_2$ if and only if

\begin{align*}
  t_{1,1} &\equiv a_4 c_2 c_4 & \text{(A1)} \\
  -t_{1,1} &\equiv a_4 c_5 & \text{(A2)} \\
  t_{2,0} &\equiv a_4 c_4 & \text{(A3)} \\
  s_{2,2} &\equiv t_{2,0} - c_2 t_{1,1} & \text{(A4)} \\
  s_{1,2} &\equiv t_{10} + t_{2,0} + c_2 s_{2,1} - c_2 a_4 & \text{(A5)}.
\end{align*}

We wish to identify values $b_1, b_2, b_3, b_4 \in \mathbb{Z}_9$ such that $\partial_1 x \equiv b_1 m_1 + b_2 m_2 + b_3 m_3 + b_4 m_4 \pmod{I}$. We see that $\partial_2 x \in W_2$ if and only if

\begin{align*}
  t_{0,2} &\equiv b_4 c_2 c_4 & \text{(B1)} \\
  s_{2,2} &\equiv b_4 c_5 & \text{(B2)} \\
  t_{1,1} &\equiv b_4 c_4 & \text{(B3)} \\
  t_{1,1} &\equiv c_2 s_{2,2} & \text{(B4)} \\
  s_{1,2} &\equiv -c_2 t_{0,2} + c_2 s_{2,1} + b_4 - c_2 t_{0,1} & \text{(B5)}.
\end{align*}

It is convenient to consider the cases $c_4 = 0$ and $c_4 \neq 0$ separately.

**Case 7.3.1.3.1** Suppose $c_4 = 0$. It follows that $t_{1,1} \equiv t_{2,0} \equiv t_{0,2} \equiv s_{2,2} \equiv 0.$ (A5) becomes $s_{1,2} \equiv t_{1,0} + c_2 s_{2,1}$. (B5) becomes $t_{0,1} \equiv s_{2,1} - c_2 s_{1,2}$. Substituting (A5) into (B5) we obtain $t_{0,1} \equiv -c_2 t_{1,0}$. These are the same congruences as those in Case
7.3 therefore $\partial^{-1}W_2 = \partial^{-1}W_1$. Thus $\text{rank}(\partial^{-1}W_2/W_2) = \text{rank}(\partial^{-1}W_1/W_2)$, $W_2$ is terminal, and $W_2 \not\in \hat{L}_2$.

**Case 7.3.1.3.2** Suppose $c_4 \neq 0$. Then (A3) and (B3) become $a_4 \equiv c_4t_{2,0}$ and $b_4 \equiv c_4t_{1,1}$. Substituting $a_4$ and $b_4$ into (A1), (A2), (B1), and (B2) we obtain $t_{1,1} \equiv c_2t_{2,0}$, $-t_{1,1} \equiv c_4c_5t_{2,0}$, $t_{0,2} \equiv t_{2,0}$ and $s_{2,2} \equiv c_4c_5t_{1,1}$. Combining (A1) and (A2) we obtain $(c_2 + c_4c_5)t_{2,0}$. Combining (A5) and (B5) and substituting $a_4, b_4$ we obtain $t_{0,1} \equiv -c_2t_{1,0} + (-c_2 - c_4 - 1)t_{2,0}$ which we will denote as our new (B5). The congruence (A5) becomes $s_{1,2} \equiv t_{1,0} + (1 - c_2c_4)t_{2,0} + c_2s_{2,1}$.

Hence $x \in \partial^{-1}W_2$ if and only if

$$t_{1,1} \equiv c_2t_{2,0}$$

$$(c_2 + c_4c_5)t_{2,0} \equiv 0$$

$$t_{0,2} \equiv t_{2,0}$$

$$s_{2,2} \equiv c_2c_4c_5t_{2,0}$$

$$s_{1,2} \equiv t_{1,0} + (1 - c_2c_4)t_{2,0} + c_2s_{2,1}$$

$$t_{0,1} \equiv -c_2t_{1,0} + (-c_2 - c_4 - 1)t_{2,0}$$

If $c_2 + c_4c_5 \neq 0$ then we obtain $t_{2,0} \equiv 0$. It then follows that $t_{1,1} \equiv t_{0,2} \equiv s_{2,2} \equiv 0$. The congruences (A5) and (B5) become $t_{0,1} \equiv -c_2t_{1,0}$ and $s_{1,2} \equiv t_{1,0} + c_2s_{2,1}$ respectively. These are the same congruences as those in Case 7.3 therefore $\partial^{-1}W_2 = \partial^{-1}W_1$. Thus $\text{rank}(\partial^{-1}W_2/W_2) = \text{rank}(\partial^{-1}W_1/W_2)$, $W_2$ is terminal, and $W_2 \not\in \hat{L}_2$. 

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Assume $c_2 + c_4c_5 \equiv 0$. Then $c_5 \equiv -c_2c_4$ and $t_{2,0}$ is a free variable. Hence

$x \in \partial^{-1}W_2$ if and only if

\[
\begin{align*}
t_{0,2} & \equiv t_{2,0} \\
t_{1,1} & \equiv c_2t_{2,0} \\
s_{2,2} & \equiv -t_{2,0} \\
s_{1,2} & \equiv t_{1,0} + (1 - c_2c_4)t_{2,0} + c_2s_{2,1} \\
t_{0,1} & \equiv -c_2t_{1,0} + (-c_2 - c_4 - 1)t_{2,0}.
\end{align*}
\]

We regard $t_{1,0}$, $s_{2,1}$, and $t_{2,0}$ as free variables. Taking $t_{1,0} \equiv 1$, $s_{2,1} = 0$, and $t_{2,0} = 0$, the matrix $x$ becomes

\[
v_1 = \begin{bmatrix} 0 & c_2 & 0 \\ 1 & 0 & 3 \\ 0 & 0 & 0 \end{bmatrix}.
\]

Taking $s_{2,1} \equiv 1$, $t_{1,0} = 0$, and $t_{2,0} = 0$, the matrix $x$ becomes

\[
v_2 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 3c_2 \\ 0 & 3 & 0 \end{bmatrix}.
\]

Taking $t_{2,0} \equiv 1$, $t_{1,0} = 0$, and $s_{2,1} = 0$, the matrix $x$ becomes

\[
v_3 = \begin{bmatrix} 0 & 0 & 1 \\ -c_2 - c_4 - 1 & c_2 & 3(1 - c_2c_4) \\ 1 & 0 & -3 \end{bmatrix}.
\]
We see that neither $v_3$ nor $3v_3$ is contained in $W_2$ and that $\partial^{-1}W_2 = \langle \partial^{-1}W_0, v_1, v_2, v_3 \rangle$. We know that $v_3 \in \partial^{-1}W_2$ and since $3v_3 \notin W_2$ then $v_3 + W_2$ is an element of the group $\partial^{-1}W_2/W_2$ whose order is larger than 3. Recall that $|\partial^{-1}W_2/W_2| = 3^3$ and that $\partial^{-1}W_1/W_2$ has rank 2. We obtain $\partial^{-1}W_2/W_2 \cong \mathbb{Z}_9 \times \mathbb{Z}_3$. Since rank($\partial^{-1}W_2/W_2$) = rank($\partial^{-1}W_1/W_2$). Hence $W_2$ is terminal and $W_2 \notin \hat{\mathcal{L}}_2$.

10.3.2 Case 7.3.2

We consider the 2-dimensional subspaces $W_2/W_1$. Let $d_1, d_2, d_3, e_1, e_2, e_3$ be unspecified variables. Let $m_4 = d_1y_2 + d_2v_1 + d_3v_2$ and $m_5 = e_1y_2 + e_2v_1 + e_3v_2$. In all the cases we consider the value of $e_1 = 0$ therefore we may exclude it from our expression of $m_5$. A formal expression for $m_4$ is

$$m_4 = d_1 \begin{bmatrix} 0 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 0 \end{bmatrix} + d_2 \begin{bmatrix} 0 & c_2 & 0 \\ 1 & 0 & 3 \\ 0 & 0 & 0 \end{bmatrix} + d_3 \begin{bmatrix} 0 & 0 & 3c_2 \\ 0 & 0 & 0 \\ 0 & 3 & 0 \end{bmatrix} = \begin{bmatrix} 0 & c_2d_2 & 0 \\ d_2 & 3d_1 & 3(d_2 + c_2d_3) \\ 0 & 3d_3 & 0 \end{bmatrix}.$$ 

A formal expression for $m_5$ is

$$m_5 = e_2 \begin{bmatrix} 0 & c_2 & 0 \\ 1 & 0 & 3 \\ 0 & 0 & 0 \end{bmatrix} + e_3 \begin{bmatrix} 0 & 0 & 3c_2 \\ 0 & 0 & 0 \\ 0 & 3 & 0 \end{bmatrix} = \begin{bmatrix} 0 & c_2e_2 & 0 \\ e_2 & 0 & 3(e_2 + c_2e_3) \\ 0 & 3e_3 & 0 \end{bmatrix}.$$ 

Let $W_2 = \langle W_1, m_4, m_5 \rangle \in \mathcal{L}_2$. We now calculate the pullback $\partial^{-1}W_2$. The subgroup $W_2$ is contained in the pattern subgroup

$$\begin{bmatrix} 2 & 2 & 1 \\ 2 & 1 & 1 \\ 1 & 1 & 0 \end{bmatrix}.$$ 

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Thus the pullback $\partial^{-1}W_2$ is contained in the pattern subgroup

$$\begin{bmatrix} 2 & 2 & 2 \\ 2 & 2 & 1 \\ 2 & 1 & 1 \end{bmatrix} \cap \begin{bmatrix} 2 & 2 & 2 \\ 2 & 2 & 1 \\ 2 & 1 & 1 \end{bmatrix} = \begin{bmatrix} 2 & 2 & 2 \\ 2 & 2 & 1 \\ 2 & 1 & 1 \end{bmatrix}.$$ 

Let

$$x = \begin{bmatrix} 3s_{0,0} + t_{0,0} & 3s_{0,1} + t_{0,1} & 3s_{0,2} + t_{0,2} \\ 3s_{1,0} + t_{1,0} & 3s_{1,1} + t_{1,1} & 3s_{1,2} \\ 3s_{2,0} + t_{2,0} & 3s_{2,1} & 3s_{2,2} \end{bmatrix} \in \begin{bmatrix} 2 & 2 & 2 \\ 2 & 2 & 1 \end{bmatrix}.$$ 

Thus

$$\partial_1 x = \begin{bmatrix} 3s_{1,0} + t_{1,0} & 3s_{1,1} + t_{1,1} & 3s_{1,2} \\ 3s_{2,0} + t_{2,0} & 3s_{2,1} & 3s_{2,2} \\ -3(t_{1,0} + t_{2,0}) & -3t_{1,1} & 0 \end{bmatrix} \quad \text{and}$$

$$\partial_2 x = \begin{bmatrix} 3s_{0,1} + t_{0,1} & 3s_{0,2} + t_{0,2} & -3(t_{0,1} + t_{0,2}) \\ 3s_{1,1} + t_{1,1} & 3s_{1,2} & -3t_{1,1} \\ 3s_{2,1} & 3s_{2,2} & 0 \end{bmatrix}.$$ 

The variables that are in play are those appearing in the matrix

$$\begin{bmatrix} 0 & t_{0,1} & t_{0,2} \\ t_{1,0} & t_{1,1} & t_{1,2} \\ t_{2,0} & 3s_{2,1} & 3s_{2,2} \end{bmatrix}.$$ 

We wish to identify values $a_1, a_2, a_3, a_4, a_5 \in \mathbb{Z}_9$ such that $\partial_1 x \equiv a_1 m_1 + a_2 m_2 + a_3 m_3 + a_4 m_4 + a_5 m_5 \pmod{I}$. A formal expression for $a_1 m_1 + a_2 m_2 + a_3 m_3 + a_4 m_4 + a_5 m_5 \pmod{I}$. A formal expression for $a_1 m_1 + a_2 m_2 + a_3 m_3 + a_4 m_4 + a_5 m_5 \pmod{I}$.
$a_4 m_4 + a_5 m_5$ is

$$
\begin{bmatrix}
2 & 0 & 3 \\
0 & 0 & 0 \\
-3 & 0 & 0
\end{bmatrix} +
\begin{bmatrix}
c_2 & 0 & -3c_2 \\
0 & 3 & 0 \\
-3c_2 & 0 & 0
\end{bmatrix} +
\begin{bmatrix}
a_3 & 0 & 0 \\
0 & 0 & 0 \\
3 & 0 & 0
\end{bmatrix}
+a_4
\begin{bmatrix}
d_2 & 3d_1 & 3(d_2 + c_2d_3) \\
0 & 3d_3 & 0 \\
0 & 3d_3 & 0
\end{bmatrix}
+a_5
\begin{bmatrix}
e_2 & 0 & 3(e_2 + c_2e_3) \\
e_2 & 0 & 3e_3 \\
e_2 & 0 & 3e_3
\end{bmatrix}
$$

Comparing (2,1)-entries, we get $-t_{1,1} \equiv a_4d_3 + a_5e_3$.

Comparing (1,0)-entries, we get $t_{2,0} \equiv a_4d_2 + a_5e_2$.

Comparing (0,1)-entries, we get $t_{1,1} \equiv c_2(a_4d_2 + a_5e_2)$. Substituting $t_{2,0} \equiv a_4d_2 + a_5e_2$ we obtain $t_{1,1} \equiv c_2t_{2,0}$. Now substituting $t_{1,1} \equiv c_2t_{2,0}$ we obtain $s_{2,2} \equiv 0$.

Comparing (1,2)-entries, we get $s_{2,2} \equiv a_4(d_2 + a_4c_2d_3 + a_5e_2 + a_5c_2e_3)$. Substituting $a_4d_2 + a_5e_2$ and $a_4d_3 + a_5e_3$ we obtain $s_{2,2} \equiv t_{2,0} - c_2 t_{1,1}$.

Comparing (1,1)-entries, we get $a_2 \equiv s_{2,1} - a_4d_1$.

Comparing (0,0)-entries, we get $a_1 \equiv -c_2a_2 - a_3 - t_{1,0}$. Substituting $a_2$ we obtain $a_1 \equiv c_2s_{2,1} - c_2a_4d_1 - a_3 - t_{1,0}$.

Comparing (2,0)-entries, we get $a_3 \equiv -t_{1,0} - t_{2,0} + a_1 + a_2c_2$. Substituting $a_1, a_2$ we obtain $a_3 \equiv -t_{1,0} + t_{2,0} + c_2 s_{2,1} - c_2 a_4d_1$.
Comparing $(0, 2)$-entries, we get $s_{1,2} \equiv a_1 - a_2c_2 - a_3$. Substituting $a_1, a_2, a_3$ we obtain $s_{1,2} \equiv t_{1,0} + t_{2,0} + c_2s_{2,1} - c_2a_4d_1$.

We see that $\partial_1 x \in W_2$ if and only if

$$-t_{1,1} \equiv a_4d_3 + a_5e_3 \quad \text{(A1)}$$
$$t_{2,0} \equiv a_4d_2 + a_5e_2 \quad \text{(A2)}$$
$$t_{1,1} \equiv c_2t_{2,0} \quad \text{(A3)}$$
$$s_{2,2} \equiv 0 \quad \text{(A4)}$$
$$s_{1,2} \equiv t_{1,0} + t_{2,0} + c_2s_{2,1} - c_2a_4d_1 \quad \text{(A4)}.$$

We wish to identify values $b_1, b_2, b_3, b_4, b_5 \in \mathbb{Z}_9$ such that $\partial_2 x \equiv b_1m_1 + b_2m_2 + b_3m_3 + b_4m_4 + b_5m_5 \pmod{I}$. A formal expression for $b_1m_1 + b_2m_2 + b_3m_3 = b_4m_4 + b_5m_5$ is

$$\begin{bmatrix}
2 & 0 & 3 \\
0 & 0 & 0 \\
-3 & 0 & 0
\end{bmatrix}
+ b_1
\begin{bmatrix}
c_2 & 0 & -3c_2 \\
0 & 0 & 0 \\
-3c_2 & 0 & 0
\end{bmatrix}
+ b_3
\begin{bmatrix}
2 & 0 & -3 \\
0 & 0 & 0 \\
3 & 0 & 0
\end{bmatrix}
+ b_2
\begin{bmatrix}
0 & c_2d_2 & 0 \\
0 & 0 & 0 \\
0 & c_2e_3 & 0
\end{bmatrix}
+ b_4
\begin{bmatrix}
d_2 & 3d_1 & 3(d_2 + c_2d_3) \\
0 & 3d_3 & 0 \\
0 & 3e_3 & 0
\end{bmatrix}
+ b_5
\begin{bmatrix}
e_2 & 0 & 3(e_2 + c_2e_3) \\
0 & 0 & 0 \\
0 & 0 & 0
\end{bmatrix}
= \begin{bmatrix}
-b_1 + b_2c_2 - b_3 & c_2(b_4d_2 + b_5e_2) & 3(b_1 - b_2c_2 - b_3) \\
b_4d_2 + b_5e_2 & 3(b_2 + b_4d_1) & 3[b_4(d_2 + c_2d_3) + b_5(e_2 + c_2e_3)] \\
3(-b_1 - c_2b_2 + b_3) & 3(b_4d_3 + b_5e_3) & 0
\end{bmatrix}$$. 

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Comparing (2, 1)-entries, we get $s_{2,2} \equiv b_4d_3 + b_5e_3$.

Comparing (1, 0)-entries, we get $t_{1,1} \equiv b_4d_2 + b_5e_2$.

Comparing (0, 1)-entries, we get $t_{0,2} \equiv c_2(b_4d_2 + b_5e_2)$. Substituting $t_{1,1} \equiv b_4d_2 + b_5e_2$ we obtain $t_{0,2} \equiv c_2t_{1,1}$.

Comparing (1, 2)-entries, we get $-t_{1,1} \equiv b_4(d_2 + b_4c_2d_3 + b_5e_2 + b_5c_2e_3)$. Substituting $b_4d_2 + b_5e_2$ and $b_4d_3 + b_5e_3$ we obtain $t_{1,1} \equiv c_2s_{2,2}$.

Comparing (1, 1)-entries, we get $b_2 \equiv s_{1,2} - b_4d_1$.

Comparing (0, 0)-entries, we get $b_1 \equiv -c_2b_2 - b_3 - t_{0,1}$. Substituting $b_2$ we obtain $b_1 \equiv c_2s_{1,2} - c_2b_4d_1 - b_3 - t_{0,1}$.

Comparing (2, 0)-entries, we get $b_3 \equiv s_{2,1} + b_1 + b_2c_2$. Substituting $b_1, b_2$ we obtain $b_3 \equiv -s_{2,1} + c_2s_{1,2} - c_2b_4d_1 + t_{0,1}$.

Comparing (0, 2)-entries, we get $-t_{0,1} - t_{0,2} \equiv b_1 - b_2c_2 - b_3$. Substituting $b_1, b_2, b_3$ we obtain $s_{1,2} \equiv -c_2t_{0,2} + c_2s_{2,1} + b_4d_1 - c_2t_{0,1}$. We see that $\partial_2 x \in W_2$ if and only if

\begin{align*}
s_{2,2} & \equiv b_4d_3 + b_5e_3 \quad (B1) \\
t_{1,1} & \equiv b_4d_2 + b_5e_2 \quad (B2) \\
t_{0,2} & \equiv c_2t_{1,1} \quad (B3) \\
t_{1,1} & \equiv c_2s_{2,2} \quad (B4) \\
\quad & \\
s_{1,2} & \equiv -c_2t_{0,2} + c_2s_{2,1} + b_4d_1 - c_2t_{0,1} \quad (B5). \end{align*}

Since $s_{2,2} \equiv 0$, then (B4) becomes $t_{1,1} \equiv 0$. Then (B3) becomes $t_{0,2} \equiv 0$. Substituting $t_{1,1}$ into (A3) we obtain $0 \equiv c_2t_{2,0}$. Since $c_2 \neq 0$ then $t_{2,0} \equiv 0$. The congruences (A1) and (A2) become $0 \equiv a_4d_3 + a_5e_3$ and $0 \equiv d_2a_4 + e_2a_5$. The
congruences (B1) and (B2) become \( 0 \equiv b_4 d_3 + b_5 e_3 \) and \( 0 \equiv d_2 b_4 + e_2 b_5 \). Substituting \( t_{2,0} \) into (A5) we obtain \( s_{1,2} \equiv c_2 s_{2,1} - c_2 a_4 d_1 + t_{1,0} \). Substituting \( t_{0,2} \) into (B5) we obtain \( s_{1,2} \equiv c_2 s_{2,1} + b_4 d_1 - c_2 t_{0,1} \).

Hence \( x \in \partial^{-1} W_2 \) if and only if

\[
\begin{align*}
0 &\equiv a_4 d_3 + a_5 e_3 \quad \text{(A1)} \\
0 &\equiv d_2 a_4 + e_2 a_5 \quad \text{(A2)} \\
t_{2,0} &\equiv 0 \quad \text{(A3)} \\
s_{2,2} &\equiv 0 \quad \text{(A4)} \\
s_{1,2} &\equiv c_2 s_{2,1} - c_2 a_4 d_1 + t_{1,0} \quad \text{(A5)} \\
0 &\equiv b_4 d_3 + b_5 e_3 \quad \text{(B1)} \\
0 &\equiv d_2 b_4 + e_2 b_5 \quad \text{(B2)} \\
t_{0,2} &\equiv 0 \quad \text{(B3)} \\
t_{1,1} &\equiv 0 \quad \text{(B4)} \\
s_{1,2} &\equiv c_2 s_{2,1} + b_4 d_1 - c_2 t_{0,1} \quad \text{(B5).}
\end{align*}
\]

Case 7.3.2.1

Let \( m_4 = v_1 \) and \( m_5 = v_2 \). Thus

\[
m_4 = \begin{bmatrix} 0 & c_2 & 0 \\ 1 & 0 & 3 \\ 0 & 0 & 0 \end{bmatrix} \quad \text{and} \quad m_5 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 3 c_2 \\ 0 & 3 & 0 \end{bmatrix}.
\]

Let \( W_2 = \langle W_1, m_4, m_5 \rangle \in \mathcal{L}_2 \). The number of subgroups \( W_2 \) of this type is

1. Since \( m_4, m_5 \notin W_1 \) and \( 3m_4, 3m_5 \in W_1 \) we have \(|W_2/W_1| = 3^2\). Since \(|W_1| = 3^6\)

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it follows that $|W_2| = 3^8$. Note that $\partial^{-1}W_1/W_2 \cong \mathbb{Z}_3$ and $\text{rank}(\partial^{-1}W_1/W_2) = 1$.

We now calculate the pullback of $\partial^{-1}W_2$. We observed in case 7.3.2 that $\partial^{-1}W_2$ is contained in the pattern subgroup

$$
\begin{bmatrix}
2 & 2 & 2 \\
2 & 2 & 1 \\
2 & 1 & 1
\end{bmatrix}.
$$

Let

$$
x = \begin{bmatrix}
3s_{0,0} + t_{0,0} & 3s_{0,1} + t_{0,1} & 3s_{0,2} + t_{0,2} \\
3s_{1,0} + t_{1,0} & 3s_{1,1} + t_{1,1} & 3s_{1,2} \\
3s_{2,0} + t_{2,0} & 3s_{2,1} & 3s_{2,2}
\end{bmatrix} \in \begin{bmatrix}
2 & 2 & 2 \\
2 & 2 & 1 \\
2 & 1 & 1
\end{bmatrix}.
$$

The variables that are in play are those appearing in the matrix

$$
\begin{bmatrix}
0 & t_{0,1} & t_{0,2} \\
t_{1,0} & t_{1,1} & 3s_{1,2} \\
t_{2,0} & 3s_{2,1} & 3s_{2,2}
\end{bmatrix}.
$$

In the notation of Case 7.3.2, we are taking $d_1 = 0, d_2 = 1, d_3 = 0, e_2 = 0, e_3 = 1$. We wish to identify values $a_1, a_2, a_3, a_4, a_5 \in \mathbb{Z}_9$ such that $\partial_1 x \equiv a_1 m_1 + a_2 m_2 + a_3 m_3 + a_4 m_4 + a_5 m_5 \pmod{I}$.  

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We see that $\partial_1 x \in W_2$ if and only if

\[
\begin{align*}
0 &\equiv a_5 \quad (A1) \\
0 &\equiv a_4 \quad (A2) \\
t_{2,0} &\equiv 0 \quad (A3) \\
s_{2,2} &\equiv 0 \quad (A4) \\
s_{1,2} &\equiv c_2 s_{2,1} + t_{1,0} \quad (A5).
\end{align*}
\]

We wish to identify values $b_1, b_2, b_3, b_4, b_5 \in \mathbb{Z}_9$ such that $\partial_1 x \equiv b_1 m_1 + b_2 m_2 + b_3 m_3 + b_4 m_4 + b_5 m_5 \pmod{I}$.

We see that $\partial_2 x \in W_2$ if and only if

\[
\begin{align*}
0 &\equiv b_5 \quad (B1) \\
0 &\equiv b_4 \quad (B2) \\
t_{0,2} &\equiv 0 \quad (B3) \\
t_{1,1} &\equiv 0 \quad (B4) \\
s_{1,2} &\equiv c_2 s_{2,1} - c_2 t_{0,1} \quad (B5).
\end{align*}
\]
Hence $x \in \partial^{-1}W_2$ if and only if

\[
\begin{align*}
s_{2,2} &\equiv 0 \\
t_{1,1} &\equiv 0 \\
t_{0,2} &\equiv 0 \\
t_{2,0} &\equiv 0 \\
s_{1,2} &\equiv c_2 s_{2,1} + t_{1,0} \\
s_{1,2} &\equiv c_2 s_{2,1} - c_2 t_{0,1}.
\end{align*}
\]

These are the same congruences as those in Case 7.3 therefore $\partial^{-1}W_2 = \partial^{-1}W_1$. Thus $\text{rank}(\partial^{-1}W_2/W_2) = \text{rank}(\partial^{-1}W_1/W_2)$, $W_2$ is terminal, and $W_2 \notin \hat{L}_2$.

Case 7.3.2.2

We fix arbitrary value $c_4 \in \{1, 2\}$. There is 2 ways to choose the value $c_4$. Let $m_4 = y_2 + c_4 v_1$ and $m_5 = v_2$. Thus

\[
m_4 = \begin{bmatrix} 0 & c_2 c_4 & 0 \\ c_4 & 3 & 3 c_4 \\ 0 & 0 & 0 \end{bmatrix} \quad \text{and} \quad m_5 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 3 c_2 \\ 0 & 3 & 0 \end{bmatrix}.
\]

Let $W_2 = \langle W_1, m_4, m_5 \rangle \in \mathcal{L}_2$. The number of subgroups $W_2$ of this type is 2. Since $m_4, m_5 \notin W_1$ and $3m_4, 3m_5 \in W_1$ we have $|W_2/W_1| = 3^2$. Since $|W_1| = 3^6$ it follows that $|W_2| = 3^8$. Note that $\partial^{-1}W_1/W_2 \cong \mathbb{Z}_3$ and $\text{rank}(\partial^{-1}W_1/W_2) = 1$. 

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We now calculate the pullback of $\partial^{-1}W_2$. We observed in case 7.3.2 that

$\partial^{-1}W_2$ is contained in the pattern subgroup

\[
\begin{bmatrix}
2 & 2 & 2 \\
2 & 2 & 1 \\
2 & 1 & 1
\end{bmatrix}.
\]

Let

\[
x = \begin{bmatrix}
3s_{0,0} + t_{0,0} & 3s_{0,1} + t_{0,1} & 3s_{0,2} + t_{0,2} \\
3s_{1,0} + t_{1,0} & 3s_{1,1} + t_{1,1} & 3s_{1,2} \\
3s_{2,0} + t_{2,0} & 3s_{2,1} & 3s_{2,2}
\end{bmatrix} \in \begin{bmatrix}
2 & 2 & 2 \\
2 & 2 & 1 \\
2 & 1 & 1
\end{bmatrix}.
\]

The variables that are in play are those appearing in the matrix

\[
\begin{bmatrix}
0 & t_{0,1} & t_{0,2} \\
t_{1,0} & t_{1,1} & 3s_{1,2} \\
t_{2,0} & 3s_{2,1} & 3s_{2,2}
\end{bmatrix}.
\]

In the notation of Case 7.3.2, we are taking $d_1 = 1$, $d_2 = c_4$, $d_3 = 0$, $e_2 = 0$, and $e_3 = 1$. We wish to identify values $a_1, a_2, a_3, a_4, a_5 \in \mathbb{Z}_9$ such that $\partial_1 x \equiv a_1 m_1 + a_2 m_2 + a_3 m_3 + a_4 m_4 + a_5 m_5 \pmod{I}$. We see that $\partial_1 x \in W_2$ if and only if

\begin{align*}
0 & \equiv a_5 \quad \text{(A1)} \\
0 & \equiv c_4 a_4 \quad \text{(A2)} \\
t_{2,0} & \equiv 0 \quad \text{(A3)} \\
s_{2,2} & \equiv 0 \quad \text{(A4)} \\
s_{1,2} & \equiv c_2 s_{2,1} - c_2 a_4 + t_{1,0} \quad \text{(A5)}.
\end{align*}
We wish to identify values $b_1, b_2, b_3, b_4, b_5 \in \mathbb{Z}_9$ such that $\partial_1 x \equiv b_1 m_1 + b_2 m_2 + b_3 m_3 + b_4 m_4 + b_5 m_5 \pmod{I}$. We see that $\partial_2 x \in W_2$ if and only if

$$0 \equiv b_5 \quad \text{(B1)}$$

$$0 \equiv c_4 b_4 \quad \text{(B2)}$$

$$t_{0,2} \equiv 0 \quad \text{(B3)}$$

$$t_{1,1} \equiv 0 \quad \text{(B4)}$$

$$s_{1,2} \equiv c_2 s_{2,1} + b_4 - c_2 t_{0,1} \quad \text{(B5)}.$$

Since $c_4 \neq 0$, then (A2) and (B2) tell us that $a_4 \equiv 0$, and $b_4 \equiv 0$. Then (A5) and (B5) become $s_{1,2} \equiv c_2 s_{2,1} + t_{1,0}$ and $s_{1,2} \equiv c_2 s_{2,1} - c_2 t_{0,1}$. Hence $x \in \partial^{-1} W_2$ if and only if

$$s_{2,2} \equiv 0$$

$$t_{1,1} \equiv 0$$

$$t_{0,2} \equiv 0$$

$$t_{2,0} \equiv 0$$

$$s_{1,2} \equiv c_2 s_{2,1} + t_{1,0}$$

$$s_{1,2} \equiv c_2 s_{2,1} - c_2 t_{0,1}.$$

These are the same congruences as those in Case 7.3 therefore $\partial^{-1} W_2 = \partial^{-1} W_1$. Thus $\text{rank}(\partial^{-1} W_2/W_2) = \text{rank}(\partial^{-1} W_1/W_2)$, $W_2$ is terminal, and $W_2 \notin \hat{L}_2$. 

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Case 7.3.2.3

We fix arbitrary values $c_4 \in \{1, 2\}$ and $c_5 \in \{0, 1, 2\}$. There are 6 ways to choose the values of $c_4, c_5$. Let $m_4 = y_2 + c_4 v_2$ and $m_5 = v_1 + c_5 v_2$. Thus

$$m_4 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 3 & 3c_2c_4 \\ 0 & 3c_4 & 0 \end{bmatrix} \quad \text{and} \quad m_5 = \begin{bmatrix} 0 & c_2 & 0 \\ 1 & 0 & 3(1 + c_2c_5) \\ 0 & 3c_5 & 0 \end{bmatrix}.$$ 

Let $W_2 = \langle W_1, m_4, m_5 \rangle \in \mathcal{L}_2$. The number of subgroups $W_2$ of this type is 6. Since $m_4, m_5 \notin W_1$ and $3m_4, 3m_5 \in W_1$ we have $|W_2/W_1| = 3^2$. Since $|W_1| = 3^6$ it follows that $|W_2| = 3^8$. Note that $\partial^{-1}W_1/W_2 \cong \mathbb{Z}_3$ and $\text{rank}(\partial^{-1}W_1/W_2) = 1$.

We now calculate the pullback of $\partial^{-1}W_2$. We observed in case 7.3.2 that $\partial^{-1}W_2$ is contained in the pattern subgroup

$$\begin{bmatrix} 2 & 2 & 2 \\ 2 & 2 & 1 \\ 2 & 1 & 1 \end{bmatrix}.$$ 

Let

$$x = \begin{bmatrix} 3s_{0,0} + t_{0,0} & 3s_{0,1} + t_{0,1} & 3s_{0,2} + t_{0,2} \\ 3s_{1,0} + t_{1,0} & 3s_{1,1} + t_{1,1} & 3s_{1,2} \\ 3s_{2,0} + t_{2,0} & 3s_{2,1} & 3s_{2,2} \end{bmatrix} \in \begin{bmatrix} 2 & 2 & 2 \\ 2 & 2 & 1 \\ 2 & 1 & 1 \end{bmatrix}.$$ 

The variables that are in play are those appearing in the matrix

$$\begin{bmatrix} 0 & t_{0,1} & t_{0,2} \\ t_{1,0} & t_{1,1} & 3s_{1,2} \\ t_{2,0} & 3s_{2,1} & 3s_{2,2} \end{bmatrix}.$$ 

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In the notation of Case 7.3.2, we are taking \( d_1 = 1, \ d_2 = 0, \ d_3 = 0 \), \( e_2 = 1 \), and \( e_3 = c_5 \). We wish to identify values \( a_1, a_2, a_3, a_4, a_5 \in \mathbb{Z}_9 \) such that

\[
\partial_1 x \equiv a_1 m_1 + a_2 m_2 + a_3 m_3 + a_4 m_4 + a_5 m_5 \pmod{I}.
\]

We see that \( \partial_1 x \in W_2 \) if and only if

\[
0 \equiv a_4 c_3 + a_5 c_5 \quad \text{(A1)}
\]

\[
0 \equiv a_5 \quad \text{(A2)}
\]

\[
t_{2,0} \equiv 0 \quad \text{(A3)}
\]

\[
s_{2,2} \equiv 0 \quad \text{(A4)}
\]

\[
s_{1,2} \equiv c_2 s_{2,1} - c_2 a_4 + t_{1,0} \quad \text{(A5)}.
\]

We wish to identify values \( b_1, b_2, b_3, b_4, b_5 \in \mathbb{Z}_9 \) such that

\[
\partial_2 x \equiv b_1 m_1 + b_2 m_2 + b_3 m_3 + b_4 m_4 + b_5 m_5 \pmod{I}.
\]

We see that \( \partial_2 x \in W_2 \) if and only if

\[
0 \equiv b_4 c_4 + b_5 c_5 \quad \text{(B1)}
\]

\[
0 \equiv b_5 \quad \text{(B2)}
\]

\[
t_{0,2} \equiv 0 \quad \text{(B3)}
\]

\[
t_{1,1} \equiv 0 \quad \text{(B4)}
\]

\[
s_{1,2} \equiv c_2 s_{2,1} + b_4 - c_2 t_{0,1} \quad \text{(B5)}.
\]

Since \( c_4 \neq 0 \), then (A1) and (B1) become \( a_4 \equiv 0 \) and \( b_4 \equiv 0 \). Then (A5) and (B5) become \( s_{1,2} \equiv c_2 s_{2,1} + t_{1,0} \) and \( s_{1,2} \equiv c_2 s_{2,1} - c_2 t_{0,1} \).
Hence \( x \in \partial^{-1}W_2 \) if and only if

\[
\begin{align*}
s_{2,2} & \equiv 0 \\
t_{1,1} & \equiv 0 \\
t_{0,2} & \equiv 0 \\
t_{2,0} & \equiv 0 \\
s_{1,2} & \equiv c_2s_{2,1} + t_{1,0} \\
s_{1,2} & \equiv c_2s_{2,1} - c_2t_{0,1}.
\end{align*}
\]

These are the same congruences as those in Case 7.3 therefore \( \partial^{-1}W_2 = \partial^{-1}W_1 \). Thus \( \text{rank}(\partial^{-1}W_2/W_2) = \text{rank}(\partial^{-1}W_1/W_2) \), \( W_2 \) is terminal, and \( W_2 \not\in \hat{L}_2 \).

### 10.4 Case 7.4

Suppose \((c_1, c_3) = (1, 1)\). Then

\[
m_1 = \begin{bmatrix} 1 & 0 & -3 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad m_2 = \begin{bmatrix} c_2 & 0 & -3c_2 \\ 0 & 3 & 0 \\ -3c_2 & 0 & 0 \end{bmatrix}, \quad \text{and } m_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ -3 & 0 & 0 \end{bmatrix}.
\]

Hence \( x \in \partial^{-1}W_2 \) if and only if

\[
\begin{align*}
s_{1,2} & \equiv -c_2s_{2,1} \quad \text{(A1)} \\
s_{1,2} & \equiv -c_2s_{2,1} \quad \text{(B1)}.
\end{align*}
\]

Hence \( x \in \partial^{-1}W_2 \) if and only if

\[
s_{1,2} \equiv -c_2s_{2,1}.
\]
We regard $t_{1,0}, t_{0,1}$, and $s_{2,1}$ as free variables. Taking $t_{1,0} \equiv 1$, $t_{0,1} = 0$, and $s_{2,1} = 0$, the matrix $x$ becomes

$$v_1 = \begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$  

Taking $t_{0,1} \equiv 1$, $t_{1,0} = 0$, and $s_{2,1} = 0$, the matrix $x$ becomes

$$v_2 = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$  

Taking $s_{2,1} \equiv 1$, $t_{0,1} = 0$, and $t_{1,0} = 0$, the matrix $x$ becomes

$$v_3 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & -3c_2 \\ 0 & 3 & 0 \end{bmatrix}.$$  

We see that $\partial^{-1}W_1 = < \partial^{-1}W_0, v_1, v_2, v_3 >$. Since $v_1 \notin \partial^{-1}W_0$, $v_2 \notin \partial^{-1}W_0, v_1 >$, $v_3 \notin \partial^{-1}W_0, v_1, v_2 >$, $3v_1 \in < \partial^{-1}W_0 >$, $3v_2 \in < \partial^{-1}W_0, v_1 >$, and $3v_3 \in \partial^{-1}W_0, v_1, v_2 >$ we know $|\partial^{-1}W_1/\partial^{-1}W_0| = 3^3$. Since $|\partial^{-1}W_0| = 3^7$ then $|\partial^{-1}W_1| = 3^{10}$. Because $|W_1| = 3^6$ then $|\partial^{-1}W_1/W_1| = 3^4$. Also, $v_1,v_2,v_3,y_2 \in \partial^{-1}W_1$ but are not contained in $W_1$. Each of $3v_1,3v_2, 3v_3, 3y_2$ is contained in $W_1$. So $v_1 + W_1, v_2 + W_1, v_3 + W_1, y_2 + W_1$ are elements of order 3 in the group $\partial^{-1}W_1/W_1$. These four elements form a generating set for the group $\partial^{-1}W_1/W_1$. Recall that $|\partial^{-1}W_1/W_1| = 3^4$, we obtain $\partial^{-1}W_1/W_1 \cong \mathbb{Z}_3 \times \mathbb{Z}_3 \times \mathbb{Z}_3 \times \mathbb{Z}_3$. Since
\[
\text{rank}(\partial^{-1}W_1/W_1) = 4 \quad \text{and} \quad \text{rank}(\partial^{-1}W_0/W_1) = 1, \quad \text{then} \quad \text{rank}(\partial^{-1}W_0/W_1) < \text{rank}(\partial^{-1}W_1/W_1). \]

Hence \( W_1 \) is nonterminal and \( W_1 \in \hat{\mathcal{L}}_1 \). A basis for \( \partial^{-1}W_1/W_1 \) is \( y_2 + W_1, v_1 + W_1, v_2 + W_1, v_3 + W_1 \). A basis for \( \partial^{-1}W_0/W_1 \) is \( y_2 + W_1 \).

The subgroups \( W_2 \) belonging to \( \mathcal{L}(W_1) \) correspond to the nontrivial proper subspaces \( W_2/W_1 \) of \( \partial^{-1}W_1/W_1 \) for which the intersection \( W_2/W_1 \cap \partial^{-1}W_0/W_1 \) is trivial. Since \( \partial^{-1}W_1/W_1 \) has dimension 4 while its subspace \( \partial^{-1}W_0/W_1 \) has dimension 1, every such subspace \( W_2/W_1 \) has dimension either 1 or 2 or 3.

To help us define the subgroups \( W_2 \) belonging to \( \mathcal{L}_2(W_1) \), it will be convenient to identify each element of the vector space \( \partial^{-1}W_1/W_1 \) with its coordinate vector with respect to the ordered basis \( y_2 + W_1, v_1 + W_1, v_2 + W_1, v_3 + W_1 \). In this way, we identify \( \partial^{-1}W_1/W_1 \) with the vector space \( \mathbb{Z}_3 \times \mathbb{Z}_3 \times \mathbb{Z}_3 \times \mathbb{Z}_3 \) consisting of row vectors. Under this identification, the elements \( y_2 + W_1, v_1 + W_1, v_2 + W_1, v_3 + W_1 \) are associated with the so-called standard basis vectors \([1, 0, 0, 0], [0, 1, 0, 0], [0, 0, 1, 0], [0, 0, 0, 1] \) in \( \mathbb{Z}_3 \times \mathbb{Z}_3 \times \mathbb{Z}_3 \times \mathbb{Z}_3 \).

The subgroups \( W_2 \) belonging to \( \mathcal{L}(W_1) \) are in one-to-one correspondence with the nontrivial proper subgroup \( W_2/W_1 \) of \( \partial^{-1}W_1/W_1 \) for which the intersection \( W_2/W_1 \cap \partial^{-1}W_0/W_1 \) is trivial. Note that \( \partial^{-1}W_0/W_1 \) is the 1-dimensional subspace generated by the element \( y_2 + W_1 \). Under our identification, each such subspace \( W_2/W_1 \) is associated with a subspace \( S \) of \( \mathbb{Z}_3 \times \mathbb{Z}_3 \times \mathbb{Z}_3 \times \mathbb{Z}_3 \) that does not contain the standard basis vector \([1, 0, 0, 0] \). Let \( m \) denote the unique matrix in reduced row-echelon form that is row equivalent to the matrix whose rows are the members of an arbitrarily-chosen basis of such a subspace \( S \).
In Case 7.4.1 we consider the 1-dimensional subspaces $W_2/W_1$. There are four possible forms for the matrix $m$. The first form is

$$m = [0, 0, 0, 1]$$

(1 possibility), which is considered in Case 7.4.1.1. The second form is

$$m = [0, 0, 1, c_4] \quad \text{for } c_4 \in \{0, 1, 2\}$$

(3 possibilities), which is considered in Case 7.4.1.2. The third form is

$$m = [0, 1, c_4, c_5] \quad \text{for } c_4, c_5 \in \{0, 1, 2\}$$

(9 possibilities), which is considered in Case 7.4.1.3. The fourth form is

$$m = [1, c_4, c_5, c_6] \quad \text{for } c_4, c_5, c_6 \in \{0, 1, 2\} \quad \text{with } (c_4, c_5, c_6) \neq (0, 0, 0)$$

(26 possibilities), which is considered in Case 7.4.1.4.

In Case 7.4.2 we consider the 2-dimensional subspaces $W_2/W_1$. There are six possible forms for the matrix $m$. The first form is

$$m = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

(1 possibility), which is considered in Case 7.4.2.1. The second form is

$$m = \begin{bmatrix} 0 & 1 & c_4 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \quad \text{for } c_4 \in \{0, 1, 2\}$$

(3 possibilities), which is considered in Case 7.4.2.2. The third form is

$$m = \begin{bmatrix} 0 & 1 & 0 & c_4 \\ 0 & 0 & 1 & c_5 \end{bmatrix} \quad \text{for } c_4, c_5 \in \{0, 1, 2\}$$

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(9 possibilities), which is considered in Case 7.4.2.3. The fourth form is

\[
m = \begin{bmatrix}
1 & c_4 & c_5 & 0 \\
0 & 0 & 0 & 1
\end{bmatrix}
\]  

for \(c_4, c_5 \in \{0, 1, 2\} \) with \((c_4, c_5) \neq (0, 0)\)

(8 possibilities), which is considered in Case 7.4.2.4. The fifth form is

\[
m = \begin{bmatrix}
1 & c_4 & 0 & c_5 \\
0 & 0 & 1 & c_6
\end{bmatrix}
\]  

for \(c_4, c_5, c_6 \in \{0, 1, 2\} \) with \((c_4, c_5) \neq (0, 0)\)

(24 possibilities), which is considered in Case 7.4.2.5. The sixth form is

\[
m = \begin{bmatrix}
1 & 0 & c_4 & c_5 \\
0 & 1 & c_6 & c_7
\end{bmatrix}
\]  

for \(c_4, c_5, c_6, c_7 \in \{0, 1, 2\} \) with \((c_4, c_5) \neq (0, 0)\)

(72 possibilities), which is considered in Case 7.4.2.6.

In Case 7.4.3 we consider the 3-dimensional subspaces \(W_2/W_1\). There are four possible forms for the matrix \(m\). The first form is

\[
m = \begin{bmatrix}
0 & 1 & 0 & 0 \\
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1
\end{bmatrix}
\]  

(1 possibilities), which is considered in Case 7.4.3.1. The second form is

\[
m = \begin{bmatrix}
1 & c_4 & 0 & 0 \\
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1
\end{bmatrix}
\]  

for \(c_4 \in \{1, 2\}\)
(2 possibilities), which is considered in Case 7.4.3.2. The third form is

\[
m = \begin{bmatrix} 1 & 0 & c_4 & 0 \\ 0 & 1 & c_5 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}
\]

for \(c_4 \in \{1, 2\}\) and \(c_5 \in \{0, 1, 2\}\)

(6 possibilities), which is considered in Case 7.4.3.3. The fourth form is

\[
m = \begin{bmatrix} 1 & 0 & 0 & c_4 \\ 0 & 1 & 0 & c_5 \\ 0 & 0 & 1 & c_6 \end{bmatrix}
\]

for \(c_4 \in \{1, 2\}\) and \(c_5, c_6 \in \{0, 1, 2\}\)

(18 possibilities), which is considered in Case 7.4.3.4.

**Lemma 10.4.1.** Let

\[
w = \begin{bmatrix} 0 & 0 & 3r_3 \\ 0 & 3r_2 & 0 \\ 3r_1 & 0 & 0 \end{bmatrix}
\]

for unknowns \(r_1, r_2, r_3 \in \{0, 1, 2\}\), \(w \in W_1\)

if and only if \(r_1 + c_2 r_2 + r_3 \equiv 0\).

**Proof.** Let

\[
w = \begin{bmatrix} 0 & 0 & 3r_3 \\ 0 & 3r_2 & 0 \\ 3r_1 & 0 & 0 \end{bmatrix}
\]

for unknowns \(r_1, r_2, r_3 \in \{0, 1, 2\}\). We want to find conditions on \(r_1, r_2, r_3\) for which \(w \in W_1\) or equivalently \(w \equiv a_1 m_1 + a_2 m_2 + a_3 m_3 (\text{mod } I)\). Recall that \(a_1 m_1 + a_2 m_2 + a_3 m_3 (\text{mod } I)\)

\[
= \begin{bmatrix} a_1 + c_2 a_2 + a_3 & 0 & 3(-a_1 - c_2 a_2) \\ 0 & 3a_2 & 0 \\ 3(-c_2 a_2 - a_3) & 0 & 0 \end{bmatrix}
\]

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Comparing $(1,1)$-entries we get, $r_2 \equiv a_2$. Comparing $(2,0)$-entries, we get $a_3 \equiv -r_1 - c_2r_2$. Comparing $(0,2)$-entries, we get $a_1 \equiv -r_3 - c_2r_2$. Comparing $(0,0)$-entries, we get $a_1 + c_2a_1 + a_3 \equiv 0$. Substituting $a_1, a_2, a_3$ we obtain $r_1 + c_2r_2 + r_3 \equiv 0$.

Thus we conclude that $w \in W_1$ if and only if $r_1 + c_2r_2 + r_3 \equiv 0$.

\[ \square \]

**Lemma 10.4.2.** $w \in W_2$ if and only if $w \in W_1$.

**Proof.** It is clear that $w \in \begin{bmatrix} 2 & 1 & 1 \\ 1 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix} = \partial^{-1}W_0$. We have $\partial^{-1}W_0 \cap W_2 = W_1$.

Hence we conclude that $w \in W_2$ if and only if $w \in W_1$.

\[ \square \]

10.4.1 Case 7.4.1

We consider the 1-dimensional subspaces $W_2/W_1$. Let $d_1, d_2, d_3, d_4$ be unspecified variables. Let $m_4 = d_1y_2 + d_2v_1 + d_3v_2 + d_4v_3$. A formal expression for $m_4$ is

\[
\begin{align*}
    m_4 &= d_1 \begin{bmatrix} 0 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 0 \end{bmatrix} + d_2 \begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} + d_3 \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} + d_4 \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 3 & 0 \end{bmatrix} \\
    &= \begin{bmatrix} 0 & d_3 & 0 \\ d_2 & 3d_1 & -3c_2d_4 \\ 0 & 3d_4 & 0 \end{bmatrix}.
\end{align*}
\]
Let $W_2 = \langle W_1, m_4 \rangle \in \mathcal{L}_2$. We now calculate the pullback $\partial^{-1} W_2$. The subgroup $W_2$ is contained in the pattern subgroup

$$
\begin{bmatrix}
2 & 2 & 1 \\
2 & 1 & 1 \\
1 & 1 & 0
\end{bmatrix}
$$

Thus the pullback $\partial^{-1} W_2$ is contained in the pattern subgroup

$$
\begin{bmatrix}
2 & 2 & 2 \\
2 & 2 & 1 \\
2 & 1 & 1
\end{bmatrix} \cap \begin{bmatrix}
2 & 2 & 2 \\
2 & 2 & 1 \\
2 & 1 & 1
\end{bmatrix} = \begin{bmatrix}
2 & 2 & 2 \\
2 & 2 & 1 \\
2 & 1 & 1
\end{bmatrix}.
$$

Let

$$
x = \begin{bmatrix}
3s_{0,0} + t_{0,0} & 3s_{0,1} + t_{0,1} & 3s_{0,2} + t_{0,2} \\
3s_{1,0} + t_{1,0} & 3s_{1,1} + t_{1,1} & 3s_{1,2} \\
3s_{2,0} + t_{2,0} & 3s_{2,1} & 3s_{2,2}
\end{bmatrix} \in \begin{bmatrix}
2 & 2 & 2 \\
2 & 2 & 1 \\
2 & 1 & 1
\end{bmatrix}.
$$

Thus

$$
\partial_1 x = \begin{bmatrix}
3s_{1,0} + t_{1,0} & 3s_{1,1} + t_{1,1} & 3s_{1,2} \\
3s_{2,0} + t_{2,0} & 3s_{2,1} & 3s_{2,2} \\
-3(t_{1,0} + t_{2,0}) & -3t_{1,1} & 0
\end{bmatrix}
$$

and

$$
\partial_2 x = \begin{bmatrix}
3s_{0,1} + t_{0,1} & 3s_{0,2} + t_{0,2} & -3(t_{0,1} + t_{0,2}) \\
3s_{1,1} + t_{1,1} & 3s_{1,2} & -3t_{1,1} \\
3s_{2,1} & 3s_{2,2} & 0
\end{bmatrix}.$$

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The variables that are in play are those appearing in the matrix

\[
\begin{bmatrix}
0 & t_{0,1} & t_{0,2} \\
t_{1,0} & t_{1,1} & 3s_{1,2} \\
t_{2,0} & 3s_{2,1} & 3s_{2,2}
\end{bmatrix}.
\]

We wish to identify values \(a_1, a_2, a_3, a_4 \in \mathbb{Z}_9\) such that \(\partial_1 x \equiv a_1 m_1 + a_2 m_2 + a_3 m_3 + a_4 m_4 \pmod{I}\). A formal expression for \(a_1 m_1 + a_2 m_2 + a_3 m_3 + a_4 m_4\) is

\[
a_1 \begin{bmatrix}
1 & 0 & -3 \\
0 & 0 & 0 \\
0 & 0 & 0
\end{bmatrix} + a_2 \begin{bmatrix}
c_2 & 0 & -3c_2 \\
0 & 3 & 0 \\
-3c_2 & 0 & 0
\end{bmatrix} + a_3 \begin{bmatrix}
1 & 0 & 0 \\
0 & 0 & 0 \\
-3 & 0 & 0
\end{bmatrix} + a_4 \begin{bmatrix}
0 & d_3 & 0 \\
d_2 & 3d_1 & -3c_2d_4 \\
0 & 3d_4 & 0
\end{bmatrix}
\]

\[
= \begin{bmatrix}
a_1 + a_2c_2 + a_3 & d_3a_4 & -3(a_1 + a_2c_2) \\
a_4d_2 & 3(a_2 + a_4d_1) & -3a_4c_2d_4 \\
-3(a_2c_2 + a_3) & 3d_4a_4 & 0
\end{bmatrix}.
\]

Comparing (0,1)-entries, we get \(t_{1,1} \equiv d_3 a_4\).

Comparing (2,1)-entries, we get \(-t_{1,1} \equiv d_4 a_4\).

Comparing (1,0)-entries, we get \(t_{2,0} \equiv d_2 a_4\).

Comparing (1,2)-entries, we get \(s_{2,2} \equiv -d_4 c_2 a_4\). Substituting \(d_4 a_4 \equiv -t_{1,1}\) we obtain \(s_{2,2} \equiv c_2 t_{1,1}\).

Comparing (1,1)-entries, we get \(a_2 \equiv s_{2,1} - d_1 a_4\).

Comparing (0,2)-entries, we get \(a_1 \equiv -s_{1,2} - c_2 a_2\). Substituting \(a_2\) we obtain \(a_1 \equiv -s_{1,2} - c_2 s_{2,1} + c_2 d_1 a_4\).
Comparing \((2,0)\)-entries, we get \(a_3 \equiv t_{1,0} + t_{2,0} - c_2a_2\). Substituting \(a_2\) we obtain
\[a_3 \equiv t_{1,0} + t_{2,0} - c_2s_{2,1} + c_2d_1a_4.\]
Comparing \((0,0)\)-entries, we get \(t_{1,0} \equiv a_1 + c_2a_2 + a_3\). Substituting \(a_1\), \(a_2\), and \(a_3\) we obtain \(s_{1,2} \equiv -c_2s_{2,1} + c_2d_1a_4 + t_{2,0}\).

We see that \(\partial_1x \in W_2\) if and only if
\[
t_{1,1} \equiv d_3a_4 \quad (A1)
\]
\[-t_{1,1} \equiv d_4a_4 \quad (A2)
\]
\[t_{2,0} \equiv d_2a_4 \quad (A3)
\]
\[s_{2,2} \equiv c_2t_{1,1} \quad (A4)
\]
\[s_{1,2} \equiv -c_2s_{2,1} + c_2d_1a_4 + t_{2,0} \quad (A5).
\]

We wish to identify values \(b_1, b_2, b_3, b_4 \in \mathbb{Z}_9\) such that \(\partial_2x \equiv b_1m_1 + b_2m_2 + b_3m_3 + b_4m_4 \pmod{I}\). A formal expression for \(b_1m_1 + b_2m_2 + b_3m_3 + b_4m_4\) is
\[
\begin{bmatrix}
1 & 0 & -3 \\
0 & 0 & 0 \\
0 & 0 & 0
\end{bmatrix}
+ b_2
\begin{bmatrix}
c_2 & 0 & -3c_2 \\
0 & 3 & 0 \\
-3c_2 & 0 & 0
\end{bmatrix}
+ b_3
\begin{bmatrix}
1 & 0 & 0 \\
0 & 0 & 0 \\
-3 & 0 & 0
\end{bmatrix}
+ b_4
\begin{bmatrix}
0 & d_3 & 0 \\
d_2 & 3d_1 & -3c_2d_4 \\
0 & 3d_4 & 0
\end{bmatrix}
\]
\[
= b_1 + b_2c_2 + b_3 + d_3b_4 - 3(b_1 + b_2c_2)
\begin{bmatrix}
b_4d_2 & 3(b_2 + b_4d_1) & -3b_4c_2d_4 \\
-3(b_2c_2 + b_3) & 3d_4b_4 & 0
\end{bmatrix}.
\]

Comparing \((0,1)\)-entries, we get \(t_{0,2} \equiv d_3b_4\).

Comparing \((2,1)\)-entries, we get \(s_{2,2} \equiv d_4b_4\).
Comparing (1, 0)-entries, we get $t_{1,1} \equiv d_2 b_4$.

Comparing (1, 2)-entries, we get $t_{1,1} \equiv d_4 c_2 b_4$. Substituting $d_4 b_4 \equiv s_{2,2}$, we obtain $t_{1,1} \equiv c_2 s_{2,2}$.

Comparing (1, 1)-entries, we get $b_2 \equiv s_{1,2} - d_1 b_4$.

Comparing (0, 2)-entries, we get $b_1 \equiv t_{0,1} + t_{0,2} - c_2 b_2$. Substituting $b_2$ we obtain

$b_1 \equiv t_{0,1} + t_{0,2} - c_2 s_{1,2} + c_2 d_1 b_4$.

Comparing (2, 0)-entries, we get $b_3 \equiv -c_2 b_2 - s_{2,1}$. Substituting $b_2$ we obtain $b_3 \equiv -c_2 s_{1,2} + c_2 d_1 b_4 - s_{2,1}$.

Comparing (0, 0)-entries, we get $t_{0,1} \equiv b_1 + b_2 c_2 + b_3$. Substituting $b_1$, $b_2$, and $b_3$ we obtain $s_{1,2} \equiv c_2 t_{0,2} - c_2 s_{2,1} + d_1 b_4$.

We see that $\partial_2 x \in W_2$ if and only if

\begin{align*}
  t_{0,2} &\equiv d_3 b_4 \quad \text{(B1)} \\
  s_{2,2} &\equiv d_4 b_4 \quad \text{(B2)} \\
  t_{1,1} &\equiv d_2 b_4 \quad \text{(B3)} \\
  t_{1,1} &\equiv c_2 s_{2,2} \quad \text{(B4)} \\
  s_{1,2} &\equiv c_2 t_{0,2} - c_2 s_{2,1} + d_1 b_4 \quad \text{(B5)}. 
\end{align*}

Case 7.4.1.1

Let $m_4 = v_3$. Thus

$$m_4 = \begin{bmatrix}
  0 & 0 & 0 \\
  0 & 0 & -3 c_2 \\
  0 & 3 & 0
\end{bmatrix}.$$
Let $W_2 = \langle W_1, m_4 \rangle \in \mathcal{L}_2$. The number of subgroups $W_2$ of this type is 1. Since $m_4 \notin W_1$ and $3m_4 \in W_1$ we have $|W_2/W_1| = 3$. Since $|W_1| = 3^6$ it follows that $|W_2| = 3^7$. Note that $\partial^{-1}W_1/W_2 \cong \mathbb{Z}_3 \times \mathbb{Z}_3 \times \mathbb{Z}_3$ and $\text{rank}(\partial^{-1}W_1/W_2) = 3$. We now calculate the pullback of $\partial^{-1}W_2$. We observed in case 7.4.1 that $\partial^{-1}W_2$ is contained in the pattern subgroup

$$\begin{bmatrix} 2 & 2 & 2 \\ 2 & 2 & 1 \\ 2 & 1 & 1 \end{bmatrix}.$$ 

Let 

$$x = \begin{bmatrix} 3s_{0,0} + t_{0,0} & 3s_{0,1} + t_{0,1} & 3s_{0,2} + t_{0,2} \\ 3s_{1,0} + t_{1,0} & 3s_{1,1} + t_{1,1} & 3s_{1,2} \\ 3s_{2,0} + t_{2,0} & 3s_{2,1} & 3s_{2,2} \end{bmatrix} \in \begin{bmatrix} 2 & 2 & 2 \\ 2 & 2 & 1 \\ 2 & 1 & 1 \end{bmatrix}.$$

The variables that are in play are those appearing in the matrix

$$\begin{bmatrix} 0 & t_{0,1} & t_{0,2} \\ t_{1,0} & t_{1,1} & 3s_{1,2} \\ t_{2,0} & 3s_{2,1} & 3s_{2,2} \end{bmatrix}.$$ 

In the notation of Case 7.4.1, we are taking $d_1 = 0$, $d_2 = 0$, $d_3 = 0$, and $d_4 = 1$. We wish to identify values $a_1, a_2, a_3, a_4 \in \mathbb{Z}_9$ such that $\partial_1 x \equiv a_1 m_1 + a_2 m_2 + a_3 m_3 + a_4 m_4 \pmod{I}$. 

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We see that $\partial_1 x \in W_2$ if and only if

\[
t_{1,1} \equiv 0 \quad \text{(A1)}
\]
\[
-t_{1,1} \equiv a_4 \quad \text{(A2)}
\]
\[
t_{2,0} \equiv 0 \quad \text{(A3)}
\]
\[
s_{2,2} \equiv c_2 t_{1,1} \quad \text{(A4)}
\]
\[
s_{1,2} \equiv -c_2 s_{2,1} \quad \text{(A5)}.
\]

We wish to identify values $b_1, b_2, b_3, b_4 \in \mathbb{Z}_9$ such that $\partial_1 x \equiv b_1 m_1 + b_2 m_2 + b_3 m_3 + b_4 m_4 \pmod{I}$. We see that $\partial_2 x \in W_2$ if and only if

\[
t_{0,2} \equiv 0 \quad \text{(B1)}
\]
\[
s_{2,2} \equiv b_4 \quad \text{(B2)}
\]
\[
t_{1,1} \equiv 0 \quad \text{(B3)}
\]
\[
t_{1,1} \equiv c_2 s_{2,2} \quad \text{(B4)}
\]
\[
s_{1,2} \equiv -c_2 s_{2,1} \quad \text{(B5)}.
\]

Comparing (A1) and (A2) we obtain $a_4 \equiv 0$. Substituting $t_{1,1} \equiv 0$ into (A4) we obtain $s_{2,2} \equiv 0$. From this we obtain $b_4 \equiv 0$. The congruence (B4) holds automatically. (B5) is redundant with (A5).
Hence $x \in \partial^{-1}W_2$ if and only if
\[
t_{1,1} \equiv 0 \\
t_{2,0} \equiv 0 \\
t_{0,2} \equiv 0 \\
s_{2,2} \equiv 0 \\
s_{1,2} \equiv -c_2s_{2,1}.
\]

This is the same congruence as that in Case 7.4 therefore $\partial^{-1}W_2 = \partial^{-1}W_1$. Thus $\text{rank}(\partial^{-1}W_2/W_2) = \text{rank}(\partial^{-1}W_1/W_2)$, $W_2$ is terminal, and $W_2 \notin \hat{L}_2$.

Case 7.4.1.2

We fix arbitrary value $c_4 \in \{0, 1, 2\}$. There are 3 ways to choose the value $c_4$. Let $m_4 = v_2 + c_4 v_3$. Thus
\[
m_4 = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & -3c_2c_4 \\ 0 & 3c_4 & 0 \end{bmatrix}.
\]

Let $W_2 = \langle W_1, m_4 \rangle \in \mathcal{L}_2$. The number of subgroups $W_2$ of this type is 3. Since $m_4 \notin W_1$ and $3m_4 \in W_1$ we have $|W_2/W_1| = 3$. Since $|W_1| = 3^6$ it follows that $|W_2| = 3^7$. Note that $\partial^{-1}W_1/W_2 \cong \mathbb{Z}_3 \times \mathbb{Z}_3 \times \mathbb{Z}_3$ and $\text{rank}(\partial^{-1}W_1/W_2) = 3$. We now calculate the pullback of $\partial^{-1}W_2$. We observed in case 7.4.1 that $\partial^{-1}W_2$ is contained
in the pattern subgroup

\[
\begin{bmatrix}
2 & 2 & 2 \\
2 & 2 & 1 \\
2 & 1 & 1
\end{bmatrix}
\]

Let

\[
x = \begin{bmatrix}
3s_{0,0} + t_{0,0} & 3s_{0,1} + t_{0,1} & 3s_{0,2} + t_{0,2} \\
3s_{1,0} + t_{1,0} & 3s_{1,1} + t_{1,1} & 3s_{1,2} \\
3s_{2,0} + t_{2,0} & 3s_{2,1} & 3s_{2,2}
\end{bmatrix}
\in \begin{bmatrix}
2 & 2 & 2 \\
2 & 2 & 1 \\
2 & 1 & 1
\end{bmatrix}.
\]

The variables that are in play are those appearing in the matrix

\[
\begin{bmatrix}
0 & t_{0,1} & t_{0,2} \\
t_{1,0} & t_{1,1} & 3s_{1,2} \\
t_{2,0} & 3s_{2,1} & 3s_{2,2}
\end{bmatrix}
\]

In the notation of Case 7.4.1, we are taking \(d_1 = 0\), \(d_2 = 0\), \(d_3 = 1\), and \(d_4 = c_4\). We wish to identify values \(a_1, a_2, a_3, a_4 \in \mathbb{Z}_9\) such that \(\partial_1 x \equiv a_1 m_1 + a_2 m_2 + a_3 m_3 + a_4 m_4 \pmod{I}\). We see that \(\partial_1 x \in W_2\) if and only if

\[
\begin{align*}
t_{1,1} &\equiv a_4 \quad (A1) \\
-t_{1,1} &\equiv c_4 a_4 \quad (A2) \\
t_{2,0} &\equiv 0 \quad (A3) \\
s_{2,2} &\equiv c_2 t_{1,1} \quad (A4) \\
s_{1,2} &\equiv -c_2 s_{2,1} \quad (A5).
\end{align*}
\]
We wish to identify values $b_1, b_2, b_3, b_4 \in \mathbb{Z}_9$ such that $\partial_1 x \equiv b_1 m_1 + b_2 m_2 + b_3 m_3 + b_4 m_4 \pmod{I}$. We see that $\partial_2 x \in W_2$ if and only if

$$t_{0,2} \equiv b_4 \quad \text{(B1)}$$

$$s_{2,2} \equiv c_4 b_4 \quad \text{(B2)}$$

$$t_{1,1} \equiv 0 \quad \text{(B3)}$$

$$t_{1,1} \equiv c_2 s_{2,2} \quad \text{(B4)}$$

$$s_{1,2} \equiv c_2 t_{0,2} - c_2 s_{2,1} \quad \text{(B5)}.$$

Substituting $t_{1,1} \equiv 0$ we obtain $a_4 \equiv 0$ from congruence (A1), $s_{2,2} \equiv 0$ from congruence (A4), and congruence (A2) automatically holds. Substituting $s_{2,2} \equiv 0$ into (B2) we obtain $c_4 b_4 \equiv 0$ and (B4) holds. Combining (A5) and (B5) we obtain $c_2 t_{0,2} \equiv 0$. Since $c_2 \neq 0$, then $t_{0,2} \equiv 0$.

Hence $x \in \partial^{-1}W_2$ if and only if

$$t_{1,1} \equiv 0$$

$$t_{2,0} \equiv 0$$

$$t_{0,2} \equiv 0$$

$$s_{2,2} \equiv 0$$

$$s_{1,2} \equiv -c_2 s_{2,1}.$$

This is the same congruence as that in Case 7.4. Therefore $\partial^{-1}W_2 = \partial^{-1}W_1$.

Thus $\text{rank}(\partial^{-1}W_2/W_2) = \text{rank}(\partial^{-1}W_1/W_2)$, $W_2$ is terminal, and $W_2 \notin \hat{\mathcal{L}}_2$.  

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Case 7.4.1.3

We fix arbitrary values $c_4, c_5 \in \{0, 1, 2\}$. There are 9 ways to choose the values $c_4$ and $c_5$. Let $m_4 = v_1 + c_4 v_2 + c_5 v_3$. Thus

$$m_4 = \begin{bmatrix} 0 & c_4 & 0 \\ 1 & 0 & -3c_2 c_5 \\ 0 & 3c_5 & 0 \end{bmatrix}.$$ 

Let $W_2 = < W_1, m_4 > \in \mathcal{L}_2$. The number of subgroups $W_2$ of this type is 9. Since $m_4 \notin W_1$ and $3m_4 \in W_1$ we have $|W_2/W_1| = 3$. Since $|W_1| = 3^6$ it follows that $|W_2| = 3^7$. Note that $\partial^{-1}W_1/W_2 \cong \mathbb{Z}_3 \times \mathbb{Z}_3 \times \mathbb{Z}_3$ and rank$(\partial^{-1}W_1/W_2) = 3$. We now calculate the pullback of $\partial^{-1}W_2$. We observed in case 7.4.1 that $\partial^{-1}W_2$ is contained in the pattern subgroup

$$\begin{bmatrix} 2 & 2 & 2 \\ 2 & 2 & 1 \\ 2 & 1 & 1 \end{bmatrix}.$$ 

Let

$$x = \begin{bmatrix} 3s_{0,0} + t_{0,0} & 3s_{0,1} + t_{0,1} & 3s_{0,2} + t_{0,2} \\ 3s_{1,0} + t_{1,0} & 3s_{1,1} + t_{1,1} & 3s_{1,2} \\ 3s_{2,0} + t_{2,0} & 3s_{2,1} & 3s_{2,2} \end{bmatrix} \in \begin{bmatrix} 2 & 2 & 2 \\ 2 & 2 & 1 \\ 2 & 1 & 1 \end{bmatrix}.$$ 

The variables that are in play are those appearing in the matrix

$$\begin{bmatrix} 0 & t_{0,1} & t_{0,2} \\ t_{1,0} & t_{1,1} & 3s_{1,2} \\ t_{2,0} & 3s_{2,1} & 3s_{2,2} \end{bmatrix}.$$
In the notation of Case 7.4.1, we are taking $d_1 = 0$, $d_2 = 1$, $d_3 = c_4$, and $d_4 = c_5$. We wish to identify values $a_1, a_2, a_3, a_4 \in \mathbb{Z}_9$ such that $\partial x \equiv a_1 m_1 + a_2 m_2 + a_3 m_3 + a_4 m_4 \pmod{I}$. We see that $\partial x \in W_2$ if and only if

$$t_{1,1} \equiv c_4 a_4 \quad (A1)$$
$$-t_{1,1} \equiv c_5 a_4 \quad (A2)$$
$$t_{2,0} \equiv a_4 \quad (A3)$$
$$s_{2,2} \equiv c_2 t_{1,1} \quad (A4)$$
$$s_{1,2} \equiv -c_2 s_{2,1} + t_{2,0} \quad (A5).$$

We wish to identify values $b_1, b_2, b_3, b_4 \in \mathbb{Z}_9$ such that $\partial x \equiv b_1 m_1 + b_2 m_2 + b_3 m_3 + b_4 m_4 \pmod{I}$. We see that $\partial x \in W_2$ if and only if

$$t_{0,2} \equiv c_4 b_4 \quad (B1)$$
$$s_{2,2} \equiv c_5 b_4 \quad (B2)$$
$$t_{1,1} \equiv b_4 \quad (B3)$$
$$t_{1,1} \equiv c_2 s_{2,2} \quad (B4)$$
$$s_{1,2} \equiv c_2 t_{0,2} - c_2 s_{2,1} \quad (B5).$$

Substituting $a_4$ into (A1) and (A2) we obtain $t_{1,1} \equiv c_4 t_{2,0}$ and $t_{1,1} \equiv -c_5 t_{2,0}$ respectively. Substituting $b_4$ into (B1) and (B2) we obtain $t_{0,2} \equiv c_4 t_{1,1}$ and $s_{2,2} \equiv c_5 t_{1,1}$ respectively. Combining (A5) and (B5) we obtain $t_{0,2} \equiv c_2 t_{2,0}$ which we will denote as our new (A5). Multiplying (B4) by $c_2$ we obtain $s_{2,2} \equiv c_2 t_{1,1}$ which is redundant with (A4). Combining (A1) and (A2) we obtain $(c_4 + c_5) t_{2,0} \equiv 0$ which we we denote as
our new (A1). Combining (A1) and (B1) we obtain $t_{0,2} \equiv c_4^2 t_{2,0}$ which we will denote as our new (B1). Substituting (B1) into (A5) we obtain $(c_2 - c_4^4) t_{2,0} \equiv 0$. Combining (A4) and (B2) we obtain $(c_2 - c_5) t_{1,1} \equiv 0$ which we will denote as our new (B2).

Hence $x \in \partial^{-1} W_2$ if and only if

\begin{align*}
(c_4 + c_5) t_{2,0} &\equiv 0 \quad \text{(A1)} \\
s_{2,2} &\equiv c_2 t_{1,1} \quad \text{(A4)} \\
(c_2 - c_4^2) t_{2,0} &\equiv 0 \quad \text{(A5)} \\
t_{0,2} &\equiv c_4^2 t_{2,0} \quad \text{(B1)} \\
(c_2 - c_5) t_{1,1} &\equiv 0 \quad \text{(B2)} \\
s_{1,2} &\equiv c_2 t_{0,2} - c_2 s_{2,1} \quad \text{(B5)}.
\end{align*}

If either $c_4 + c_5 \neq 0$ or $c_2 - c_4^2 \neq 0$ then $t_{2,0} = 0$. It then follows that $t_{0,2} \equiv t_{1,1} \equiv s_{2,2} \equiv t_{2,0} \equiv 0$.

If $c_2 - c_5 \neq 0$ then $t_{1,1} = 0$. It then follows that $t_{0,2} \equiv t_{1,1} \equiv s_{2,2} \equiv t_{2,0} \equiv 0$.

Thus we will assume that $c_4 + c_5 \equiv 0$, $c_2 - c_4^2 \equiv 0$, and $c_2 - c_5 \equiv 0$. Since $c_4^2 \equiv c_2 \in \{1,2\}$ we get $c_4 \neq 0$. Hence $c_4 \in \{1,2\}$. It follows that $c_2 \equiv 1$. Hence $c_5 \equiv 1$. Since $c_4 + c_5 \equiv 0$, and $c_5 \equiv 1$, we get $c_4 \equiv 2$.

**Case 7.4.1.3.1** Suppose $c_4 \neq 2$ or $c_2 \neq 1$ or $c_5 \neq 1$.

If we suppose $c_4 \neq 2$. Then $c_4 \in \{0,1\}$. If $c_4 \equiv 0$, then because $c_2 \in \{1,2\}$ we get $c_2 - c_4^2 \neq 0$ and by $(c_2 - c_4) t_{2,0} \equiv 0$ we get $t_{2,0} \equiv 0$. It then follows that $t_{0,2} \equiv t_{1,1} \equiv s_{2,2} \equiv t_{2,0} \equiv 0$ and $s_{1,2} \equiv -c_2 s_{2,1}$. This is the same congruence as in
Case 7.4. Hence $W_2$ is terminal, and $W_2 \not\in \hat{L}_2$. If $c_4 \equiv 1$ then because $c_2 \in \{1, 2\}$ we get $c_2 \equiv 1$ and $(c_2 - c_4^2) \equiv 0$. Also since $c_2 \equiv c_5$ we have $c_5 \equiv 1$. Then $c_4 + c_5 \not\equiv 0$ and by $(c_4 + c_5)t_{2,0} \equiv 0$ we get $t_{2,0} \equiv 0$. It then follows that $t_{0,2} \equiv t_{1,1} \equiv s_{2,2} \equiv t_{2,0} \equiv 0$ and $s_{1,2} \equiv -c_2s_{2,1}$. This is the same congruence as in Case 7.4. Hence $W_2$ is terminal, and $W_2 \not\in \hat{L}_2$. 

Suppose $c_2 \not\equiv 1$. Then $c_2 \equiv 2$ since $c_2 \in \{1, 2\}$. Since 2 is not a square mod 3 it follows that $c_2 - c_4^2 \not\equiv 0$ and by $(c_2 - c_4^2)t_{2,0} \equiv 0$ we get $t_{2,0} \equiv 0$. It then follows that $t_{0,2} \equiv t_{1,1} \equiv s_{2,2} \equiv t_{2,0} \equiv 0$ and $s_{1,2} \equiv -c_2s_{2,1}$. This is the same congruence as in Case 7.4. Hence $W_2$ is terminal, and $W_2 \not\in \hat{L}_2$. 

Suppose $c_5 \not\equiv 1$. Then $c_5 \in \{0, 2\}$. If $c_5 \equiv 0$ then $c_2 - c_5 \not\equiv 0$ and by $(c_2 - c_5)t_{1,1} \equiv 0$ we get $t_{1,1} \equiv 0$. It then follows that $t_{0,2} \equiv t_{1,1} \equiv s_{2,2} \equiv t_{2,0} \equiv 0$ and $s_{1,2} \equiv -c_2s_{2,1}$. This is the same congruence as in Case 7.4. Hence $W_2$ is terminal, and $W_2 \not\in \hat{L}_2$. If $c_5 \equiv 2$, then $c_2 \equiv 2$. Then since 2 is not a square mod 3 it follows that $c_2 - c_4^2 \not\equiv 0$ and by $(c_2 - c_4^2)t_{2,0}$ we get $t_{2,0} \equiv 0$. It then follows that $t_{0,2} \equiv t_{1,1} \equiv s_{2,2} \equiv t_{2,0} \equiv 0$ and $s_{1,2} \equiv -c_2s_{2,1}$. This is the same congruence as in Case 7.4. Hence $W_2$ is terminal, and $W_2 \not\in \hat{L}_2$.

Case 7.4.1.3.2 Now assume $c_4 \equiv 2$, $c_2 \equiv 1$, and $c_5 \equiv 1$. Then $t_{2,0}$ and $t_{1,1}$ become free variables.
Hence \( x \in \partial^{-1}W_2 \) if and only if

\[
\begin{align*}
  s_{2,2} & \equiv t_{1,2} \quad \text{(A4)} \\
  t_{0,2} & \equiv t_{2,0} \quad \text{(B1)} \\
  s_{1,2} & \equiv t_{2,0} - s_{2,1} \quad \text{(B5)}.
\end{align*}
\]

We regard \( t_{0,1}, t_{1,0}, s_{2,1}, t_{2,0}, t_{1,1} \) as free variables. Taking \( t_{1,0} \equiv 1, t_{0,1} = 0, s_{2,1} = 0, t_{2,0} = 0, \) and \( t_{1,1} = 0, \) the matrix \( x \) becomes

\[
v_1 = \begin{bmatrix}
0 & 0 & 0 \\
1 & 0 & 0 \\
0 & 0 & 0
\end{bmatrix}.
\]

Taking \( t_{0,1} \equiv 1, t_{1,0} = 0, s_{2,1} = 0, t_{2,0} = 0, \) and \( t_{1,1} = 0, \) the matrix \( x \) becomes

\[
v_2 = \begin{bmatrix}
0 & 1 & 0 \\
0 & 0 & 0 \\
0 & 0 & 0
\end{bmatrix}.
\]

Taking \( s_{2,1} \equiv 1, t_{0,1} = 0, t_{1,0} = 0, t_{2,0} = 0, \) and \( t_{1,1} = 0, \) the matrix \( x \) becomes

\[
v_3 = \begin{bmatrix}
0 & 0 & 0 \\
0 & 0 & -3 \\
0 & 3 & 0
\end{bmatrix}.
\]

Taking \( t_{2,0} \equiv 1, t_{0,1} = 0, t_{1,0} = 0, s_{2,1} = 0, \) and \( t_{1,1} = 0, \) the matrix \( x \) becomes

\[
v_4 = \begin{bmatrix}
0 & 0 & 1 \\
0 & 0 & 0 \\
1 & 3 & 0
\end{bmatrix}.
\]
Taking $t_{1,1} \equiv 1$, $t_{0,1} = 0$, $t_{1,0} = 0$, $s_{2,1} = 0$, and $t_{2,0} = 0$, the matrix $x$ becomes

$$v_5 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 3 \end{bmatrix}.$$

We see that neither $v_4, v_5$ nor $3v_4, 3v_5$ is contained in $W_2$ and that $\partial^{-1}W_2 = < \partial^{-1}W_0, v_1, v_2, v_3, v_4, v_5 >$. We know that $v_4 \in \partial^{-1}W_2, v_5 \in \partial^{-1}W_2$ and since $3v_4 \notin W_2, 3v_5 \notin W_2$ then $v_4 + W_2$ and $v_5 + W_2$ are elements of the group $\partial^{-1}W_2/W_2$ whose order is larger than 3. Recall that $|\partial^{-1}W_2/W_2| = 3^5$ and that $\partial^{-1}W_1/W_2$ has rank 3. We obtain $\partial^{-1}W_2/W_2 \cong \mathbb{Z}_9 \times \mathbb{Z}_9 \times \mathbb{Z}_3$. Since $\text{rank}(\partial^{-1}W_2/W_2) = \text{rank}(\partial^{-1}W_1/W_2)$. Hence $W_2$ is terminal and $W_2 \notin \mathbf{\hat{L}}_2$.

**Case 7.4.1.4**

We fix arbitrary values $c_4, c_5, c_6 \in \{0, 1, 2\}$ and $(c_4, c_5, c_6) \neq (0, 0, 0)$. There are 26 ways to choose the values $c_4, c_5, c_6$. Let $m_4 = y_2 + c_4v_1 + c_5v_2 + c_6v_3$. Thus

$$m_4 = \begin{bmatrix} 0 & c_5 & 0 \\ c_4 & 3 & -3c_2c_6 \\ 0 & 3c_6 & 0 \end{bmatrix}.$$

Let $W_2 = < W_1, m_4 > \in \mathbf{L}_2$. The number of subgroups $W_2$ of this type is 26. Since $m_4 \notin W_1$ and $3m_4 \in W_1$ we have $|W_2/W_1| = 3$. Since $|W_1| = 3^6$ it follows that $|W_2| = 3^7$. Note that $\partial^{-1}W_1/W_2 \cong \mathbb{Z}_3 \times \mathbb{Z}_3 \times \mathbb{Z}_3$ and $\text{rank}(\partial^{-1}W_1/W_2) = 3$. We now calculate the pullback of $\partial^{-1}W_2$. We observed in case 7.4.1 that $\partial^{-1}W_2$ is contained
in the pattern subgroup

\[
\begin{bmatrix}
2 & 2 & 2 \\
2 & 2 & 1 \\
2 & 1 & 1 \\
\end{bmatrix}.
\]

Let

\[
x = \begin{bmatrix}
3s_{0,0} + t_{0,0} & 3s_{0,1} + t_{0,1} & 3s_{0,2} + t_{0,2} \\
3s_{1,0} + t_{1,0} & 3s_{1,1} + t_{1,1} & 3s_{1,2} \\
3s_{2,0} + t_{2,0} & 3s_{2,1} & 3s_{2,2}
\end{bmatrix} \in \begin{bmatrix}
2 & 2 & 2 \\
2 & 2 & 1 \\
2 & 1 & 1 \\
\end{bmatrix}.
\]

The variables that are in play are those appearing in the matrix

\[
\begin{bmatrix}
0 & t_{0,1} & t_{0,2} \\
t_{1,0} & t_{1,1} & 3s_{1,2} \\
t_{2,0} & 3s_{2,1} & 3s_{2,2}
\end{bmatrix}.
\]

In the notation of Case 7.4.1, we are taking \(d_1 = 1, d_2 = c_4, d_3 = c_5,\) and \(d_4 = c_6.\) We wish to identify values \(a_1, a_2, a_3, a_4 \in \mathbb{Z}_9\) such that \(\partial_1 x \equiv a_1 m_1 + a_2 m_2 + a_3 m_3 + a_4 m_4 \pmod{I}.\) We see that \(\partial_1 x \in W_2\) if and only if

\[
\begin{align*}
t_{1,1} &\equiv c_5 a_4 \quad (A1) \\
-t_{1,1} &\equiv c_6 a_4 \quad (A2) \\
t_{2,0} &\equiv c_4 a_4 \quad (A3) \\
s_{2,2} &\equiv c_2 t_{1,1} \quad (A4) \\
s_{1,2} &\equiv -c_2 s_{2,1} + c_2 a_4 + t_{2,0} \quad (A5). \]
\]
We wish to identify values \(b_1, b_2, b_3, b_4 \in \mathbb{Z}_9\) such that \(\partial_1 x \equiv b_1 m_1 + b_2 m_2 + b_3 m_3 + b_4 m_4 \pmod{I}\). We see that \(\partial_2 x \in W_2\) if and only if

\[
\begin{align*}
t_{0,2} &\equiv c_5 b_4 \quad \text{(B1)} \\
s_{2,2} &\equiv c_6 b_4 \quad \text{(B2)} \\
t_{1,1} &\equiv c_4 b_4 \quad \text{(B3)} \\
t_{1,1} &\equiv c_2 s_{2,2} \quad \text{(B4)} \\
s_{1,2} &\equiv c_2 t_{0,2} - c_2 s_{2,1} + b_4 \quad \text{(B5)}.
\end{align*}
\]

Multiplying (B4) by \(c_2\) we obtain \(s_{2,2} \equiv c_2 t_{1,1}\) which is redundant with (A4). Combining (A5) and (B5) we obtain \(c_2 a_4 + t_{2,0} \equiv c_2 t_{0,2} + b_4\) which we will denote as our new (B5).

Hence \(x \in \partial^{-1} W_2\) if and only if

\[
\begin{align*}
t_{1,1} &\equiv c_5 a_4 \quad \text{(A1)} \\
-t_{1,1} &\equiv c_6 a_4 \quad \text{(A2)} \\
t_{2,0} &\equiv c_4 a_4 \quad \text{(A3)} \\
s_{2,2} &\equiv c_2 t_{1,1} \quad \text{(A4)} \\
s_{1,2} &\equiv -c_2 s_{2,1} + c_2 a_4 + t_{2,0} \quad \text{(A5)} \\
t_{0,2} &\equiv c_5 b_4 \quad \text{(B1)} \\
s_{2,2} &\equiv c_6 b_4 \quad \text{(B2)} \\
t_{1,1} &\equiv c_4 b_4 \quad \text{(B3)} \\
c_2 a_4 + t_{2,0} &\equiv c_2 t_{0,2} + b_4 \quad \text{(B5)}.
\end{align*}
\]

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Recall that \((c_4, c_5, c_6) \neq (0, 0, 0)\). Therefore it is convenient to consider the following three cases. The first case is when \(c_4 \neq 0\). The second case is when \(c_4 = 0\) and \(c_5 \neq 0\). The third case is when \(c_4 = c_5 = 0\), and \(c_6 \neq 0\).

**Case 7.4.1.4.1** First we consider the case \(c_4 \neq 0\). Then \(c_4 \in \{1, 2\} \) and \(c_4^2 \equiv 1\). Solving (A3) for \(a_4\) and (B3) for \(b_4\) we obtain \(a_4 \equiv c_4 t_{2,0}\) and \(b_4 \equiv c_4 t_{1,1}\). Substituting \(a_4\) into (A1), (A2), and (A5) we obtain \(t_{1,1} \equiv c_4 c_5 t_{2,0}\), \(-t_{1,1} \equiv c_4 c_6 t_{2,0}\), and \(s_{1,2} \equiv -c_2 s_{2,1} + (c_2 c_4 + 1) t_{2,0}\). Substituting \(b_4\) into (B1) and (B2) we obtain \(t_{0,2} \equiv c_4 c_5 t_{1,1}\) and \(s_{2,2} \equiv c_4 c_6 t_{1,1}\). Substituting \(a_4\) and \(b_4\) into (B5) we obtain \(c_2 c_4 t_{2,0} + t_{2,0} \equiv c_2 t_{0,2} + c_4 t_{1,1}\). Combining (A1) and (A2) we get \((c_5 + c_6) t_{2,0} \equiv 0\). Combining (A4) and (B2) we get \((c_2 - c_4 c_6) t_{1,1} \equiv 0\). Substituting (B1) and (A1) into (B5) we obtain \((c_2 c_4 + 1 - c_2 c_5^2 - c_5) t_{2,0} \equiv 0\).

Hence \(x \in \partial^{-1} W_2\) if and only if

\[
\begin{align*}
(c_5 + c_6) t_{2,0} &\equiv 0 \\
(c_2 - c_4 c_6) t_{1,1} &\equiv 0 \\
t_{1,1} &\equiv c_4 c_5 t_{2,0} \\
s_{1,2} &\equiv -c_2 s_{2,1} + (c_2 c_4 + 1) t_{2,0} \\
t_{0,2} &\equiv c_4 c_5 t_{1,1} \\
s_{2,2} &\equiv c_4 c_6 t_{1,1} \\
(c_2 c_4 + 1 - c_2 c_5^2 - c_5) t_{2,0} &\equiv 0
\end{align*}
\]
If either $c_5 + c_6 \neq 0$ or $c_2c_4 + 1 - c_2c_5^2 - c_5 \neq 0$ then $t_{2,0} \equiv 0$. It then follows that $t_{2,0} \equiv t_{1,1} \equiv s_{2,2} \equiv t_{0,2} \equiv 0$.

If $c_2 - c_4c_6 \neq 0$ then $t_{1,1} \equiv 0$. It then follows that $t_{2,0} \equiv t_{1,1} \equiv s_{2,2} \equiv t_{0,2} \equiv 0$.

Thus we will assume that $c_5 + c_6 \equiv 0$, $c_2c_4 - c_6 \equiv 0$, and $c_2c_4 + 1 - c_2c_5^2 - c_5 \equiv 0$. This implies that $c_5 \equiv -c_6$ and $c_2c_4 \equiv c_6$. Substituting these congruences into $c_2c_4 + 1 - c_2c_5^2 - c_5 \equiv 0$ we obtain $1 - c_2c_6^2 - c_6 \equiv 0$. Since $c_2c_4 \equiv c_6$ and $c_2$, $c_4 \in \{1, 2\}$ then $c_2c_4 \neq 0$ and hence $c_6 \neq 0$. Then $c_5 \neq 0$. We know $c_6^2 \equiv 1$ so $1 - c_2c_6^2 - c_6 \equiv 0$ becomes $c_2 \equiv 1 - c_6$. Since $c_2 \neq 0$ then $c_6 \equiv 2$. This implies that $c_2 \equiv 2$ and $c_5 \equiv 1$ since $c_5 \equiv -c_6$. Then from $c_2c_4 \equiv c_6$ we obtain $c_4 \equiv 1$.

**Case 7.4.1.4.1.1** Assume that $c_2 \neq 2$ or $c_4 \neq 1$ or $c_5 \neq 1$ or $c_6 \neq 2$.

Suppose that $c_2 \neq 2$. Then since $c_2 \in \{1, 2\}$ we know that $c_2 \equiv 1$. Then $c_4 \equiv c_6$. Since $c_4 \neq 0$ then $c_6 \neq 0$. If $c_4 \equiv 1$, then $c_6 \equiv 1$ and $c_5 \equiv 2$. Then $1 - c_2c_6^2 - c_6 \neq 0$ and by $(1 - c_2c_6^2 - c_6)t_{2,0} \equiv 0$, we get $t_{2,0} \equiv 0$. It then follows that $t_{0,2} \equiv t_{1,1} \equiv s_{2,2} \equiv t_{2,0} \equiv 0$ and $s_{1,2} \equiv -c_2s_{2,1}$. This is the same congruence as in Case 7.4. Hence $W_2$ is terminal, and $W_2 \not\in \hat{L}_2$. If $c_4 \equiv 2$, then $c_6 \equiv 2$ and $c_5 \equiv 2$. Then $1 - c_2c_6^2 - c_6 \neq 0$ and by $(1 - c_2c_6^2 - c_6)t_{2,0} \equiv 0$, we get $t_{2,0} \equiv 0$. It then follows that $t_{0,2} \equiv t_{1,1} \equiv s_{2,2} \equiv t_{2,0} \equiv 0$ and $s_{1,2} \equiv -c_2s_{2,1}$. This is the same congruence as in Case 7.4. Hence $W_2$ is terminal, and $W_2 \not\in \hat{L}_2$.

Suppose $c_4 \neq 1$, then $c_4 \equiv 2$. If $c_2 \equiv 1$ then $c_6 \equiv 2$ and $c_5 \equiv 1$. Then $1 - c_2c_6^2 - c_6 \neq 0$ and by $(1 - c_2c_6^2 - c_6)t_{2,0} \equiv 0$, we get $t_{2,0} \equiv 0$. It then follows that $t_{0,2} \equiv t_{1,1} \equiv s_{2,2} \equiv t_{2,0} \equiv 0$ and $s_{1,2} \equiv -c_2s_{2,1}$. This is the same congruence as in
Case 7.4. Hence $W_2$ is terminal, and $W_2 \not\in \hat{L}_2$. If $c_2 \equiv 2$, then $c_6 \equiv 1$ and $c_5 \equiv 2$.

Then $1 - c_2 c_6^2 - c_6 \not\equiv 0$ and by $(1 - c_2 c_6^2 - c_6) t_{2,0} \equiv 0$, we get $t_{2,0} \equiv 0$. It then follows that $t_{0,2} \equiv t_{1,1} \equiv s_{2,2} \equiv t_{2,0} \equiv 0$ and $s_{1,2} \equiv -c_2 s_{2,1}$. This is the same congruence as in Case 7.4. Hence $W_2$ is terminal, and $W_2 \not\in \hat{L}_2$.

Suppose $c_5 \not\equiv 1$. If $c_5 \equiv 0$ then $c_6 \equiv 0$. We see that $c_2 c_4 - c_6 \not\equiv 0$ since $c_2, c_4 \not\equiv 0$. Then by $(c_2 c_4 - c_6) t_{1,1} \not\equiv 0$ and we get $t_{1,1} \equiv 0$. It then follows that $t_{0,2} \equiv t_{1,1} \equiv s_{2,2} \equiv t_{2,0} \equiv 0$ and $s_{1,2} \equiv -c_2 s_{2,1}$. This is the same congruence as in Case 7.4. Hence $W_2$ is terminal, and $W_2 \not\in \hat{L}_2$. If $c_5 \equiv 2$ then $c_6 \equiv 1$. Then $1 - c_2 c_6^2 - c_6 \not\equiv 0$ and by $(1 - c_2 c_6^2 - c_6) t_{2,0} \equiv 0$, we get $t_{2,0} \equiv 0$. It then follows that $t_{0,2} \equiv t_{1,1} \equiv s_{2,2} \equiv t_{2,0} \equiv 0$ and $s_{1,2} \equiv -c_2 s_{2,1}$. This is the same congruence as in Case 7.4. Hence $W_2$ is terminal, and $W_2 \not\in \hat{L}_2$.

Suppose $c_6 \not\equiv 2$. If $c_6 \equiv 0$ then $c_5 \equiv 0$. We see that $c_2 c_4 - c_6 \not\equiv 0$ since $c_2, c_4 \not\equiv 0$. Then by $(c_2 c_4 - c_6) t_{1,1} \not\equiv 0$ and we get $t_{1,1} \equiv 0$. It then follows that $t_{0,2} \equiv t_{1,1} \equiv s_{2,2} \equiv t_{2,0} \equiv 0$ and $s_{1,2} \equiv -c_2 s_{2,1}$. This is the same congruence as in Case 7.4. Hence $W_2$ is terminal, and $W_2 \not\in \hat{L}_2$. Then $1 - c_2 c_6^2 - c_6 \not\equiv 0$ and by $(1 - c_2 c_6^2 - c_6) t_{2,0} \equiv 0$, we get $t_{2,0} \equiv 0$. It then follows that $t_{0,2} \equiv t_{1,1} \equiv s_{2,2} \equiv t_{2,0} \equiv 0$ and $s_{1,2} \equiv -c_2 s_{2,1}$. This is the same congruence as in Case 7.4. Hence $W_2$ is terminal, and $W_2 \not\in \hat{L}_2$.

**Case 7.4.1.4.1.2** Now we assume that $c_2 \equiv 2$, $c_4 \equiv 1$, $c_5 \equiv 1$, and $c_6 \equiv 2$. Then $t_{2,0}$ is a free variable.
Hence $x \in \partial^{-1}W_2$ if and only if

$$t_{1,1} \equiv t_{2,0}$$

$$s_{1,2} \equiv s_{2,1}$$

$$t_{0,2} \equiv t_{2,0}$$

$$s_{2,2} \equiv -t_{2,0}.$$  

We regard $t_{1,0}, t_{0,1}, s_{2,1}, t_{2,0}$ as free variables. Taking $t_{1,0} \equiv 1, t_{0,1} = 0, s_{2,1} = 0,$ and $t_{2,0} = 0$, the matrix $x$ becomes

$$v_1 = \begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$  

Taking $t_{0,1} \equiv 1, t_{1,0} = 0, s_{2,1} = 0,$ and $t_{2,0} = 0$, the matrix $x$ becomes

$$v_2 = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$  

Taking $s_{2,1} \equiv 1, t_{0,1} = 0, t_{1,0} = 0,$ and $t_{2,0} = 0$, the matrix $x$ becomes

$$v_3 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 3 \\ 0 & 3 & 0 \end{bmatrix}.$$
Taking \( t_{2,0} \equiv 1, t_{0,1} = 0, t_{1,0} = 0, \) and \( s_{2,1} = 0, \) the matrix \( x \) becomes

\[
v_4 = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & -3 \end{bmatrix}.
\]

We see that neither \( v_4 \) nor \( 3v_4 \) is contained in \( W_2 \) and that

\[
\partial^{-1}W_2 = \langle \partial^{-1}W_0, v_1, v_2, v_3, v_4 \rangle.
\]

We know that \( v_4 \in \partial^{-1}W_2 \) and since \( 3v_4 \notin W_2 \) then \( v_4 + W_2 \) is an element of the group \( \partial^{-1}W_2/W_2 \) whose order is larger than 3.

Recall that \( |\partial^{-1}W_2/W_2| = 3^4 \) and that \( \partial^{-1}W_1/W_2 \) has rank 3. Since \( \partial^{-1}W_2 \) has an element of order larger than 3 then \( \text{rank}(\partial^{-1}W_2/W_2) \) is not greater than 3 since \( \partial^{-1}W_2/W_2 \cong \mathbb{Z}_9 \times \mathbb{Z}_3 \times \mathbb{Z}_3 \) and therefore \( W_2 \) is terminal and \( W_2 \notin \hat{L}_2 \).

**Case 7.4.1.4.2** Now we consider the case \( c_4 = 0, c_5 \neq 0. \) Then \( c_5 \in \{1, 2\} \) and \( c_5^2 \equiv 1. \) Solving (A1) for \( a_4 \) and (B1) for \( b_4 \) we obtain \( a_4 \equiv c_5t_{1,1} \) and \( b_4 \equiv c_5t_{0,2}. \)

Since \( c_4 = 0 \) the congruence (A3) becomes \( t_{2,0} \equiv 0 \) and the congruence (B3) becomes \( t_{1,1} \equiv 0. \) Therefore \( a_4 \equiv 0. \) Hence \( s_{2,2} \equiv 0 \) and congruences (A5), (B2), and (B5) become \( s_{1,2} \equiv -c_2s_{2,1}, \) \( 0 \equiv c_5c_6t_{0,2}, \) and \( 0 \equiv (c_2 + c_5)t_{0,2}. \)
Hence \( x \in \partial^{-1}W_2 \) if and only if

\[
\begin{align*}
    t_{1,1} &\equiv 0 \\
    t_{2,0} &\equiv 0 \\
    s_{2,2} &\equiv 0 \\
    s_{1,2} &\equiv -c_2s_{2,1} \\
    c_5c_6t_{0,2} &\equiv 0 \\
    (c_2 + c_5)t_{0,2} &\equiv 0.
\end{align*}
\]

If either \( c_5c_6 \neq 0 \) or \( c_2 + c_5 \neq 0 \) then \( t_{0,2} \equiv 0 \). It follows that we have the congruence \( s_{1,2} \equiv -c_2s_{2,1} \). This is the same congruence as that in Case 7.4. Therefore \( \partial^{-1}W_2 = \partial^{-1}W_1 \). Thus \( \text{rank}(\partial^{-1}W_2/W_2) = \text{rank}(\partial^{-1}W_1/W_2) \), \( W_2 \) is terminal, and \( W_2 \not\in \hat{L}_2 \).

We will assume that \( c_5c_6 \equiv 0 \) and \( c_2 + c_5 \equiv 0 \). Since \( c_5 \neq 0 \) then \( c_6 \equiv 0 \). Also \( c_2 \equiv -c_5 \). Then \( t_{0,2} \) is a free variable. We regard \( t_{1,0}, t_{0,1}, s_{2,1}, t_{2,0} \) as free variables.

Taking \( t_{1,0} \equiv 1, t_{0,1} = 0, s_{2,1} = 0, \) and \( t_{2,0} = 0 \), the matrix \( x \) becomes

\[
\begin{bmatrix}
    0 & 0 & 0 \\
    1 & 0 & 0 \\
    0 & 0 & 0
\end{bmatrix}.
\]

Taking \( t_{0,1} \equiv 1, t_{1,0} = 0, s_{2,1} = 0, \) and \( t_{2,0} = 0 \), the matrix \( x \) becomes

\[
\begin{bmatrix}
    0 & 1 & 0 \\
    0 & 0 & 0 \\
    0 & 0 & 0
\end{bmatrix}.
\]
Taking \( s_{2,1} \equiv 1, t_{0,1} = 0, t_{1,0} = 0 \), and \( t_{2,0} = 0 \), the matrix \( x \) becomes
\[
v_3 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 3c_2 \\ 0 & -3 & 0 \end{bmatrix}.
\]
Taking \( t_{2,0} \equiv 1, t_{0,1} = 0, t_{1,0} = 0 \), and \( s_{2,1} = 0 \), the matrix \( x \) becomes
\[
v_4 = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}.
\]
We see that neither \( v_4 \) nor \( 3v_4 \) is contained in \( W_2 \) and that \( \partial^{-1}W_2 = \langle \partial^{-1}W_0, v_1, v_2, v_3, v_4 \rangle \). We know that \( v_4 \in \partial^{-1}W_2 \) and since \( 3v_4 \notin W_2 \) then \( v_4 + W_2 \) is an element of the group \( \partial^{-1}W_2/W_2 \) whose order is larger than 3. Recall that \( |\partial^{-1}W_2/W_2| = 3^4 \) and that \( \partial^{-1}W_1/W_2 \) has rank 3. Since \( \partial^{-1}W_2 \) has an element of order larger than 3 then \( \operatorname{rank}(\partial^{-1}W_2/W_2) \) is not greater than 3 since \( \partial^{-1}W_2/W_2 \cong \mathbb{Z}_9 \times \mathbb{Z}_3 \times \mathbb{Z}_3 \) and therefore \( W_2 \) is terminal and \( W_2 \notin \hat{L}_2 \).

**Case 7.4.1.4.3** Now we consider the case \( c_4 = c_5 = 0, c_6 \neq 0 \). Then \( c_6 \in \{1, 2\} \) and \( c_6^2 \equiv 1 \). Solving (A2) for \( a_4 \) and (B2) for \( b_4 \) we obtain \( a_4 \equiv -c_6 t_{1,1} \) and \( b_4 \equiv c_2 s_{2,2} \).

Since \( c_5 = 0 \), from (A1) we obtain \( t_{1,1} \equiv 0 \). Therefore \( a_4 \equiv 0 \). Then from (A3) \( t_{2,0} \equiv 0 \). Additionally since \( t_{1,1} \equiv 0 \) then (A4) becomes \( s_{2,2} \equiv 0 \) which implies that \( b_4 \equiv 0 \). Therefore from (B1) we obtain \( t_{0,2} \equiv 0 \). The congruence (A5) becomes \( s_{1,2} \equiv -c_2 s_{2,1} \).
Hence $x \in \partial^{-1}W_2$ if and only if

\begin{align*}
t_{1,1} & \equiv 0 \\
t_{2,0} & \equiv 0 \\
s_{2,2} & \equiv 0 \\
t_{0,2} & \equiv 0 \\
s_{1,2} & \equiv -c_2s_{2,1}.
\end{align*}

This is the same congruence as that is Case 7.4. Therefore $\partial^{-1}W_2 = \partial^{-1}W_1$. Thus $\text{rank}(\partial^{-1}W_2/W_2) = \text{rank}(\partial^{-1}W_1/W_2)$, $W_2$ is terminal, and $W_2 \not\in \hat{L}_2$.

10.4.2 Case 7.4.2

We consider the 2-dimensional subspaces $W_2/W_1$. Let $d_1, d_2, d_3, d_4, e_1, e_2, e_3, e_4$ be unspecified variables. Let $m_4 = d_1y_2 + d_2v_1 + d_3v_2 + d_4v_3$ and $m_5 = e_1y_2 + e_2v_1 + e_3v_2 + e_4v_3$. In all the cases we consider the value of $e_1 = 0$ therefore we may exclude it from our expression of $m_5$. A formal expression for $m_4$ is

\[
m_4 = d_1 \begin{bmatrix} 0 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 0 \end{bmatrix} + d_2 \begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} + d_3 \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} + d_4 \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & -3c_2 \\ 0 & 3 & 0 \end{bmatrix}
\]

\[
= \begin{bmatrix} 0 & d_3 & 0 \\ d_2 & 3d_1 & -3c_2d_4 \\ 0 & 3d_4 & 0 \end{bmatrix}.
\]
A formal expression for $m_5$ is

$$m_5 = e_2 \begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} + e_3 \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} + e_4 \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 3 \\ 0 & 3 & 0 \end{bmatrix} = \begin{bmatrix} 0 & e_3 & 0 \\ e_2 & 0 & -3e_2 e_4 \\ 0 & 3e_4 & 0 \end{bmatrix}.$$ 

Let $W_2 = \langle W_1, m_4, m_5 \rangle \in \mathcal{L}_2$. We now calculate the pullback $\partial^{-1}W_2$. The subgroup $W_2$ is contained in the pattern subgroup

$$\begin{bmatrix} 2 & 2 & 1 \\ 2 & 1 & 1 \\ 1 & 1 & 0 \end{bmatrix}.$$ 

Thus the pullback $\partial^{-1}W_2$ is contained in the pattern subgroup

$$\begin{bmatrix} 2 & 2 & 2 \\ 2 & 2 & 1 \\ 2 & 1 & 1 \end{bmatrix} \cap \begin{bmatrix} 2 & 2 & 2 \\ 2 & 2 & 1 \\ 2 & 1 & 1 \end{bmatrix} = \begin{bmatrix} 2 & 2 & 1 \\ 2 & 1 & 1 \end{bmatrix}.$$

Let

$$x = \begin{bmatrix} 3s_{0,0} + t_{0,0} & 3s_{0,1} + t_{0,1} & 3s_{0,2} + t_{0,2} \\ 3s_{1,0} + t_{1,0} & 3s_{1,1} + t_{1,1} & 3s_{1,2} \\ 3s_{2,0} + t_{2,0} & 3s_{2,1} & 3s_{2,2} \end{bmatrix} \in \begin{bmatrix} 2 & 2 & 2 \\ 2 & 2 & 1 \\ 2 & 1 & 1 \end{bmatrix}.$$ 

Thus

$$\partial_1 x = \begin{bmatrix} 3s_{1,0} + t_{1,0} & 3s_{1,1} + t_{1,1} & 3s_{1,2} \\ 3s_{2,0} + t_{2,0} & 3s_{2,1} & 3s_{2,2} \\ -3(t_{1,0} + t_{2,0}) & -3t_{1,1} & 0 \end{bmatrix}$$ and

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\[ \partial_2 x = \begin{bmatrix} 3s_{0,1} + t_{0,1} & 3s_{0,2} + t_{0,2} & -3(t_{0,1} + t_{0,2}) \\ 3s_{1,1} + t_{1,1} & 3s_{1,2} & -3t_{1,1} \\ 3s_{2,1} & 3s_{2,2} & 0 \end{bmatrix}. \]

The variables that are in play are those appearing in the matrix
\[ \begin{bmatrix} 0 & t_{0,1} & t_{0,2} \\ t_{1,0} & t_{1,1} & 3s_{1,2} \\ t_{2,0} & 3s_{2,1} & 3s_{2,2} \end{bmatrix}. \]

We wish to identify values \( a_1, a_2, a_3, a_4, a_5 \in \mathbb{Z}_9 \) such that \( \partial x \equiv a_1 m_1 + a_2 m_2 + a_3 m_3 + a_4 m_4 + a_5 m_5 \pmod{I} \). A formal expression for \( a_1 m_1 + a_2 m_2 + a_3 m_3 + a_4 m_4 + a_5 m_5 \) is
\[ a_1 \begin{bmatrix} 1 & 0 & -3 \\ 0 & 0 & 0 \end{bmatrix} + a_2 \begin{bmatrix} c_2 & 0 & -3c_2 \\ 0 & 3 & 0 \end{bmatrix} + a_3 \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} + a_4 \begin{bmatrix} d_2 & 3d_1 & -3c_2d_4 \\ 0 & 3d_4 & 0 \end{bmatrix} + a_5 \begin{bmatrix} 0 & e_3 & 0 \\ e_2 & 0 & -3c_2e_4 \end{bmatrix}. \]

Comparing \((0,1)\)-entries, we get \( t_{1,1} \equiv a_4d_3 + a_5e_3 \).

Comparing \((2,1)\)-entries, we get \( -t_{1,1} \equiv a_4d_4 + a_5e_4 \).

Comparing \((1,0)\)-entries, we get \( t_{2,0} \equiv a_4d_2 + a_5e_2 \).

Comparing \((1,2)\)-entries, we get \( s_{2,2} \equiv -c_2(a_4d_4 + a_5e_4) \). Substituting \( a_4d_4 + a_5e_4 \equiv -t_{1,1} \) we obtain \( s_{2,2} \equiv c_2 t_{1,1} \).
Comparing \((1,1)\)-entries, we get \(a_2 \equiv s_{2,1} - a_4 d_1\).

Comparing \((0,2)\)-entries, we get \(s_{1,2} \equiv -(a_1 + a_2 c_2)\). Substituting \(a_2\) we obtain \(a_1 \equiv c_2 a_4 d_1 - c_2 s_{2,1} - s_{1,2}\).

Comparing \((2,0)\)-entries, we get \(c_2 a_2 + a_3 \equiv t_{1,0} + t_{2,0}\). Substituting \(a_2\) we obtain \(a_3 \equiv t_{1,0} + t_{2,0} - c_2 s_{2,1} + c_2 a_4 d_1\).

Comparing \((0,0)\)-entries, we get \(a_1 + a_2 c_2 + a_3 \equiv t_{1,0}\). Substituting \(a_1, a_2, a_3\), we obtain \(s_{1,2} \equiv c_2 a_4 d_1 + t_{2,0} - c_2 s_{2,1}\).

We see that \(\partial_1 x \in W_2\) if and only if

\[
\begin{align*}
t_{1,1} &\equiv a_4 d_3 + a_5 e_3 \quad (A1) \\
-t_{1,1} &\equiv a_4 d_4 + a_5 e_4 \quad (A2) \\
t_{2,0} &\equiv a_4 d_2 + a_5 e_2 \quad (A3) \\
s_{2,2} &\equiv c_2 t_{1,1} \quad (A4) \\
s_{1,2} &\equiv c_2 a_4 d_1 + t_{2,0} - c_2 s_{2,1} \quad (A5).
\end{align*}
\]

We wish to identify values \(b_1, b_2, b_3, b_4, b_5 \in \mathbb{Z}_9\) such that \(\partial_2 x \equiv b_1 m_1 + b_2 m_2 + b_3 m_3 + b_4 m_4 + b_5 m_5 \pmod{I}\). A formal expression for \(b_1 m_1 + b_2 m_2 + b_3 m_3 = b_4 m_4 + b_5 m_5\) is

\[
\begin{bmatrix}
1 & 0 & -3 \\
0 & 0 & 0 \\
0 & 0 & 0
\end{bmatrix}
+ \begin{bmatrix}
c_2 & 0 & -3c_2 \\
0 & 3 & 0 \\
-3c_2 & 0 & 0
\end{bmatrix}
+ \begin{bmatrix}
1 & 0 & 0 \\
0 & 0 & 0 \\
0 & 0 & 0
\end{bmatrix}
\begin{bmatrix}
0 & d_3 & 0 \\
d_2 & 3d_4 & -3c_2 d_4 \\
0 & 3d_4 & 0
\end{bmatrix}
\]

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\[
\begin{bmatrix}
0 & e_3 & 0 \\
e_2 & 0 & -3c_2 e_4 \\
0 & 3e_4 & 0
\end{bmatrix} + b_5 
\begin{bmatrix}
0 & e_3 & 0 \\
e_2 & 0 & -3c_2 e_4 \\
0 & 3e_4 & 0
\end{bmatrix} = 
\begin{bmatrix}
b_1 + b_2 c_2 + b_3 & b_4 d_3 + b_5 e_3 & -3(b_1 + b_2 c_2) \\
b_4 d_2 + b_5 e_2 & 3(b_2 + b_4 d_1) & -3c_2(b_4 d_4 + b_5 e_4) \\
-3(c_2 b_2 + b_3) & 3(b_4 d_4 + b_5 e_4) & 0
\end{bmatrix}.
\]

Comparing (0,1)-entries, we get \( t_{0,2} \equiv b_4 d_3 + b_5 e_3 \).

Comparing (2,1)-entries, we get \( s_{2,2} \equiv b_4 d_4 + b_5 e_4 \).

Comparing (1,0)-entries, we get \( t_{1,1} \equiv b_4 d_2 + b_5 e_2 \).

Comparing (1,2)-entries, we get \( t_{1,1} \equiv c_2(b_4 d_4 + b_5 e_4) \). Substituting \( b_4 d_4 + b_5 e_4 \equiv s_{2,2} \) we obtain \( t_{1,1} \equiv c_2 s_{2,2} \).

Comparing (1,1)-entries, we get \( b_2 \equiv s_{1,2} - b_4 d_1 \).

Comparing (0,2)-entries, we get \( b_1 + b_2 c_2 \equiv t_{0,1} + t_{0,2} \). Substituting \( b_2 \) we obtain \( b_1 \equiv t_{0,1} + t_{0,2} - c_2 s_{1,2} + c_2 b_4 d_1 \).

Comparing (2,0)-entries, we get \( -c_2 b_2 - b_3 \equiv s_{2,1} \). Substituting \( b_2 \) we obtain \( b_3 \equiv -c_2 s_{1,2} + c_2 b_4 d_1 - s_{2,1} \).

Comparing (0,0)-entries, we get \( b_1 + b_2 c_2 + b_3 \equiv t_{0,1} \). Substituting \( b_1, b_2, b_3 \), we obtain \( s_{1,2} \equiv -c_2 s_{2,1} + c_2 t_{0,2} + b_4 d_1 \).
We see that $\partial_2 x \in W_2$ if and only if

$$t_{0,2} \equiv b_4 d_3 + b_5 e_3 \quad \text{(B1)}$$

$$s_{2,2} \equiv b_4 d_4 + b_5 e_4 \quad \text{(B2)}$$

$$t_{1,1} \equiv b_4 d_2 + b_5 e_2 \quad \text{(B3)}$$

$$t_{1,1} \equiv c_2 s_{2,2} \quad \text{(B4)}$$

$$s_{1,2} \equiv -c_2 s_{2,1} + c_2 t_{0,2} + b_4 d_1 \quad \text{(B5)}.$$  

Case 7.4.2.1

Let $m_4 = v_2$ and $m_5 = v_3$. Thus

$$m_4 = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \quad \text{and} \quad m_5 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & -3c_2 \\ 0 & 3 & 0 \end{bmatrix}.$$  

Let $W_2 = < W_1, m_4, m_5 > \in L_2$. The number of subgroups $W_2$ of this type is 1. Since $m_4, m_5 \not\in W_1$ and $3m_4, 3m_5 \in W_1$ we have $|W_2/W_1| = 3^2$. Since $|W_1| = 3^6$ it follows that $|W_2| = 3^8$. Note that $\partial^{-1}W_1/W_2 \cong \mathbb{Z}_3 \times \mathbb{Z}_3$ and rank($\partial^{-1}W_1/W_2$) = 2.

We now calculate the pullback of $\partial^{-1}W_2$. We observed in case 7.4.2 that $\partial^{-1}W_2$ is contained in the pattern subgroup

$$\begin{bmatrix} 2 & 2 & 2 \\ 2 & 2 & 1 \\ 2 & 1 & 1 \end{bmatrix}.$$
Let
\[ x = \begin{bmatrix} 3s_{0,0} + t_{0,0} & 3s_{0,1} + t_{0,1} & 3s_{0,2} + t_{0,2} \\ 3s_{1,0} + t_{1,0} & 3s_{1,1} + t_{1,1} & 3s_{1,2} \\ 3s_{2,0} + t_{2,0} & 3s_{2,1} & 3s_{2,2} \end{bmatrix} \in \begin{bmatrix} 2 & 2 & 2 \\ 2 & 2 & 1 \\ 2 & 1 & 1 \end{bmatrix}. \]

The variables that are in play are those appearing in the matrix
\[
\begin{bmatrix} 0 & t_{0,1} & t_{0,2} \\ t_{1,0} & t_{1,1} & 3s_{1,2} \\ t_{2,0} & 3s_{2,1} & 3s_{2,2} \end{bmatrix}.
\]

In the notation of Case 7.4.2, we are taking \( d_1 = 0, d_2 = 0, d_3 = 1, d_4 = 0, \)
\( e_2 = 0, e_3 = 0, \) and \( e_4 = 1. \) We wish to identify values \( a_1, a_2, a_3, a_4, a_5 \in \mathbb{Z}_9 \) such that
\[ \partial_1 x \equiv a_1 m_1 + a_2 m_2 + a_3 m_3 + a_4 m_4 + a_5 m_5 \pmod{I}. \]

Hence \( \partial_1 x \in W_2 \) if and only if
\[
\begin{align*}
t_{1,1} & \equiv a_4 \quad \text{(A1)} \\
-t_{1,1} & \equiv a_5 \quad \text{(A2)} \\
t_{2,0} & \equiv 0 \quad \text{(A3)} \\
s_{2,2} & \equiv c_2 t_{1,1} \quad \text{(A4)} \\
s_{1,2} & \equiv -c_2 s_{2,1} \quad \text{(A5)}. \end{align*}
\]

We wish to identify values \( b_1, b_2, b_3, b_4, b_5 \in \mathbb{Z}_9 \) such that \( \partial_1 x \equiv b_1 m_1 + b_2 m_2 + b_3 m_3 + b_4 m_4 + b_5 m_5 \pmod{I}. \)
Hence $\partial_2 x \in W_2$ if and only if

\[
\begin{align*}
t_{0,2} & \equiv b_4 \quad (B1) \\
s_{2,2} & \equiv b_5 \quad (B2) \\
t_{1,1} & \equiv 0 \quad (B3) \\
t_{1,1} & \equiv c_2 s_{2,2} \quad (B4) \\
s_{1,2} & \equiv -c_2 s_{2,1} + c_2 t_{0,2} \quad (B5).
\end{align*}
\]

Since $t_{1,1} \equiv 0$ then $s_{2,2} \equiv 0$. Setting (A5) and (B5) equal we obtain $c_2 t_{0,2} \equiv 0$. Since $c_2 \in \{1, 2\}$ then $t_{0,2} \equiv 0$. Hence (B5) becomes $s_{1,2} \equiv -c_2 s_{2,1}$ which is redundant with (A5).

Hence $x \in \partial^{-1}W_2$ if and only if

\[
\begin{align*}
t_{1,1} & \equiv 0 \\
t_{2,0} & \equiv 0 \\
t_{0,2} & \equiv 0 \\
s_{2,2} & \equiv 0 \\
s_{1,2} & \equiv -c_2 s_{2,1}.
\end{align*}
\]

This is the same congruence as that in Case 7.4 therefore $\partial^{-1}W_2 = \partial^{-1}W_1$.

Thus $\text{rank}(\partial^{-1}W_2/W_2) = \text{rank}(\partial^{-1}W_1/W_2)$, $W_2$ is terminal, and $W_2 \notin \hat{L}_2$. 

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Case 7.4.2.2

We fix arbitrary value \( c_4 \in \{0, 1, 2\} \). There are 3 ways to choose the value \( c_4 \). Let \( m_4 = v_1 + c_4 v_2 \) and \( m_5 = v_3 \). Thus

\[
\begin{bmatrix}
0 & c_4 & 0 \\
1 & 0 & 0 \\
0 & 0 & 0 \\
\end{bmatrix}
\quad \text{and} \quad
\begin{bmatrix}
0 & 0 & 0 \\
0 & 0 & -3c_2 \\
0 & 3 & 0 \\
\end{bmatrix}
\]

Let \( W_2 = \langle W_1, m_4, m_5 \rangle \in \mathcal{L}_2 \). The number of subgroups \( W_2 \) of this type is 3. Since \( m_4, m_5 \notin W_1 \) and \( 3m_4, 3m_5 \in W_1 \) we have \( |W_2/W_1| = 3^2 \). Since \( |W_1| = 3^6 \) it follows that \( |W_2| = 3^8 \). Note that \( \partial^{-1}W_1/W_2 \cong \mathbb{Z}_3 \times \mathbb{Z}_3 \) and \( \text{rank}(\partial^{-1}W_1/W_2) = 2 \).

We now calculate the pullback of \( \partial^{-1}W_2 \). We observed in case 7.4.2 that \( \partial^{-1}W_2 \) is contained in the pattern subgroup

\[
\begin{bmatrix}
2 & 2 & 2 \\
2 & 2 & 1 \\
2 & 1 & 1 \\
\end{bmatrix}
\]

Let

\[
x = \begin{bmatrix}
3s_{0,0} + t_{0,0} & 3s_{0,1} + t_{0,1} & 3s_{0,2} + t_{0,2} \\
3s_{1,0} + t_{1,0} & 3s_{1,1} + t_{1,1} & 3s_{1,2} \\
3s_{2,0} + t_{2,0} & 3s_{2,1} & 3s_{2,2} \\
\end{bmatrix}
\in \begin{bmatrix}
2 & 2 & 2 \\
2 & 2 & 1 \\
2 & 1 & 1 \\
\end{bmatrix}
\]

The variables that are in play are those appearing in the matrix

\[
\begin{bmatrix}
0 & t_{0,1} & t_{0,2} \\
t_{1,0} & t_{1,1} & 3s_{1,2} \\
t_{2,0} & 3s_{2,1} & 3s_{2,2} \\
\end{bmatrix}
\]

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In the notation of Case 7.4.2, we are taking \(d_1 = 0, d_2 = 1, d_3 = c_4, d_4 = 0, e_2 = 0, e_3 = 0, \) and \(e_4 = 1.\) We wish to identify values \(a_1, a_2, a_3, a_4, a_5 \in \mathbb{Z}_9\) such that \(\partial_1 x \equiv a_1 m_1 + a_2 m_2 + a_3 m_3 + a_4 m_4 + a_5 m_5 \pmod{I}.\) We see that \(\partial_1 x \in W_2\) if and only if

\[
t_{1,1} \equiv a_4 c_4 \quad \text{(A1)}
\]

\[
-t_{1,1} \equiv a_5 \quad \text{(A2)}
\]

\[
t_{2,0} \equiv a_4 \quad \text{(A3)}
\]

\[
s_{2,2} \equiv c_2 t_{1,1} \quad \text{(A4)}
\]

\[
s_{1,2} \equiv t_{2,0} - c_2 s_{2,1} \quad \text{(A5)}.
\]

We wish to identify values \(b_1, b_2, b_3, b_4, b_5 \in \mathbb{Z}_9\) such that \(\partial_1 x \equiv b_1 m_1 + b_2 m_2 + b_3 m_3 + b_4 m_4 + b_5 m_5 \pmod{I}.\) We see that \(\partial_2 x \in W_2\) if and only if

\[
t_{0,2} \equiv b_4 c_4 \quad \text{(B1)}
\]

\[
s_{2,2} \equiv b_5 \quad \text{(B2)}
\]

\[
t_{1,1} \equiv b_4 \quad \text{(B3)}
\]

\[
t_{1,1} \equiv c_2 s_{2,2} \quad \text{(B4)}
\]

\[
s_{1,2} \equiv -c_2 s_{2,1} + c_2 t_{0,2} \quad \text{(B5)}.
\]

Multiplying (B4) by \(c_2\) we obtain \(s_{2,2} \equiv c_2 t_{1,1}\) which is redundant with (A4). Substituting \(a_4\) and \(b_4\) into (A1) and (B1) we obtain \(t_{1,1} \equiv c_4 t_{2,0}\) and \(t_{0,2} \equiv c_4 t_{1,1}\) respectively. Hence we deduce that \(t_{0,2} \equiv c_4^2 t_{2,0}\) which we will denote as (B1). Combining (A5) and (B5) we obtain \(t_{0,2} \equiv c_2 t_{2,0}\) which we will denote as our new (B5).
Comparing this with (B1) we obtain \(0 \equiv (c_2 - c_4^2)t_{2,0}\). Substituting (A1) into (A4) we obtain \(s_{2,2} \equiv c_2c_4t_{2,0}\).

Hence \(x \in \partial^{-1}W_2\) if and only if

\[
\begin{align*}
t_{1,1} &\equiv c_4t_{2,0} \quad \text{(A1)} \\
s_{2,2} &\equiv c_2c_4t_{2,0} \quad \text{(A4)} \\
s_{1,2} &\equiv -c_2s_{2,1} + t_{2,0} \quad \text{(A5)} \\
t_{0,2} &\equiv c_4^2t_{2,0} \quad \text{(B1)} \\
(c_2 - c_4^2)t_{2,0} &\equiv 0 \quad \text{(B5)}.
\end{align*}
\]

If \(c_2 - c_4^2 \neq 0\), then \(t_{2,0} \equiv 0\). It then follows that \(t_{1,1} \equiv s_{2,1} \equiv t_{0,2} \equiv t_{2,0} \equiv 0\) and \(s_{1,2} \equiv c_2s_{2,1}\). This is the same congruence as that in Case 7.4 therefore \(\partial^{-1}W_2 = \partial^{-1}W_1\). Thus \(\text{rank}(\partial^{-1}W_2/W_2) = \text{rank}(\partial^{-1}W_1/W_2)\), \(W_2\) is terminal, and \(W_2 \notin \hat{L}_2\).

Therefore we will assume that \(c_2 - c_4^2 \equiv 0\). Then \(t_{2,0}\) is a free variable. Hence \(c_4^2 \equiv c_2\). Since \(c_2 \in \{1, 2\}\) then \(c_4^2 \equiv 1\), \(c_4 \in \{1, 2\}\), and \(c_2 \equiv 1\).

Hence \(x \in \partial^{-1}W_2\) if and only if

\[
\begin{align*}
t_{1,1} &\equiv c_4t_{2,0} \quad \text{(A1)} \\
s_{2,2} &\equiv c_4t_{2,0} \quad \text{(A4)} \\
s_{1,2} &\equiv -s_{2,1} + t_{2,0} \quad \text{(A5)} \\
t_{0,2} &\equiv t_{2,0} \quad \text{(B1)}.
\end{align*}
\]
We regard $t_{1,0}$, $t_{0,1}$, $s_{2,1}$, $t_{2,0}$ as the free variables. Taking $t_{1,0} \equiv 1$, $t_{0,1} = 0$, $s_{2,1} = 0$, and $t_{2,0} = 0$, the matrix $x$ becomes

$$v_1 = \begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

Taking $t_{0,1} \equiv 1$, $t_{1,0} = 0$, $s_{2,1} = 0$, and $t_{2,0} = 0$, the matrix $x$ becomes

$$v_2 = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

Taking $s_{2,1} \equiv 1$, $t_{0,1} = 0$, $t_{1,0} = 0$, and $t_{2,0} = 0$, the matrix $x$ becomes

$$v_3 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & -3 \\ 0 & 3 & 0 \end{bmatrix}.$$

Taking $t_{2,0} \equiv 1$, $t_{0,1} = 0$, $t_{1,0} = 0$, and $s_{2,1} = 0$, the matrix $x$ becomes

$$v_4 = \begin{bmatrix} 0 & 0 & 1 \\ 0 & c_4 & 3 \\ 1 & 0 & 3c_4 \end{bmatrix}.$$

We see that neither $v_4$ nor $3v_4$ is contained in $W_2$ and that $\partial^{-1}W_2 =< \partial^{-1}W_0, v_1, v_2, v_3, v_4 >$. We know that $v_4 \in \partial^{-1}W_2$ and since $3v_4 \not\in W_2$ then $v_4 + W_2$ in an element of the group $\partial^{-1}W_2/W_2$ whose order is larger than 3. Recall that $|\partial^{-1}W_2/W_2| = 3^3$ and that $\partial^{-1}W_1/W_2$ has rank 2. We obtain $\partial^{-1}W_2/W_2 \cong \mathbb{Z}_9 \times \mathbb{Z}_3$. Since rank($\partial^{-1}W_2/W_2$) = rank($\partial^{-1}W_1/W_2$). Hence $W_2$ is terminal and $W_2 \not\in \hat{L}_2$. 446
Case 7.4.2.3

We fix arbitrary values $c_4, c_5 \in \{0, 1, 2\}$. There are 9 ways to choose the values $c_4$ and $c_5$. Let $m_4 = v_1 + c_4 v_3$ and $m_5 = v_2 + c_5 v_3$. Thus

$$m_4 = \begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & -3c_2c_4 \\ 0 & 3c_4 & 0 \end{bmatrix} \quad \text{and} \quad m_5 = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & -3c_2c_5 \\ 0 & 3c_5 & 0 \end{bmatrix}.$$ 

Let $W_2 = \langle W_1, m_4, m_5 \rangle \in L_2$. The number of subgroups $W_2$ of this type is 9. Since $m_4, m_5 \notin W_1$ and $3m_4, 3m_5 \in W_1$ we have $|W_2/W_1| = 3^2$. Since $|W_1| = 3^6$ it follows that $|W_2| = 3^8$. Note that $\partial^{-1}W_1/W_2 \cong Z_3 \times Z_3$ and $\text{rank}(\partial^{-1}W_1/W_2) = 2.$

We now calculate the pullback of $\partial^{-1}W_2$. We observed in case 7.4.2 that $\partial^{-1}W_2$ is contained in the pattern subgroup

$$\begin{bmatrix} 2 & 2 & 2 \\ 2 & 2 & 1 \\ 2 & 1 & 1 \end{bmatrix}.$$ 

Let

$$x = \begin{bmatrix} 3s_{0,0} + t_{0,0} & 3s_{0,1} + t_{0,1} & 3s_{0,2} + t_{0,2} \\ 3s_{1,0} + t_{1,0} & 3s_{1,1} + t_{1,1} & 3s_{1,2} \\ 3s_{2,0} + t_{2,0} & 3s_{2,1} & 3s_{2,2} \end{bmatrix} \in \begin{bmatrix} 2 & 2 & 2 \\ 2 & 2 & 1 \\ 2 & 1 & 1 \end{bmatrix}.$$ 

The variables that are in play are those appearing in the matrix

$$\begin{bmatrix} 0 & t_{0,1} & t_{0,2} \\ t_{1,0} & t_{1,1} & 3s_{1,2} \\ t_{2,0} & 3s_{2,1} & 3s_{2,2} \end{bmatrix}.$$ 

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In the notation of Case 7.4.2, we are taking $d_1 = 0$, $d_2 = 1$, $d_3 = 0$, $d_4 = c_4$, $e_2 = 0$, $e_3 = 1$, and $e_4 = c_5$. We wish to identify values $a_1, a_2, a_3, a_4, a_5 \in \mathbb{Z}_9$ such that $\partial_1 x \equiv a_1 m_1 + a_2 m_2 + a_3 m_3 + a_4 m_4 + a_5 m_5 \pmod{I}$. We see that $\partial_1 x \in W_2$ if and only if

$$ t_{1,1} \equiv a_5 \quad (A1) $$

$$ -t_{1,1} \equiv a_4 c_4 + a_5 c_5 \quad (A2) $$

$$ t_{2,0} \equiv a_4 \quad (A3) $$

$$ s_{2,2} \equiv c_2 t_{1,1} \quad (A4) $$

$$ s_{1,2} \equiv t_{2,0} - c_2 s_{2,1} \quad (A5). $$

We wish to identify values $b_1, b_2, b_3, b_4, b_5 \in \mathbb{Z}_9$ such that $\partial_2 x \equiv b_1 m_1 + b_2 m_2 + b_3 m_3 + b_4 m_4 + b_5 m_5 \pmod{I}$. We see that $\partial_2 x \in W_2$ if and only if

$$ t_{0,2} \equiv b_5 \quad (B1) $$

$$ s_{2,2} \equiv b_4 c_4 + b_5 c_5 \quad (B2) $$

$$ t_{1,1} \equiv b_4 \quad (B3) $$

$$ t_{1,1} \equiv c_2 s_{2,2} \quad (B4) $$

$$ s_{1,2} \equiv -c_2 s_{2,1} + c_2 t_{0,2} \quad (B5). $$

Multiplying (B4) by $c_2$ we obtain $s_{2,2} \equiv c_2 t_{1,1}$ which is redundant with (A4).

Substituting $a_4, a_5, b_4, b_5$ into (A2) and (B2) we obtain $0 \equiv c_4 t_{2,0} + (c_5 + 1) t_{1,1}$ and $s_{2,2} \equiv c_4 t_{1,1} + c_5 t_{0,2}$ respectively. Combining (A5) and (B5) we obtain $t_{0,2} \equiv c_2 t_{2,0}$
which we will denote as our new (B5). Substituting (A4) and (B5) into (B2) we obtain $0 \equiv (c_4 - c_2)t_{1,1} + c_2c_5t_{2,0}$.

Hence $x \in \partial^{-1}W_2$ if and only if

$$0 \equiv c_4t_{2,0} + (c_5 + 1)t_{1,1} \quad (A2)$$

$$s_{2,2} \equiv c_2t_{1,1} \quad (A4)$$

$$s_{1,2} \equiv -c_2s_{2,1} + t_{2,0} \quad (A5)$$

$$0 \equiv c_2c_5t_{2,0} + (c_4 - c_2)t_{1,1} \quad (B2)$$

$$t_{0,2} \equiv c_2t_{2,0} \quad (B5).$$

It is convenient to consider the cases $c_4 = 0$ and $c_4 \neq 0$ separately.

**Case 7.4.2.3.1** First we consider the case $c_4 = 0$.

Hence $x \in \partial^{-1}W_2$ if and only if

$$0 \equiv (c_5 + 1)t_{1,1} \quad (A2)$$

$$s_{2,2} \equiv c_2t_{1,1} \quad (A4)$$

$$s_{1,2} \equiv -c_2s_{2,1} + t_{2,0} \quad (A5)$$

$$0 \equiv -c_2t_{1,1} + c_2c_5t_{2,0} \quad (B2)$$

$$t_{0,2} \equiv c_2t_{2,0} \quad (B5).$$

If $c_5 + 1 \neq 0$ then $t_{1,1} \equiv 0$. It then follows that $s_{2,2} \equiv t_{1,1} \equiv 0$. 

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Hence $x \in \partial^{-1}W_2$ if and only if

\[ s_{1,2} \equiv -c_2s_{2,1} + t_{2,0} \quad \text{(A5)} \]

\[ 0 \equiv c_2c_5t_{2,0} \quad \text{(B2)} \]

\[ t_{0,2} \equiv c_2t_{2,0} \quad \text{(B5)}. \]

If $c_5 \neq 0$ then $t_{2,0} \equiv 0$. It then follows that $t_{0,2} \equiv s_{2,2} \equiv t_{1,1} \equiv t_{2,0} \equiv 0$ and $s_{1,2} \equiv -c_2s_{2,1}$. This is the same congruence as that in Case 7.4 therefore $\partial^{-1}W_2 = \partial^{-1}W_1$. Thus $\text{rank}(\partial^{-1}W_2/W_2) = \text{rank}(\partial^{-1}W_1/W_2)$, $W_2$ is terminal, and $W_2 \notin \hat{L}_2$.

If $c_5 = 0$ then $t_{2,0}$ is a free variable. We regard $t_{1,0}$, $t_{0,1}$, $s_{2,1}$, $t_{2,0}$ as the free variables. Taking $t_{1,0} \equiv 1$, $t_{0,1} = 0$, $s_{2,1} = 0$, and $t_{2,0} = 0$, the matrix $x$ becomes

\[
\begin{bmatrix}
0 & 0 & 0 \\
1 & 0 & 0 \\
0 & 0 & 0
\end{bmatrix}.
\]

Taking $t_{0,1} \equiv 1$, $t_{1,0} = 0$, $s_{2,1} = 0$, and $t_{2,0} = 0$, the matrix $x$ becomes

\[
\begin{bmatrix}
0 & 1 & 0 \\
0 & 0 & 0 \\
0 & 0 & 0
\end{bmatrix}.
\]

Taking $s_{2,1} \equiv 1$, $t_{0,1} = 0$, $t_{1,0} = 0$, and $t_{2,0} = 0$, the matrix $x$ becomes

\[
\begin{bmatrix}
0 & 0 & 0 \\
0 & 0 & -3c_2 \\
0 & 3 & 0
\end{bmatrix}.
\]
Taking $t_{2,0} \equiv 1$, $t_{0,1} = 0$, $t_{1,0} = 0$, and $s_{2,1} = 0$, the matrix $x$ becomes

$$v_4 = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 3 \\ 1 & 0 & 0 \end{bmatrix}.$$

We see that neither $v_4$ nor $3v_4$ is contained in $W_2$ and that $\partial^{-1}W_2 = < \partial^{-1}W_0, v_1, v_2, v_3, v_4 >$. We know that $v_4 \in \partial^{-1}W_2$ and since $3v_4 \not\in W_2$ then $v_4 + W_2$ in an element of the group $\partial^{-1}W_2/W_2$ whose order is larger than 3. Recall that $|\partial^{-1}W_2/W_2| = 3^3$ and that $\partial^{-1}W_1/W_2$ has rank 2. We obtain $\partial^{-1}W_2/W_2 \cong \mathbb{Z}_9 \times \mathbb{Z}_3$. Since $\text{rank}(\partial^{-1}W_2/W_2) = \text{rank}(\partial^{-1}W_1/W_2)$. Hence $W_2$ is terminal and $W_2 \not\in \hat{L}_2$.

Now we assume that $c_5 + 1 = 0$. Then $c_5 \equiv 2$ and $t_{1,1}$ is a free variable. (B2) becomes $0 \equiv -c_2t_{1,1} - c_2t_{2,0}$. Rewriting this we obtain $t_{2,0} \equiv -t_{1,1}$. Hence $x \in \partial^{-1}W_2$ if and only if

$$s_{2,2} \equiv c_2t_{1,1} \quad \text{(A4)}$$

$$s_{1,2} \equiv -c_2s_{2,1} - t_{1,1} \quad \text{(A5)}$$

$$t_{2,0} \equiv -t_{1,1} \quad \text{(B2)}$$

$$t_{0,2} \equiv -c_2t_{1,1} \quad \text{(B5)}.$$

We regard $t_{1,0}$, $t_{0,1}$, $s_{2,1}$, $t_{1,1}$ as the free variables. Taking $t_{1,0} \equiv 1$, $t_{0,1} = 0$, $s_{2,1} = 0$, and $t_{1,1} = 0$, the matrix $x$ becomes

$$v_1 = \begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$
Taking $t_{0,1} \equiv 1$, $t_{1,0} = 0$, $s_{2,1} = 0$, and $t_{1,1} = 0$, the matrix $x$ becomes
\[
v_2 = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}.
\]

Taking $s_{2,1} \equiv 1$, $t_{0,1} = 0$, $t_{1,0} = 0$, and $t_{2,0} = 0$, the matrix $x$ becomes
\[
v_3 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & -3c_2 \\ 0 & 3 & 0 \end{bmatrix}.
\]

Taking $t_{1,1} \equiv 1$, $t_{0,1} = 0$, $t_{1,0}$, and $s_{2,1} = 0$, the matrix $x$ becomes
\[
v_4 = \begin{bmatrix} 0 & 0 & -c_2 \\ 0 & 1 & -3 \\ -1 & 0 & 3c_2 \end{bmatrix}.
\]

We see that neither $v_4$ nor $3v_4$ is contained in $W_2$ and that
\[
\partial^{-1}W_2 = \langle \partial^{-1}W_0, v_1, v_2, v_3, v_4 \rangle. \quad \text{We know that } v_4 \in \partial^{-1}W_2 \text{ and since } 3v_4 \not\in W_2 \text{ then }
\]
$v_4 + W_2$ in an element of the group $\partial^{-1}W_2/W_2$ whose order is larger than 3. Recall that
\[
|\partial^{-1}W_2/W_2| = 3^3 \text{ and that } \partial^{-1}W_1/W_2 \text{ has rank } 2. \quad \text{We obtain } \partial^{-1}W_2/W_2 \cong \mathbb{Z}_9 \times \mathbb{Z}_3.
\]
Since rank$(\partial^{-1}W_2/W_2) = \text{rank}(\partial^{-1}W_1/W_2)$. Hence $W_2$ is terminal and $W_2 \not\in \hat{\mathcal{L}}_2$.

**Case 7.4.2.3.2** Now we consider the case when $c_4 \neq 0$. Then from (A2) we obtain
\[
t_{2,0} \equiv (-c_4c_5 - c_4)t_{1,1}. \quad \text{The congruence (A5) and (B5) become } s_{1,2} \equiv c_2s_{2,1} + (-c_4c_5 - c_4)t_{1,1} \text{ and } t_{0,2} \equiv c_2(-c_4c_5 - c_4)t_{1,1}. \quad \text{The congruence (B2) becomes } (-c_2c_4c_5^2 - c_2c_4c_5 + c_4 - c_2)t_{1,1} \equiv 0.
\]
Hence \( x \in \partial^{-1}W_2 \) if and only if

\[
t_{2,0} \equiv (-c_4c_5 - c_4)t_{1,1} \quad \text{(A2)}
\]

\[
s_{2,2} \equiv c_2t_{1,1} \quad \text{(A4)}
\]

\[
s_{1,2} \equiv -c_2s_{2,1} + (-c_4c_5 - c_4)t_{1,1} \quad \text{(A5)}
\]

\[
(-c_2c_4c_5^2 - c_2c_4c_5 + c_4 - c_2)t_{1,1} \equiv 0 \quad \text{(B2)}
\]

\[
t_{0,2} \equiv c_2(-c_4c_5 - c_4)t_{1,1} \quad \text{(B5)}.
\]

Let \( q = -c_2c_4c_5^2 - c_2c_4c_5 + c_4 - c_2 \). If \( q \neq 0 \) then \( t_{1,1} \equiv 0 \). (A1) becomes

\[0 \equiv c_4t_{2,0}.\] Since \( c_4 \neq 0 \) then \( t_{2,0} \equiv 0 \). It follows that \( t_{1,1} \equiv s_{2,2} \equiv t_{2,0} \equiv t_{0,2} \equiv 0 \) and

\[s_{1,2} \equiv -c_2s_{2,1}.\] This is the same congruence as that in Case 7.4 therefore \( \partial^{-1}W_2 = \partial^{-1}W_1 \). Thus \( \text{rank}(\partial^{-1}W_2/W_2) = \text{rank}(\partial^{-1}W_1/W_2) \), \( W_2 \) is terminal, and \( W_2 \notin \hat{\mathcal{L}}_2 \).

Now we assume that \( q = 0 \). Then \( t_{1,1} \) is a free variable. \( q = 0 \) if and only if \( (c_2, c_4, c_5) \) is one of these: \( (1, 1, 0), (1, 1, 2), (1, 2, 1), (1, 2, 2), (2, 2, 0), (2, 2, 2) \).

We regard \( t_{1,0}, t_{0,1}, s_{2,1}, t_{1,1} \) as the free variables. Taking \( t_{1,0} \equiv 1, t_{1,0} = 0, t_{0,1} = 0, \) and \( s_{2,1} = 0 \), the matrix \( x \) becomes

\[
v_1 = \begin{bmatrix}
0 & 0 & 0 \\
1 & 0 & 0 \\
0 & 0 & 0
\end{bmatrix}.
\]

Taking \( t_{0,1} \equiv 1, t_{1,0} = 0, s_{2,1} = 0, \) and \( t_{1,1} = 0 \), the matrix \( x \) becomes

\[
v_2 = \begin{bmatrix}
0 & 1 & 0 \\
0 & 0 & 0 \\
0 & 0 & 0
\end{bmatrix}.
\]

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Taking \( s_{2,1} \equiv 1, t_{0,1} = 0, t_{1,0} = 0, \) and \( t_{1,1} = 0, \) the matrix \( x \) becomes

\[
v_3 = \begin{bmatrix}
0 & 0 & 0 \\
0 & 0 & -3c_2 \\
0 & 3 & 0
\end{bmatrix}.
\]

Taking \( t_{1,1} \equiv 1, t_{0,1} = 0, t_{1,0} = 0, \) and \( s_{2,1} = 0, \) the matrix \( x \) becomes

\[
v_4 = \begin{bmatrix}
0 & 0 & c_2(-c_4c_5 - c_4) \\
0 & 1 & 3(-c_4c_5 - c_4) \\
-c_4c_5 - c_4 & 0 & 3c_2
\end{bmatrix}.
\]

We see that neither \( v_4 \) nor \( 3v_4 \) is contained in \( W_2 \) and that

\[
\partial^{-1}W_2 = < \partial^{-1}W_0, v_1, v_2, v_3, v_4 >. \]

We know that \( v_4 \in \partial^{-1}W_2 \) and since \( 3v_4 \not\in W_2 \) then \( v_4 + W_2 \) in an element of the group \( \partial^{-1}W_2/W_2 \) whose order is larger than 3. Recall that \( |\partial^{-1}W_2/W_2| = 3^3 \) and that \( \partial^{-1}W_1/W_2 \) has rank 2. We obtain \( \partial^{-1}W_2/W_2 \cong \mathbb{Z}_9 \times \mathbb{Z}_3 \).

Since \( \text{rank}(\partial^{-1}W_2/W_2) = \text{rank}(\partial^{-1}W_1/W_2) \). Hence \( W_2 \) is terminal and \( W_2 \not\in \hat{L}_2 \).

Case 7.4.2.4

We fix arbitrary values \( c_4, c_5 \in \{0, 1, 2\} \) with \( (c_4, c_5) \neq (0, 0) \). There are 8 ways to choose the values \( c_4 \) and \( c_5 \). Let \( m_4 = y_2 + c_4v_1 + c_5v_2 \) and \( m_5 = v_3 \). Thus

\[
m_4 = \begin{bmatrix}
0 & c_5 & 0 \\
c_4 & 3 & 0 \\
0 & 0 & 0
\end{bmatrix} \quad \text{and} \quad m_5 = \begin{bmatrix}
0 & 0 & 0 \\
0 & 3 & 0 \\
0 & 0 & 0
\end{bmatrix}.
\]

Let \( W_2 = < W_1, m_4, m_5 > \in L_2 \). The number of subgroups \( W_2 \) of this type is

8. Since \( m_4, m_5 \not\in W_1 \) and \( 3m_4, 3m_5 \in W_1 \) we have \( |W_2/W_1| = 3^2 \). Since \( |W_1| = 3^6 \) it
follows that $|W_2| = 3^8$. Note that $\partial^{-1}W_1/W_2 \cong \mathbb{Z}_3 \times \mathbb{Z}_3$ and rank($\partial^{-1}W_1/W_2$) = 2.

We now calculate the pullback of $\partial^{-1}W_2$. We observed in case 7.4.2 that $\partial^{-1}W_2$ is contained in the pattern subgroup

$$\begin{bmatrix}
2 & 2 & 2 \\
2 & 2 & 1 \\
2 & 1 & 1 \\
\end{bmatrix}.$$

Let

$$x = \begin{bmatrix}
3s_{0,0} + t_{0,0} & 3s_{0,1} + t_{0,1} & 3s_{0,2} + t_{0,2} \\
3s_{1,0} + t_{1,0} & 3s_{1,1} + t_{1,1} & 3s_{1,2} \\
3s_{2,0} + t_{2,0} & 3s_{2,1} & 3s_{2,2} \\
\end{bmatrix} \in \begin{bmatrix}
2 & 2 & 2 \\
2 & 2 & 1 \\
2 & 1 & 1 \\
\end{bmatrix}.$$

The variables that are in play are those appearing in the matrix

$$\begin{bmatrix}
0 & t_{0,1} & t_{0,2} \\
t_{1,0} & t_{1,1} & 3s_{1,2} \\
t_{2,0} & 3s_{2,1} & 3s_{2,2} \\
\end{bmatrix}.$$

In the notation of Case 7.4.2, we are taking $d_1 = 1$, $d_2 = c_4$, $d_3 = c_5$, $d_4 = 0$, $e_2 = 0$, $e_3 = 0$, and $e_4 = 1$. We wish to identify values $a_1, a_2, a_3, a_4, a_5 \in \mathbb{Z}_9$ such that $\partial_1 x \equiv a_1m_1 + a_2m_2 + a_3m_3 + a_4m_4 + a_5m_5 \pmod{I}$.  

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We see that $\partial_1 x \in W_2$ if and only if

\[
t_{1,1} \equiv a_4 c_5 \quad \text{(A1)}
\]
\[
-t_{1,1} \equiv a_5 \quad \text{(A2)}
\]
\[
t_{2,0} \equiv a_4 c_4 \quad \text{(A3)}
\]
\[
s_{2,2} \equiv c_2 t_{1,1} \quad \text{(A4)}
\]
\[
s_{1,2} \equiv c_2 a_4 + t_{2,0} - c_2 s_{2,1} \quad \text{(A5)}.
\]

We wish to identify values $b_1, b_2, b_3, b_4, b_5 \in \mathbb{Z}_9$ such that $\partial_1 x \equiv b_1 m_1 + b_2 m_2 + b_3 m_3 + b_4 m_4 + b_5 m_5 \pmod{I}$. We see that $\partial_2 x \in W_2$ if and only if

\[
t_{0,2} \equiv b_4 c_5 \quad \text{(B1)}
\]
\[
s_{2,2} \equiv b_5 \quad \text{(B2)}
\]
\[
t_{1,1} \equiv b_4 c_4 \quad \text{(B3)}
\]
\[
t_{1,1} \equiv c_2 s_{2,2} \quad \text{(B4)}
\]
\[
s_{1,2} \equiv -c_2 s_{2,1} + c_2 t_{0,2} + b_4 \quad \text{(B5)}.
\]

Multiplying (B4) by $c_2$ we obtain $s_{2,2} \equiv c_2 t_{1,1}$ which is redundant with (A4).

Combining (A5) and (B5) we obtain $c_2 a_4 + t_{2,0} \equiv b_4 + c_2 t_{0,2}$ which we will denote as our new (B5).
Hence $x \in \partial^{-1}W_2$ if and only if

\[
\begin{align*}
  t_{1,1} &\equiv a_4 c_5 \quad \text{(A1)} \\
  t_{2,0} &\equiv a_4 c_4 \quad \text{(A3)} \\
  s_{2,2} &\equiv c_2 t_{1,1} \quad \text{(A4)} \\
  s_{1,2} &\equiv c_2 a_4 + t_{2,0} - c_2 s_{2,1} \quad \text{(A5)} \\
  t_{0,2} &\equiv b_4 c_5 \quad \text{(B1)} \\
  t_{1,1} &\equiv b_4 c_4 \quad \text{(B3)} \\
  c_2 a_4 + t_{2,0} &\equiv c_2 t_{0,2} + b_4 \quad \text{(B5)}.
\end{align*}
\]

It is convenient to consider the cases $c_4 \neq 0$ and $c_4 = 0$ separately.

**Case 7.4.2.4.1** First we consider the case $c_4 \neq 0$. Then $c_4 \in \{1, 2\}$ and $c_4^2 \equiv 1$. Then from (A3) and (B3) we obtain $a_4 \equiv c_4 t_{2,0}$ and $b_4 \equiv c_4 t_{1,1}$. Substituting $a_4$ and $b_4$ into (A1) and (B1) we obtain $t_{1,1} \equiv c_4 c_5 t_{2,0}$ and $t_{0,2} \equiv c_4 c_5 t_{1,1}$. Substituting (A1) into (B1) we obtain $t_{0,2} \equiv c_4^2 t_{2,0}$. The congruence (A4) becomes $s_{2,2} \equiv c_2 c_4 c_5 t_{2,0}$. Substituting $a_4$ into (A5) we obtain $s_{1,2} \equiv (c_2 c_4 + 1) t_{2,0} - c_2 s_{2,1}$. Substituting $a_4$, $b_4$, and $t_{0,2}$ into (B5) we obtain $(c_4 + c_2 - c_2 c_5 - c_4^2) t_{2,0} \equiv 0$. 

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Hence $x \in \partial^{-1}W_2$ if and only if

\begin{align*}
  t_{1,1} &\equiv c_4 c_5 t_{2,0} \quad \text{(A1)} \\
  s_{2,2} &\equiv c_2 c_4 c_5 t_{2,0} \quad \text{(A4)} \\
  s_{1,2} &\equiv (c_2 c_4 + 1) t_{2,0} - c_2 s_{2,1} \quad \text{(A5)} \\
  t_{0,2} &\equiv c_5^2 t_{2,0} \quad \text{(B1)} \\
 (c_4 + c_2 - c_2 c_5 - c_5^2) t_{2,0} &\equiv 0 \quad \text{(B5)}.
\end{align*}

Let $q = c_4 + c_2 - c_2 c_5 - c_5^2$. If $q \neq 0$ then $t_{2,0} \equiv 0$. It then follows that $t_{1,1} \equiv s_{2,2} \equiv t_{0,2} \equiv t_{2,0} \equiv 0$, and $s_{1,2} \equiv -c_2 s_{2,1}$. This is the same congruence as that in Case 7.4 therefore $\partial^{-1}W_2 = \partial^{-1}W_1$. Thus $\text{rank}(\partial^{-1}W_2/W_2) = \text{rank}(\partial^{-1}W_1/W_2)$, $W_2$ is terminal, and $W_2 \not\in \tilde{L}_2$.

Therefore we will assume that $q = 0$. Then $t_{2,0}$ is a free variable. $q = 0$ if and only if $(c_2, c_4, c_5)$ is one of these: $(1, 2, 0), (1, 1, 1), (1, 2, 2), (2, 1, 0), (2, 1, 1)$.

We regard $t_{1,0}, t_{0,1}, s_{2,1}, t_{2,0}$ as the free variables. Taking $t_{1,0} \equiv 1, t_{0,1} = 0, s_{2,1} = 0, t_{2,0} = 0$, the matrix $x$ becomes

\[ v_1 = \begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \cdot \]

Taking $t_{0,1} \equiv 1, t_{1,0} = 0, s_{2,1} = 0, t_{2,0} = 0$, the matrix $x$ becomes

\[ v_2 = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \cdot \]

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Taking \( s_{2,1} \equiv 1, t_{0,1} = 0, t_{1,0} = 0, \) and \( t_{2,0} = 0, \) the matrix \( x \) becomes

\[
v_3 = \begin{bmatrix}
0 & 0 & 0 \\
0 & 0 & -3c_2 \\
0 & 3 & 0
\end{bmatrix}.
\]

Taking \( t_{2,0} \equiv 1, t_{0,1} = 0, t_{1,0} = 0, \) and \( s_{2,1} = 0, \) the matrix \( x \) becomes

\[
v_4 = \begin{bmatrix}
0 & 0 & c_5^2 \\
0 & c_4c_5 & 3(c_2c_4 + 1) \\
1 & 0 & 3c_2c_4c_5
\end{bmatrix}.
\]

We see that neither \( v_4 \) nor \( 3v_4 \) is contained in \( W_2 \) and that

\[\partial^{-1}W_2 = \langle \partial^{-1}W_0, v_1, v_2, v_3, v_4 \rangle.\]

We know that \( v_4 \in \partial^{-1}W_2 \) and since \( 3v_4 \not\in W_2 \) then \( v_4 + W_2 \) is an element of the group \( \partial^{-1}W_2/W_2 \) whose order is larger than 3. Recall that \( |\partial^{-1}W_2/W_2| = 3^3 \) and that \( \partial^{-1}W_1/W_2 \) has rank 2. We obtain \( \partial^{-1}W_2/W_2 \cong \mathbb{Z}_9 \times \mathbb{Z}_3. \)

Since \( \text{rank}(\partial^{-1}W_2/W_2) = \text{rank}(\partial^{-1}W_1/W_2) \). Hence \( W_2 \) is terminal and \( W_2 \not\in \hat{L}_2. \)

**Case 7.4.2.4.2** Now we consider the case \( c_4 = 0. \) Then \( c_5 \neq 0, c_5 \in \{1, 2\}, \) and \( c_5^2 \equiv 1. \) Congruences (A3) and (B3) become \( t_{2,0} \equiv 0 \) and \( t_{1,1} \equiv 0. \) Congruence (A1) can be rewritten as \( a_4 \equiv c_5t_{1,1}. \) Since \( t_{1,1} \equiv 0 \) then \( a_4 \equiv 0. \) Congruence (B1) can be rewritten as \( b_4 \equiv c_5t_{0,2}. \) Substituting \( t_{1,1}, a_4, b_4, \) and \( t_{2,0} \) into (A4), (A5), and (B5) we obtain \( s_{2,2} \equiv 0, s_{1,2} \equiv -c_2s_{2,1}, \) and \( 0 \equiv (c_5 + c_2)t_{0,2}, \) respectively.
Hence $x \in \partial^{-1}W_2$ if and only if

\[
t_{1,1} \equiv 0 \\
\quad s_{2,2} \equiv 0 \\
\quad t_{2,0} \equiv 0 \\
\quad s_{1,2} \equiv -c_2s_{2,1} \\
\quad 0 \equiv (c_5 + c_2)t_{0,2}.
\]

If $c_5 + c_2 \neq 0$, then $t_{0,2} \equiv 0$. Then $t_{1,1} \equiv s_{2,2} \equiv t_{0,2} \equiv t_{2,0} \equiv 0$ and $s_{1,2} \equiv -c_2s_{2,1}$. This is the same congruence as that in Case 7.4 therefore $\partial^{-1}W_2 = \partial^{-1}W_1$. Thus $\text{rank}(\partial^{-1}W_2/W_2) = \text{rank}(\partial^{-1}W_1/W_2)$, $W_2$ is terminal, and $W_2 \not\in \hat{L}_2$.

Therefore we will assume that $c_5 + c_2 \equiv 0$. Then $c_5 \equiv -c_2$ and $t_{0,2}$ is a free variable. We regard $t_{1,0}$, $t_{0,1}$, $s_{2,1}$, and $t_{0,2}$ as the free variables. Taking $t_{1,0} \equiv 1$, $t_{0,1} = 0$, $s_{2,1} = 0$, and $t_{0,2} = 0$, the matrix $x$ becomes

\[
v_1 = \begin{bmatrix}
0 & 0 & 0 \\
1 & 0 & 0 \\
0 & 0 & 0
\end{bmatrix}.
\]

Taking $t_{0,1} \equiv 1$, $t_{1,0} = 0$, $s_{2,1} = 0$, and $t_{0,2} = 0$, the matrix $x$ becomes

\[
v_2 = \begin{bmatrix}
0 & 1 & 0 \\
0 & 0 & 0 \\
0 & 0 & 0
\end{bmatrix}.
\]
Taking \( s_{2,1} \equiv 1, t_{0,1} = 0, t_{1,0} = 0, \) and \( t_{0,2} = 0, \) the matrix \( x \) becomes
\[
v_3 = \begin{bmatrix}
  0 & 0 & 0 \\
  0 & 0 & -3c_2 \\
  0 & 3 & 0
\end{bmatrix}.
\]

Taking \( t_{0,2} \equiv 1, t_{0,1} = 0, t_{1,0} = 0, \) and \( s_{2,1} = 0, \) the matrix \( x \) becomes
\[
v_4 = \begin{bmatrix}
  0 & 0 & 1 \\
  0 & 0 & 0 \\
  0 & 0 & 0
\end{bmatrix}.
\]

We see that neither \( v_4 \) nor \( 3v_4 \) is contained in \( W_2 \) and that
\[
\partial^{-1}W_2 = \langle \partial^{-1}W_0, v_1, v_2, v_3, v_4 \rangle. \]
We know that \( v_4 \in \partial^{-1}W_2 \) and since \( 3v_4 \not\in W_2 \) then
\( v_4 + W_2 \) in an element of the group \( \partial^{-1}W_2/W_2 \) whose order is larger than 3. Recall that
\[
|\partial^{-1}W_2/W_2| = 3^3 \]
and that \( \partial^{-1}W_1/W_2 \) has rank 2. We obtain \( \partial^{-1}W_2/W_2 \cong \mathbb{Z}_9 \times \mathbb{Z}_3 \).

Since \( \text{rank}(\partial^{-1}W_2/W_2) = \text{rank}(\partial^{-1}W_1/W_2) \). Hence \( W_2 \) is terminal and \( W_2 \not\in \hat{\mathcal{L}}_2 \).

Case 7.4.2.5

We fix arbitrary values \( c_4, c_5, c_6 \in \{0, 1, 2\} \) with \( (c_4, c_5) \neq (0, 0) \). There are 24 ways to choose the values \( c_4, c_5, \) and \( c_6 \). Let \( m_4 = y_2 + c_4 v_1 + c_5 v_3 \) and \( m_5 = v_2 + c_6 v_3 \).

Thus
\[
m_4 = \begin{bmatrix}
  0 & 0 & 0 \\
  c_4 & 3 & -3c_2c_5 \\
  0 & 3c_5 & 0
\end{bmatrix} \quad \text{and} \quad m_5 = \begin{bmatrix}
  0 & 1 & 0 \\
  0 & 0 & -3c_2c_6 \\
  0 & 3c_6 & 0
\end{bmatrix}.
\]

Let \( W_2 = \langle W_1, m_4, m_5 \rangle \in \mathcal{L}_2 \). The number of subgroups \( W_2 \) of this type is
24. Since \( m_4, m_5 \not\in W_1 \) and \( 3m_4, 3m_5 \in W_1 \) we have \( |W_2/W_1| = 3^2 \). Since \( |W_1| = 3^6 \)
it follows that \(|W_2| = 3^8\). Note that \(\partial^{-1}W_1/W_2 \cong \mathbb{Z}_3 \times \mathbb{Z}_3\) and \(\text{rank}(\partial^{-1}W_1/W_2) = 2\).

We now calculate the pullback of \(\partial^{-1}W_2\). We observed in case 7.4.2 that \(\partial^{-1}W_2\) is contained in the pattern subgroup

\[
\begin{bmatrix}
2 & 2 & 2 \\
2 & 2 & 1 \\
2 & 1 & 1
\end{bmatrix}.
\]

Let

\[
x = \begin{bmatrix} 3s_{0,0} + t_{0,0} & 3s_{0,1} + t_{0,1} & 3s_{0,2} + t_{0,2} \\
3s_{1,0} + t_{1,0} & 3s_{1,1} + t_{1,1} & 3s_{1,2} \\
3s_{2,0} + t_{2,0} & 3s_{2,1} & 3s_{2,2}
\end{bmatrix} \in \begin{bmatrix} 2 & 2 & 2 \\
2 & 2 & 1 \\
2 & 1 & 1
\end{bmatrix}.
\]

The variables that are in play are those appearing in the matrix

\[
\begin{bmatrix}
0 & t_{0,1} & t_{0,2} \\
t_{1,0} & t_{1,1} & 3s_{1,2} \\
t_{2,0} & 3s_{2,1} & 3s_{2,2}
\end{bmatrix}.
\]

In the notation of Case 7.4.2, we are taking \(d_1 = 1, d_2 = c_4, d_3 = 0, d_4 = c_5, e_2 = 0, e_3 = 1, \) and \(e_4 = c_6\). We wish to identify values \(a_1, a_2, a_3, a_4, a_5 \in \mathbb{Z}_9\) such that \(\partial_1 x \equiv a_1m_1 + a_2m_2 + a_3m_3 + a_4m_4 + a_5m_5 \pmod{I}\).
We see that $\partial_1 x \in W_2$ if and only if

$$t_{1,1} \equiv a_5 \quad \text{(A1)}$$

$$-t_{1,1} \equiv a_4c_5 + a_5c_6 \quad \text{(A2)}$$

$$t_{2,0} \equiv a_4c_4 \quad \text{(A3)}$$

$$s_{2,2} \equiv c_2t_{1,1} \quad \text{(A4)}$$

$$s_{1,2} \equiv c_2a_4 + t_{2,0} - c_2s_{2,1} \quad \text{(A5)}.$$

We wish to identify values $b_1, b_2, b_3, b_4, b_5 \in \mathbb{Z}_9$ such that $\partial_1 x \equiv b_1m_1 + b_2m_2 + b_3m_3 + b_4m_4 + b_5m_5 \pmod{I}$.

We see that $\partial_2 x \in W_2$ if and only if

$$t_{0,2} \equiv b_5 \quad \text{(B1)}$$

$$s_{2,2} \equiv b_4c_5 + b_5c_6 \quad \text{(B2)}$$

$$t_{1,1} \equiv b_4c_4 \quad \text{(B3)}$$

$$t_{1,1} \equiv c_2s_{2,2} \quad \text{(B4)}$$

$$s_{1,2} \equiv -c_2s_{2,1} + c_2t_{0,2} + b_4 \quad \text{(B5)}.$$

Multiplying (B4) by $c_2$ we obtain $s_{2,2} \equiv c_2t_{1,1}$ which is redundant with (A4).

Substituting $a_5$ into (A2) we obtain $0 \equiv c_5a_4 + (c_6 + 1)t_{1,1}$. Substituting $b_5$ and (A4) into (B2) we obtain $c_2t_{1,1} \equiv c_5b_4 + c_6t_{0,2}$. Combining (A5) and (B5) we obtain

$$c_2a_4 + t_{2,0} \equiv c_2t_{0,2} + b_4$$

which we will denote as our new (B5).
Hence \( x \in \partial^{-1}W_2 \) if and only if

\[
0 \equiv c_5a_4 + (c_6 + 1)t_{1,1} \quad \text{(A2)}
\]

\[
t_{2,0} \equiv a_4c_4 \quad \text{(A3)}
\]

\[
s_{2,2} \equiv c_2t_{1,1} \quad \text{(A4)}
\]

\[
s_{1,2} \equiv c_2a_4 + t_{2,0} - c_2s_{2,1} \quad \text{(A5)}
\]

\[
c_2t_{1,1} \equiv b_4c_5 + c_6t_{0,2} \quad \text{(B2)}
\]

\[
t_{1,1} \equiv b_4c_4 \quad \text{(B3)}
\]

\[
c_2a_4 + t_{2,0} \equiv c_2t_{0,2} + b_4 \quad \text{(B5)}.
\]

It is convenient to consider the cases \( c_4 \neq 0 \) and \( c_4 = 0 \) separately.

**Case 7.4.2.5.1** First we consider the case \( c_4 \neq 0 \). Then \( c_4 \in \{1, 2\} \) and \( c_4^2 \equiv 1 \). Then from (A3) and (B3) we get \( a_4 \equiv c_4t_{2,0} \) and \( b_4 \equiv c_4t_{1,1} \). Substituting \( a_4 \) into (A2) and \( b_4 \) into (B2) we obtain \( 0 \equiv c_4c_5t_{2,0} + (c_6 + 1)t_{1,1} \) and \( 0 \equiv (c_4c_5 - c_2)t_{1,1} + c_6t_{0,2} \).

Substituting \( a_4 \) into (A5) we obtain \( s_{1,2} \equiv (c_2c_4 + 1)t_{2,0} - c_2s_{2,1} \). Substituting \( a_4 \) and \( b_4 \) into (B5) we obtain \( (c_2c_4 + 1)t_{2,0} \equiv c_2t_{0,2} + c_4t_{1,1} \) which we will rewrite as \( t_{1,1} \equiv (c_2 + c_4)t_{2,0} - c_2c_4t_{0,2} \). Substituting (B5) into (A2) and (B2) we obtain \( c_2c_4(c_6 + 1)t_{0,2} \equiv (c_4c_5+c_2c_6+c_4c_6+c_2+c_4)t_{2,0} \) and \( (c_2c_4c_5+c_5-1-c_2c_4)t_{2,0} \equiv (c_2c_5-c_4-c_6)t_{0,2} \).
Hence $x \in \partial^{-1}W_2$ if and only if
\[ c_2c_4(c_6 + 1)t_{0,2} \equiv (c_4c_5 + c_2c_6 + c_4c_6 + c_2 + c_4)t_{2,0} \quad (A2) \]
\[ s_{2,2} \equiv c_2t_{1,1} \quad (A4) \]
\[ s_{1,2} \equiv (c_2c_4 + 1)t_{2,0} - c_2s_{2,1} \quad (A5) \]
\[ (c_2c_4c_5 + c_5 - 1 - c_2c_4)t_{2,0} \equiv (c_2c_5 - c_4 - c_6)t_{0,2} \quad (B2) \]
\[ t_{1,1} \equiv (c_2 + c_4)t_{2,0} - c_2c_4t_{0,2} \quad (B5). \]

It is convenient to consider the cases $c_6 = 2$ and $c_6 \neq 2$ separately.

**Case 7.4.2.5.1.1** Suppose $c_6 = 2$. Then (A2) becomes $0 \equiv c_4c_5t_{2,0}$. Since $c_4 \neq 0$ this becomes $0 \equiv c_5t_{2,0}$. (B2) becomes $(c_2c_4c_5 + c_5 - 1 - c_2c_4)t_{0,2} \equiv (c_2c_5 - c_4 + 1)t_{2,0}$.

Hence $x \in \partial^{-1}W_2$ if and only if
\[ 0 \equiv c_5t_{2,0} \quad (A2) \]
\[ s_{2,2} \equiv c_2t_{1,1} \quad (A4) \]
\[ s_{1,2} \equiv (c_2c_4 + 1)t_{2,0} - c_2s_{2,1} \quad (A5) \]
\[ (c_2c_4c_5 + c_5 - 1 - c_2c_4)t_{2,0} \equiv (c_2c_5 - c_4 + 1)t_{0,2} \quad (B2) \]
\[ t_{1,1} \equiv (c_2 + c_4)t_{2,0} - c_2c_4t_{0,2} \quad (B5). \]

It is convenient to consider $c_5 \neq 0$ and $c_5 = 0$ separately.

**Case 7.4.2.5.1.1.1** If $c_5 \neq 0$ then $t_{2,0} \equiv 0$. (A5) becomes $s_{1,2} \equiv -c_2s_{2,1}$, (B2) becomes $0 \equiv (1 + c_2c_5 - c_4)t_{0,2}$, and (B5) becomes $t_{1,1} \equiv -c_2c_4t_{0,2}$.
If $1 + c_2c_5 - c_4 \neq 0$ then $t_{0,2} \equiv 0$. It follows that $t_{1,1} \equiv 0$, $s_{2,2} \equiv 0$, and $s_{1,2} \equiv -c_2s_{2,1}$. These are the same congruences as those in Case 7.4. Hence $\partial^{-1}W_2 = \partial^{-1}W_1$. Thus $\text{rank}(\partial^{-1}W_2/W_2) = \text{rank}(\partial^{-1}W_1/W_2)$, $W_2$ is terminal, and $W_2 \not\in \hat{\mathcal{L}}_2$.

Suppose $1 + c_2c_5 - c_4 \equiv 0$ then $t_{0,2}$ is a new free variable. Taking $t_{0,2} \equiv 1$, $t_{0,1} = 0$, $t_{1,0} = 0$, and $s_{2,1} = 0$, the matrix $x$ becomes

$$v_4 = \begin{bmatrix} 0 & 0 & 1 \\ 0 & -c_2c_4 & 0 \\ 0 & 0 & -3c_4 \end{bmatrix} \quad \text{and} \quad 3v_4 = \begin{bmatrix} 0 & 0 & 3 \\ 0 & -3c_2c_4 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$ 

Let $r_1 = 0$, $r_2 = -c_2c_4$, and $r_3 = 1$. Hence $r_1 + c_2r_2 + r_3 = -c_4 + 1$. Note that $-c_4 + 1 \equiv 0$ holds if and only if $c_4 = 1$.

If $c_4 = 2$ then by Lemma 10.4.2 $3v_4 \not\in W_2$ and $v_4 + W_2$ has order 9. Hence $\partial^{-1}W_2/W_2 \cong \mathbb{Z}_9 \times \mathbb{Z}_3$. Hence $\text{rank}(\partial^{-1}W_1/W_2) = \text{rank}(\partial^{-1}W_2/W_2)$. Hence $W_2$ is terminal and $W_2 \not\in \hat{\mathcal{L}}_2$.

If $c_4 = 1$, then by Lemmas 10.4.1 and 10.4.2 $3v_4 \equiv -c_2m_2 + m_3 \in W_1 \subseteq W_2$. Therefore $\partial^{-1}W_2/W_2 \cong \mathbb{Z}_3 \times \mathbb{Z}_3 \times \mathbb{Z}_3$. Hence $\text{rank}(\partial^{-1}W_1/W_2) < \text{rank}(\partial^{-1}W_2/W_2)$. Hence $W_2$ is nonterminal and $W_2 \in \hat{\mathcal{L}}_2$. $\partial^{-1}W_2/W_2$ has basis $v_1 + W_2, v_3 + W_2, v_4 + W_2$ while its subspace $\partial^{-1}W_1/W_2$ has basis $v_1 + W_2, v_3 + W_2$.

Let $c_7, c_8 \in \{0, 1, 2\}$. Let $m_6 = c_7v_1 + c_8v_3 + v_4$. Thus

$$m_6 = \begin{bmatrix} 0 & c_8 & 1 \\ c_7 & -c_2c_4 & 0 \\ 0 & 0 & -3c_4 \end{bmatrix}.$$
Let $W_3 = \langle W_2, m_6 \rangle \in L_3$. There are 9 subgroups of this type. By the Modified Terminal Lemma we conclude that $\partial^{-1}W_3 = \partial^{-1}W_2$. Thus $\text{rank}(\partial^{-1}W_3/W_3) = \text{rank}(\partial^{-1}W_2/W_3)$, $W_3$ is terminal, and $W_3 \notin \hat{L}_3$.

**Case 7.4.2.5.1.1.2** Suppose $c_5 = 0$. Then (A2) is automatic and (B2) becomes

$$(c_4 - 1)t_{0,2} \equiv (c_2c_4 + 1)t_{2,0}.$$  

Hence $x \in \partial^{-1}W_2$ if and only if

$$s_{2,2} \equiv c_2t_{1,1} \quad \text{(A4)}$$

$$s_{1,2} \equiv (c_2c_4 + 1)t_{2,0} - c_2s_{2,1} \quad \text{(A5)}$$

$$(c_4 - 1)t_{0,2} \equiv (c_2c_4 + 1)t_{2,0} \quad \text{(B2)}$$

$$t_{1,1} \equiv (c_2 + c_4)t_{2,0} - c_2c_4t_{0,2} \quad \text{(B5)}.$$  

Hence it is convenient to consider $c_4 = 1$ and $c_4 \neq 1$ separately.

Suppose $c_4 = 1$ then $t_{0,2}$ is a new free variable. (B2) becomes $0 \equiv (c_2 + 1)t_{2,0}$.

If $c_2 \neq 2$ then $t_{2,0} \equiv 0$. Then $t_{0,2}$ is the only new free variable. Taking $t_{0,2} \equiv 1$, $t_{0,1} = 0$, $t_{1,0} = 0$, and $s_{2,1} = 0$, the matrix $x$ becomes

$$v_4 = \begin{bmatrix} 0 & 0 & 1 \\ 0 & -c_2 & 0 \\ 0 & 0 & -3 \end{bmatrix} \quad \text{and} \quad 3v_4 = \begin{bmatrix} 0 & 0 & 3 \\ 0 & -3c_2c_4 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$  

Let $r_1 = 0$, $r_2 = -c_2$, and $r_3 = 1$. Hence $r_1 + c_2r_2 + r3 \equiv -1 + 1 \equiv 0$.

Then by Lemmas 10.4.1 and 10.4.2 it follows that $3v_4 \equiv -2m_2 + m_3 \in W_1 \subseteq W_2$. Therefore $\partial^{-1}W_2/W_2 \cong \mathbb{Z}_3 \times \mathbb{Z}_3 \times \mathbb{Z}_3$. Hence $\text{rank}(\partial^{-1}W_1/W_2) < \text{rank}(\partial^{-1}W_2/W_2)$. 

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Hence $W_2$ is nonterminal and $W_2 \in \hat{L}_2$. $\partial^{-1}W_2/W_2$ has basis $v_1 + W_2, v_3 + W_2, v_4 + W_2$ while its subspace $\partial^{-1}W_1/W_2$ has basis $v_1 + W_2, v_3 + W_2$. Let $c_7, c_8 \in \{0, 1, 2\}$. Let $m_6 = c_7 v_1 + c_8 v_3 + v_4$. Thus

$$m_6 = \begin{bmatrix} 0 & c_8 & 1 \\ c_7 & -c_2 & 0 \\ 0 & 0 & -3 \end{bmatrix}.$$  

Let $W_3 = <W_2, m_6> \in L_3$. There are 9 subgroups of this type. By the Modified Terminal Lemma we conclude that $\partial^{-1}W_3 = \partial^{-1}W_2$. Thus $\text{rank}(\partial^{-1}W_3/W_3) = \text{rank}(\partial^{-1}W_2/W_3)$, $W_3$ is terminal, and $W_3 \notin \hat{L}_3$.

If $c_2 = 2$ then $t_{2,0}$ is also a free variable. Taking $t_{0,2} \equiv 1, t_{0,1} = 0, t_{1,0} = 0, s_{2,1} = 0$, and $t_{2,0} = 0$, the matrix $x$ becomes

$$v_4 = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & -3 \end{bmatrix} \text{ and } 3v_4 = \begin{bmatrix} 0 & 0 & 3 \\ 0 & 3 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$  

Let $r_1 = 0, r_2 = 1$, and $r_3 = 0$. Hence $r_1 + c_2 r_2 + r_3 \equiv 2 + 1 \equiv 0$. Then by Lemmas 10.4.1 and 10.4.2 it follows that $3v_4 \equiv m_2 + m_3 \in W_1 \subseteq W_2$ and $v_4 + W_2$ is an element of order 3.

Taking $t_{2,0} \equiv 1, t_{0,1} = 0, t_{1,0} = 0, s_{2,1} = 0$, and $t_{0,2} = 0$, the matrix $x$ becomes

$$v_5 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix} \text{ and } 3v_5 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 3 & 0 & 0 \end{bmatrix}.$$  

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Let \( r_1 = 1 \), \( r_2 = 0 \), and \( r_3 = 0 \). Hence \( r_1 + c_2 r_2 + r_3 \equiv 1 \not\equiv 0 \). Then by Lemmas 10.4.1 and 10.4.2 it follows that \( 3v_5 \notin W_2 \). Therefore \( v_5 + W_2 \) is an element of order larger than 3.

We see that \(|\partial^{-1}W_2/\partial^{-1}W_1| = 3^2\). We know that \( \partial^{-1}W_2/W_2 \cong \mathbb{Z}_9 \times \mathbb{Z}_3 \times \mathbb{Z}_3 \) or \( \mathbb{Z}_3 \times \mathbb{Z}_3 \times \mathbb{Z}_3 \times \mathbb{Z}_3 \). Since \( v_4 + W_2 \) has order 3 and \( v_5 + W_2 \) has order 9, we see that \( \partial^{-1}W_2/W_2 \cong \mathbb{Z}_9 \times \mathbb{Z}_3 \times \mathbb{Z}_3 \). We know \( \Omega_1(\partial^{-1}W_2/W_2) \cong \mathbb{Z}_3 \times \mathbb{Z}_3 \times \mathbb{Z}_3 \) and \( \Omega_1(\partial^{-1}W_2/W_2) \cong \partial^{-1}W_1/W_1 \). Hence every \( W_3/W_2 \) has order 3 and is thus contained in \( \Omega_1(\partial^{-1}W_2/W_2) \cong \mathbb{Z}_3 \times \mathbb{Z}_3 \times \mathbb{Z}_3 \). \( \partial^{-1}W_2/W_2 \) has basis \( v_1 + W_2, v_3 + W_2, v_4 + W_2 \) while its subspace \( \partial^{-1}W_1/W_2 \) has basis \( v_1 + W_2, v_3 + W_2 \). Let \( c_7, c_8 \in \{0, 1, 2\} \). Let
\[
m_6 = c_7 v_1 + c_8 v_3 + v_4.
\]

Thus
\[
m_6 = \begin{bmatrix}
0 & c_8 & 1 \\
c_7 & 1c_4 & 0 \\
0 & 0 & -3
\end{bmatrix}.
\]

Let \( W_3 = <W_2, m_6> \in \mathcal{L}_3 \). There are 9 subgroups of this type. By the Modified Terminal Lemma we conclude that \( \partial^{-1}W_3 = \partial^{-1}W_2 \). Thus \( \text{rank}(\partial^{-1}W_3/W_3) = \text{rank}(\partial^{-1}W_2/W_3) \), \( W_3 \) is terminal, and \( W_3 \notin \hat{\mathcal{L}}_3 \).

If \( c_4 \not\equiv 1 \) then \((c_4 - 1)^2 \equiv 1\) and since \( c_4 \not\equiv 0 \) then we know that \( c_4 = 2 \). Thus (B2) becomes \( t_{0,2} \equiv (1 - c_2)t_{2,0} \).
Hence \( x \in \partial^{-1}W_2 \) if and only if

\[
s_{2,2} \equiv c_2 t_{1,1} \quad \text{(A4)}
\]

\[
s_{1,2} \equiv (1 - c_2) t_{2,0} - c_2 s_{2,1} \quad \text{(A5)}
\]

\[
t_{0,2} \equiv (1 - c_2) t_{2,0} \quad \text{(B2)}
\]

\[
t_{1,1} \equiv (1 - c_2) t_{2,0} \quad \text{(B5)}.
\]

We regard \( t_{2,0} \) as our new free variable. Taking \( t_{2,0} \equiv 1, \ t_{0,1} = 0, \ t_{1,0} = 0, \) and \( s_{2,1} = 0, \) the matrix \( x \) becomes

\[
v_4 = \begin{bmatrix}
0 & 0 & 1 - c_2 \\
0 & 1 - c_2 & 3(1 - c_2) \\
1 & 0 & 3(c_2 - 1)
\end{bmatrix}.
\]

We see that neither \( v_4 \) nor \( 3v_4 \) is contained in \( W_2 \) and that

\( \partial^{-1}W_2 = < \partial^{-1}W_0, v_1, v_2, v_3, v_4 >. \) We know that \( v_4 \in \partial^{-1}W_2 \) and since \( 3v_4 \not\in W_2 \) then \( v_4 + W_2 \) in an element of the group \( \partial^{-1}W_2/W_2 \) whose order is larger than 3. Recall that

\( |\partial^{-1}W_2/W_2| = 3^3 \) and that \( \partial^{-1}W_1/W_2 \) has rank 2. We obtain \( \partial^{-1}W_2/W_2 \cong \mathbb{Z}_9 \times \mathbb{Z}_3. \)

Since \( \text{rank}(\partial^{-1}W_2/W_2) = \text{rank}(\partial^{-1}W_1/W_2). \) Hence \( W_2 \) is terminal and \( W_2 \not\in \hat{L}_2. \)

**Case 7.4.2.5.1.2** Suppose \( c_6 \neq 2. \) Then \( c_6 + 1 \neq 0 \) and \( (c_6 + 1)^2 \equiv 1. \) Recall \( c_2, c_4 \in \{1, 2\} \) so \( c_2c_4 \neq 0. \) Therefore (A2) becomes \( t_{0,2} \equiv (c_2c_5c_6 + c_4c_6^2 + c_2c_6^2 - c_4c_6 - c_2c_6 + c_2c_5 + c_4 + c_2)t_{2,0}. \) For convenience let \( q = c_2c_5c_6 + c_4c_6^2 + c_2c_6^2 - c_4c_6 - c_2c_6 + c_2c_5 + c_4 + c_2. \) Thus \( t_{0,2} \equiv qt_{2,0}. \) It is convenient to consider the cases \( q \equiv 0 \) and \( q \not\equiv 0 \) separately.
Recall $c_2, c_4 \in \{1, 2\}$, $c_5 \in \{0, 1, 2\}$ and $c_6 \in \{0, 1\}$. Hence there are exactly $2^3 \cdot 3 = 24$ possible combinations for the quadruplet $(c_2, c_4, c_5, c_6)$. It is tedious but straightforward to show that there are exactly 8 combinations of $(c_2, c_4, c_5, c_6$ such that $q \equiv 0 \mod 3$ if and only if $(c_2, c_4, c_5, c_6)$ is one of the following: $(1, 2, 0, 0)$, $(1, 1, 1, 0)$, $(2, 1, 0, 0)$, $(2, 2, 1, 0)$, $(1, 2, 0, 1)$, $(1, 1, 0, 1)$, $(2, 1, 0, 1)$, $(2, 2, 2, 1)$. If $q \equiv 0$ then $t_{2,0}$ is a free variable and $t_{0,2} \equiv 0$. Taking $t_{2,0} \equiv 1$, $t_{0,1} = 0$, $t_{1,0} = 0$, and $s_{2,1} = 0$, the matrix $x$ becomes

$$v_4 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & c_2 + c_4 & 3(c_2c_4 + 1) \\ 1 & 0 & 3(1 + c_2c_4) \end{bmatrix}.$$  

We see that neither $v_4$ nor $3v_4$ is contained in $W_2$ and that

$$\partial^{-1}W_2 = < \partial^{-1}W_0, v_1, v_2, v_3, v_4 >.$$  

We know that $v_4 \in \partial^{-1}W_2$ and since $3v_4 \not\in W_2$ then $v_4 + W_2$ in an element of the group $\partial^{-1}W_2/W_2$ whose order is larger than 3. Recall that $|\partial^{-1}W_2/W_2| = 3^3$ and that $\partial^{-1}W_1/W_2$ has rank 2. We obtain $\partial^{-1}W_2/W_2 \cong \mathbb{Z}_9 \times \mathbb{Z}_3$. Since rank($\partial^{-1}W_2/W_2$) = rank($\partial^{-1}W_1/W_2$). Hence $W_2$ is terminal and $W_2 \not\in \hat{L}_2$.

If $q \not\equiv 0$ then we will regard $t_{2,0}$ as our new free variable. Taking $t_{2,0} \equiv 1$, $t_{0,1} = 0$, $t_{1,0} = 0$, and $s_{2,1} = 0$, the matrix $x$ becomes

$$v_4 = \begin{bmatrix} 0 & 0 & q \\ 0 & c_2 + c_4 - c_2c_4q & 3(c_2c_4 + 1) \\ 1 & 0 & 3(1 + c_2c_4 - c_4q) \end{bmatrix}.$$  

We see that neither $v_4$ nor $3v_4$ is contained in $W_2$ and that

$$\partial^{-1}W_2 = < \partial^{-1}W_0, v_1, v_2, v_3, v_4 >.$$  

We know that $v_4 \in \partial^{-1}W_2$ and since $3v_4 \not\in W_2$ then
\( v_4 + W_2 \) in an element of the group \( \partial^{-1}W_2/W_2 \) whose order is larger than 3. Recall that \( |\partial^{-1}W_2/W_2| = 3^3 \) and that \( \partial^{-1}W_1/W_2 \) has rank 2. We obtain \( \partial^{-1}W_2/W_2 \cong \mathbb{Z}_9 \times \mathbb{Z}_3 \). Since \( \text{rank}(\partial^{-1}W_2/W_2) = \text{rank}(\partial^{-1}W_1/W_2) \). Hence \( W_2 \) is terminal and \( W_2 \not\in \hat{\mathcal{L}}_2 \).

**Case 7.4.2.5.2** Now we consider the case \( c_4 = 0 \). Then \( c_5 \neq 0 \). This implies that \( c_5 \in \{1, 2\} \) and \( c_5^2 \equiv 1 \). From (A3) and (B3) we get \( t_{2,0} \equiv 0 \) and \( t_{1,1} \equiv 0 \). Since \( t_{1,1} \equiv 0 \), (A4) becomes \( s_{2,2} \equiv 0 \) and (A1) becomes \( 0 \equiv a_4 c_5 \). Since \( c_5 \neq 0 \), then \( a_4 \equiv 0 \). (B5) becomes \( b_4 \equiv -c_2 t_{0,2} \). Substituting \( b_4 \) and \( t_{1,1} \) into (B2) we obtain

\[
0 \equiv (-c_2 c_5 + c_6) t_{0,2}
\]

The congruence (A5) becomes \( s_{1,2} \equiv -c_2 s_{2,1} \).

Hence \( x \in \partial^{-1}W_2 \) if and only if

\[
\begin{align*}
t_{1,1} &\equiv 0 \\
s_{2,2} &\equiv 0 \\
t_{2,0} &\equiv 0 \\
0 &\equiv (-c_2 c_5 + c_6) t_{0,2} \\
s_{1,2} &\equiv -c_2 s_{2,1}.
\end{align*}
\]

If \( -c_2 c_5 + c_6 \neq 0 \) then \( t_{0,2} \equiv 0 \). Then \( t_{1,1} \equiv s_{2,2} \equiv t_{2,0} \equiv t_{0,2} \equiv 0 \) and \( s_{1,2} \equiv -c_2 s_{2,1} \). These are the same congruences as that in Case 7.4 therefore \( \partial^{-1}W_2 = \partial^{-1}W_1 \). Thus \( \text{rank}(\partial^{-1}W_2/W_2) = \text{rank}(\partial^{-1}W_1/W_2) \), \( W_2 \) is terminal, and \( W_2 \not\in \hat{\mathcal{L}}_2 \).

Therefore we will assume that \( -c_2 c_5 + c_6 \equiv 0 \). Then \( c_6 \equiv c_2 c_5 \) and \( t_{0,2} \) is a free variable. We regard \( t_{1,0}, t_{0,1}, s_{2,1}, t_{0,2} \) as free variables. Taking \( t_{1,0} \equiv 1, t_{0,1} = 0, \)
\( s_{2,1} = 0 \), and \( t_{0,2} = 0 \), the matrix \( x \) becomes
\[
\begin{bmatrix}
0 & 0 & 0 \\
1 & 0 & 0 \\
0 & 0 & 0
\end{bmatrix}
\].

Taking \( t_{0,1} \equiv 1 \), \( t_{1,0} = 0 \), \( s_{2,1} = 0 \), and \( t_{0,2} = 0 \), the matrix \( x \) becomes
\[
\begin{bmatrix}
0 & 1 & 0 \\
0 & 0 & 0 \\
0 & 0 & 0
\end{bmatrix}
\].

Taking \( s_{2,1} \equiv 1 \), \( t_{0,1} = 0 \), \( t_{1,0} = 0 \), and \( t_{0,2} = 0 \), the matrix \( x \) becomes
\[
\begin{bmatrix}
0 & 0 & 0 \\
0 & 0 & -3c_2 \\
0 & 3 & 0
\end{bmatrix}
\].

Taking \( t_{0,2} \equiv 1 \), \( t_{0,1} = 0 \), \( t_{1,0} = 0 \), and \( s_{2,1} = 0 \), the matrix \( x \) becomes
\[
\begin{bmatrix}
0 & 0 & 1 \\
0 & 0 & 0 \\
0 & 0 & 0
\end{bmatrix}
\].

We see that neither \( v_4 \) nor \( 3v_4 \) is contained in \( W_2 \) and that
\( \partial^{-1}W_2 =< \partial^{-1}W_0, v_1, v_2, v_3, v_4 > \). We know that \( v_4 \in \partial^{-1}W_2 \) and since \( 3v_4 \notin W_2 \) then \( v_4 + W_2 \) is an element of the group \( \partial^{-1}W_2/W_2 \) whose order is larger than 3. Recall that \( |\partial^{-1}W_2/W_2| = 3^3 \) and that \( \partial^{-1}W_1/W_2 \) has rank 2. Since \( \partial^{-1}W_2 \) has an element of order larger than 3 then rank(\( \partial^{-1}W_2/W_2 \)) is not greater than 3 since \( \partial^{-1}W_2/W_2 \cong \mathbb{Z}_9 \times \mathbb{Z}_3 \) and therefore \( W_2 \) is terminal and \( W_2 \notin \hat{L}_2 \).
Case 7.4.2.6

We fix arbitrary values \(c_4, c_5, c_6, c_7 \in \{0, 1, 2\}\) with \((c_4, c_5) \neq (0, 0)\). There are 72 ways to choose the values \(c_4, c_5, c_6,\) and \(c_7\). Let \(m_4 = y_2 + c_4 v_2 + c_5 v_3\) and \(m_5 = v_1 + c_6 v_2 + c_7 v_3\). Thus

\[
\begin{align*}
\begin{bmatrix} 0 & c_4 & 0 \\ 0 & 3 & -3c_2c_5 \\ 0 & 3c_5 & 0 
\end{bmatrix} \quad \text{and} \quad 
\begin{bmatrix} 0 & c_6 & 0 \\ 1 & 0 & -3c_2c_7 \\ 0 & 3c_7 & 0 
\end{bmatrix},
\end{align*}
\]

Let \(W_2 =< W_1, m_4, m_5 > \in \mathcal{L}_2\). The number of subgroups \(W_2\) of this type is 72. Since \(m_4, m_5 \notin W_1\) and \(3m_4, 3m_5 \in W_1\) we have \(|W_2/W_1| = 3^2\). Since \(|W_1| = 3^6\) it follows that \(|W_2| = 3^8\). Note that \(\partial^{-1}W_1/W_2 \cong \mathbb{Z}_3 \times \mathbb{Z}_3\) and \(\text{rank}(\partial^{-1}W_1/W_2) = 2\).

We now calculate the pullback of \(\partial^{-1}W_2\). We observed in case 7.4.2 that \(\partial^{-1}W_2\) is contained in the pattern subgroup

\[
\begin{bmatrix} 2 & 2 & 2 \\ 2 & 2 & 1 \\ 2 & 1 & 1 
\end{bmatrix}.
\]

Let

\[
x = \begin{bmatrix} 3s_{0,0} + t_{0,0} & 3s_{0,1} + t_{0,1} & 3s_{0,2} + t_{0,2} \\ 3s_{1,0} + t_{1,0} & 3s_{1,1} + t_{1,1} & 3s_{1,2} \\ 3s_{2,0} + t_{2,0} & 3s_{2,1} & 3s_{2,2} \end{bmatrix} \in \begin{bmatrix} 2 & 2 & 2 \\ 2 & 2 & 1 \\ 2 & 1 & 1 \end{bmatrix}.
\]
The variables that are in play are those appearing in the matrix

\[
\begin{bmatrix}
0 & t_{0,1} & t_{0,2} \\
t_{1,0} & t_{1,1} & 3s_{1,2} \\
t_{2,0} & 3s_{2,1} & 3s_{2,2}
\end{bmatrix}
\]

In the notation of Case 7.4.2, we are taking \(d_1 = 1, d_2 = 0, d_3 = c_4, d_4 = c_5, e_2 = 1, e_3 = c_6, \) and \(e_4 = c_7.\) We wish to identify values \(a_1, a_2, a_3, a_4, a_5 \in \mathbb{Z}_9\) such that \(\partial_1 x \equiv a_1 m_1 + a_2 m_2 + a_3 m_3 + a_4 m_4 + a_5 m_5 \pmod{I}.\) We see that \(\partial_1 x \in W_2\) if and only if

\[t_{1,1} \equiv a_4 c_4 + a_5 c_6 \quad \text{(A1)}\]
\[-t_{1,1} \equiv a_4 c_5 + a_5 c_7 \quad \text{(A2)}\]
\[t_{2,0} \equiv a_5 \quad \text{(A3)}\]
\[s_{2,2} \equiv c_2 t_{1,1} \quad \text{(A4)}\]
\[s_{1,2} \equiv c_2 a_4 + t_{2,0} - c_2 s_{2,1} \quad \text{(A5)}\]

We wish to identify values \(b_1, b_2, b_3, b_4, b_5 \in \mathbb{Z}_9\) such that \(\partial_1 x \equiv b_1 m_1 + b_2 m_2 + b_3 m_3 + b_4 m_4 + b_5 m_5 \pmod{I}.\) We see that \(\partial_2 x \in W_2\) if and only if

\[t_{0,2} \equiv b_4 c_4 + b_5 c_6 \quad \text{(B1)}\]
\[s_{2,2} \equiv b_4 c_5 + b_5 c_7 \quad \text{(B2)}\]
\[t_{1,1} \equiv b_5 \quad \text{(B3)}\]
\[t_{1,1} \equiv c_2 s_{2,2} \quad \text{(B4)}\]
\[s_{1,2} \equiv -c_2 s_{2,1} + c_2 t_{0,2} + b_4 \quad \text{(B5)}\]
Multiplying (B4) by \( c_2 \) we obtain \( s_{2,2} \equiv c_2 t_{1,1} \) which is redundant with (A4).

Substituting \( a_5 \) into (A1) and (A2) we obtain \( t_{1,1} \equiv a_4 c_4 + c_6 t_{2,0} \) and \( -t_{1,1} \equiv a_4 c_5 + c_7 t_{2,0} \). Substituting \( b_5 \) into (B1) and (B2) we obtain \( t_{0,2} \equiv b_4 c_4 + c_6 t_{1,1} \) and \( s_{2,2} \equiv b_4 c_5 + c_7 t_{1,1} \). Combining (A5) and (B5) we obtain \( c_2 a_4 + t_{2,0} \equiv b_4 + c_2 t_{0,2} \) which we will denote as our new (B5).

Hence \( x \in \partial^{-1}W_2 \) if and only if

\[
\begin{align*}
t_{1,1} & \equiv a_4 c_4 + c_6 t_{2,0} \quad \text{(A1)} \\
-t_{1,1} & \equiv a_4 c_5 + c_7 t_{2,0} \quad \text{(A2)} \\
s_{2,2} & \equiv c_2 t_{1,1} \quad \text{(A4)} \\
s_{1,2} & \equiv c_2 a_4 + t_{2,0} - c_2 s_{2,1} \quad \text{(A5)} \\
t_{0,2} & \equiv b_4 c_4 + c_6 t_{1,1} \quad \text{(B1)} \\
s_{2,2} & \equiv b_4 c_5 + c_7 t_{1,1} \quad \text{(B2)} \\
c_2 a_4 + t_{2,0} & \equiv c_2 t_{0,2} + b_4 \quad \text{(B5)}.
\end{align*}
\]

It is convenient to consider the cases \( c_4 = 0 \) and \( c_4 \neq 0 \) separately.

**Case 7.4.2.6.1** Suppose \( c_4 \neq 0 \). (A1) becomes \( a_4 \equiv c_4 t_{1,1} - c_4 c_6 t_{2,0} \) and (B1) becomes \( b_4 \equiv c_4 t_{0,2} - c_4 c_6 t_{2,0} \). Now we substitute \( a_4 \) and \( b_4 \) into (A2), (A5), (B2), and (B5). We obtain \( (c_4 c_5 c_6 - c_7) t_{2,0} \equiv (c_4 c_5 + 1) t_{1,1}, s_{1,2} \equiv -c_2 s_{2,1} + c_2 c_4 t_{1,1} + (1 - c_2 c_4 c_6) t_{2,0}, (c_4 c_5 c_6 - c_7 + c_2) t_{1,1} \equiv c_4 c_5 t_{0,2}, \) and \( c_4 (c_2 + c_6) t_{1,1} \equiv (c_2 c_4 c_6 - 1) t_{2,0} + (c_2 + c_4) t_{0,2} \). Since \( c_4 \neq 0 \), then we can multiply (B2) by \( c_4 \), so (B2) becomes \( [c_5 c_6 + c_4 (c_2 - c_7)] t_{1,1} \equiv c_5 t_{0,2} \).
Hence $x \in \partial^{-1}W_2$ if and only if

$$(c_4 c_5 c_6 - c_7) t_{2,0} \equiv (c_4 c_5 + 1) t_{1,1} \quad (A2)$$

$$s_{2,2} \equiv c_2 t_{1,1} \quad (A4)$$

$$s_{1,2} \equiv -c_2 s_{2,1} + c_2 c_4 t_{1,1} + (1 - c_2 c_4 c_6) t_{2,0} \quad (A5)$$

$$[c_5 c_6 + c_4 (c_2 - c_7)] t_{1,1} \equiv c_5 t_{0,2} \quad (B2)$$

$$c_4 (c_2 + c_6) t_{1,1} \equiv (c_2 c_4 c_6 - 1) t_{2,0} + (c_2 + c_4) t_{0,2} \quad (B5).$$

**Case 7.4.2.6.1.1** Suppose $c_5 \equiv 0$. Then the congruences (A2), (B2), and (B5) become as follows: (A2) $t_{1,1} \equiv -c_7 t_{2,0}$, (B2) $(c_2 - c_7) t_{1,1} \equiv 0$, and (B5) $c_4 (c_2 + c_6) t_{1,1} \equiv (c_2 c_4 c_6 - 1) t_{2,0} + (c_2 + c_4) t_{0,2}$. Recall $c_2, c_4 \in \{1, 2\}$. We now consider cases $c_4 \equiv c_2$ and $c_4 \equiv -c_2$.

**Case 7.4.2.6.1.1** Suppose $c_4 \equiv -c_2$. Thus (B5) becomes $-(c_2 c_6 + 1) t_{1,1} \equiv -(c_6 + 1) t_{2,0}$ which says $(c_2 c_6 + 1) t_{1,1} \equiv (c_6 + 1) t_{2,0}$.

Hence $x \in \partial^{-1}W_2$ if and only if

$$t_{1,1} \equiv -c_7 t_{2,0} \quad (A2)$$

$$(c_2 - c_7) t_{1,1} \equiv 0 \quad (B2)$$

$$(c_2 c_6 + 1) t_{1,1} \equiv (c_6 + 1) t_{2,0} \quad (B5).$$
Since \( t_{0,2} \) does not appear anywhere, it is a free variable with

\[
v_4 = \begin{bmatrix}
0 & 0 & 1 \\
0 & 0 & 0 \\
0 & 0 & 0
\end{bmatrix}.
\]

We consider the cases \( c_2 \equiv c_7 \) and \( c_2 \not\equiv c_7 \) separately.

**Case 7.4.2.6.1.1.1.1** Suppose \( c_2 \equiv c_7 \). Thus (B2) is automatic and \( c_7 \not\equiv 0 \). Thus (A2) becomes \( t_{1,1} \equiv -c_2 t_{2,0} \). Use (A2) to substitute for \( t_{1,1} \) in (B5) so that (B5) becomes \((c_2 - c_6 + 1)t_{2,0} \equiv 0 \). Now we consider the cases \( c_6 \not\equiv c_2 + 1 \) and \( c_6 \equiv c_2 + 1 \) separately.

**Case 7.4.2.6.1.1.1.1** Suppose \( c_6 \not\equiv c_2 + 1 \). Thus \( c_2 - c_6 + 1 \not\equiv 0 \) and (B5) yields \( t_{2,0} \equiv 0 \). Then (A2) yields \( t_{1,1} \equiv 0 \). Hence \( t_{0,2} \) is the only new free variable and we obtain

\[
v_4 = \begin{bmatrix}
0 & 0 & 1 \\
0 & 0 & 0 \\
0 & 0 & 0
\end{bmatrix}.
\]

By Lemma 10.4.1, we see that \( v_4 + W_2 \) is an element with order greater than 3. Recall that \(|\partial^{-1}W_2/W_2| = 3^3\) and that \( \partial^{-1}W_1/W_2 \) has rank 2. We obtain \( \partial^{-1}W_2/W_2 \cong \mathbb{Z}_9 \times \mathbb{Z}_3 \). Since \( \text{rank}(\partial^{-1}W_2/W_2) = \text{rank}(\partial^{-1}W_1/W_2) \). Hence \( W_2 \) is terminal and \( W_2 \not\in \hat{L}_2 \).
Case 7.4.2.6.1.1.1.2 Suppose $c_6 \equiv c_2 + 1$. Thus $c_2 - c_6 + 1 \neq 0$ and (B5) is automatic. Hence $t_{2,0}$ is a new free variable. Taking $t_{0,2} \equiv 1$, $t_{0,1} = 0$, $t_{1,0} = 0$, $t_{2,0} = 0$, and $s_{2,1} = 0$, the matrix $x$ becomes

$$v_4 = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

Taking $t_{2,0} \equiv 1$, $t_{0,1} = 0$, $t_{1,0} = 0$, $s_{2,1} = 0$, and $t_{0,2} = 0$, the matrix $x$ becomes

$$v_5 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & -c_2 & 3(-c_2 - 1) \\ 1 & 0 & -3 \end{bmatrix}.$$

Let $r_1 = 1$, $r_2 = -c_2$, $r_3 = 0$. Then $r_1 + c_2 r_2 + r_3 = 1 + c_2(-c_2) = -1 + 1 \equiv 0$.

By Lemma 10.4.1, we see that $v_4 + W_2$ has order 9 and $v_5 + W_2$ has order 3. Hence $\partial^{-1}W_2/W_2 \cong \mathbb{Z}_9 \times \mathbb{Z}_3 \times \mathbb{Z}_3$ and $W_2$ is nonterminal. As shown earlier, since $W_2$ is nonterminal we will fix $c_8, c_9 \in \{0, 1, 2\}$ and obtain 9 subgroups $W_3$, all satisfying $|W_3| = 3^{10}$. By the Modified Terminal Lemma, we will see that each of these 9 subgroups is terminal.

Case 7.4.2.6.1.1.2 Suppose $c_2 \neq c_7$. Thus $c_2 - c_7 \neq 0$ and (B2) is equivalent to $t_{1,1} \equiv 0$. Hence (A2) is equivalent to $c_7 t_{2,0} \equiv 0$ and (B5) is equivalent to $(c_6 + 1)t_{2,0} \equiv 0$. We consider the cases $c_7 \neq 0$ or $c_6 \neq 2$ and $(c_6, c_7) \equiv (2, 0)$ separately.
Case 7.4.2.6.1.1.2.1 Suppose either \( c_7 \not\equiv 0 \) or \( c_6 \not\equiv 2 \). We show \( t_{2,0} \equiv 0 \). If \( c_7 \not\equiv 0 \) then (A2) is equivalent to \( t_{2,0} \equiv 0 \) and then (B5) is automatic. If \( c_6 \not\equiv 2 \), then (B5) is equivalent to \( t_{2,0} \equiv 0 \) and then (B2) is automatic. Thus \( t_{0,2} \) is the only new free variable and we obtain
\[
v_4 = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}.
\]

By Lemma 10.4.1, we see that \( v_4 + W_2 \) is an element with order greater than 3. Recall that \( |\partial^{-1}W_2/W_2| = 3^3 \) and that \( \partial^{-1}W_1/W_2 \) has rank 2. We obtain \( \partial^{-1}W_2/W_2 \cong \mathbb{Z}_9 \times \mathbb{Z}_3 \). Since \( \text{rank}(\partial^{-1}W_2/W_2) = \text{rank}(\partial^{-1}W_1/W_2) \). Hence \( W_2 \) is terminal and \( W_2 \not\in \hat{L}_2 \).

Case 7.4.2.6.1.1.2.2 Suppose \( c_6 \equiv 2 \) and \( c_7 \equiv 0 \). Thus (A2) and (B5) are both automatic. Hence \( t_{2,0} \) is a new free variable. Taking \( t_{0,2} \equiv 1, t_{0,1} = 0, t_{1,0} = 0, s_{2,1} = 0, \) and \( t_{2,0} = 0 \), the matrix \( x \) becomes
\[
v_4 = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}.
\]

Taking \( t_{2,0} \equiv 1, t_{0,1} = 0, t_{1,0} = 0, s_{2,1} = 0, \) and \( t_{0,2} = 0 \), the matrix \( x \) becomes
\[
v_5 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix}.
\]

Let \( r_1 = 1, r_2 = 0, r_3 = 0 \). Then \( r_1 + c_2r_2 + r_3 = 1 \). By Lemma 10.4.1, we see that \( v_4 + W_2 \) has order 9 and \( v_5 + W_2 \) has order 9. Hence \( W_2 \) is nonterminal.
As shown earlier, since $W_2$ is nonterminal we will fix $c_8, c_9 \in \{0, 1, 2\}$ and obtain 9 subgroups $W_3$, all satisfying $|W_3| = 3^{10}$. By the Modified Terminal Lemma, we will see that each of these 9 subgroups is terminal.

**Case 7.4.2.6.1.1.2** Suppose $c_4 = c_2$. Thus $c_2 + c_4 \equiv -c_2$. and (B5) becomes

$$t_{0,2} \equiv c_2(c_6 - 1)t_{2,0} - (c_2 + c_6)t_{1,1}.$$  

Hence $x \in \partial^{-1}W_2$ if and only if

$$t_{1,1} \equiv -c_7 t_{2,0} \quad \text{(A2)}$$

$$(c_2 - c_7)t_{1,1} \equiv 0 \quad \text{(B2)}$$

$$c_2(c_6 - 1)t_{2,0} - (c_2 + c_6)t_{1,1} \equiv t_{0,2} \quad \text{(B5)}.$$  

Note that we can use (A2) to substitute for $t_{1,1}$ in (B5) so that (B5) becomes

$$t_{0,2} \equiv c_2(c_6 - 1)t_{2,0} + c_7(c_2 + c_6)t_{2,0}. \quad \text{We now consider cases } c_2 = c_7 \text{ and } c_2 \neq c_7 \text{ separately.}$$

**Case 7.4.2.6.1.1.2.1** Suppose $c_7 \equiv c_2$. Thus (B2) is automatic, and since $c_2 \in \{1, 2\}$ we get $c_7 \in \{1, 2\}$. (A2) becomes $t_{1,1} \equiv -c_2 t_{2,0}$. (B5) becomes $t_{0,2} \equiv (1-c_2-c_2c_6)t_{2,0}$, which says $t_{0,2} \equiv [1 - c_2(c_6 + 1)]t_{2,0}$. Now $t_{2,0}$ is the only new free variable. Taking $t_{2,0} \equiv 1$, $t_{0,1} = 0$, $t_{1,0} = 0$, and $s_{2,1} = 0$, the matrix $x$ becomes

$$v_4 = \begin{bmatrix} 0 & 0 & 1 - c_2(c_6 + 1) \\ 0 & -c_2 & 3(-c_2 + 1 - c_6) \\ 1 & 0 & -3 \end{bmatrix}.$$
Note

\[ 3v_4 = \begin{bmatrix}
0 & 0 & 3(1 - c_2(c_6 + 1)) \\
0 & -3c_2 & 0 \\
3 & 0 & 0
\end{bmatrix}. \]

Take \( r_1 = 1, r_2 = -c_2, r_3 = 1 - c_2(c_6 + 1) \). Then \( r_1 + c_2r_2 + r_3 = 1 - c_2^2 + 1 - c_2(c_6 + 1) \). Since \( c_2^2 \equiv 1 \), we see that \( r_1 + c_2r_2 + r_3 \equiv 0 \) if and only if \( c_2(c_6 + 1) \equiv 1 \).

Recall \( c_2 \in \{1, 2\} \) and \( c_6 \in \{0, 1, 2\} \). The condition holds if and only if \((c_2, c_6)\) is \((1, 0)\) or \((2, 1)\).

Suppose \((c_2, c_6)\) is either \((1, 0)\) or \((2,1)\). Then \(v_4 + W_2\) has order 3, \( \partial^{-1}W_2/W_2 \cong \mathbb{Z}_3 \times \mathbb{Z}_3 \times \mathbb{Z}_3 \), and \(W_2\) is nonterminal. As shown earlier, since \(W_2\) is nonterminal we will fix \(c_8, c_9 \in \{0, 1, 2\}\) and obtain 9 subgroups \(W_3\), all satisfying \(|W_3| = 3^9\). By the Modified Terminal Lemma, we will see that each of these 9 subgroups is terminal.

Suppose \((c_2, c_6)\) is neither \((1,0)\) nor \((2,1)\). Then \(v_4 + W_2\) has order 9 and \(\partial^{-1}W_2/W_2 \cong \mathbb{Z}_9 \times \mathbb{Z}_3\). Therefore, \(W_2\) is terminal.

**Case 7.4.2.6.1.1.2.2** Suppose \(c_7 \neq c_2\). Thus \((B2)\) is equivalent to \(t_{1,1} \equiv 0\). Thus \((A2)\) becomes \(c_7t_{2,0} \equiv 0\) and \((B5)\) becomes \(t_{0,2} \equiv c_2(c_6 - 1)t_{2,0}\). We now consider the cases \(c_7 = 0\) and \(c_7 \neq 0\) separately.

**Case 7.4.2.6.1.1.2.2.1** Suppose \(c_7 \neq 0\). Then \((A2)\) is equivalent to \(t_{2,0} \equiv 0\) and \((B5)\) is equivalent to \(t_{0,2} \equiv 0\). Thus \(\partial^{-1}W_2 = \partial^{-1}W_1\) and \(W_2\) is terminal.
Case 7.4.2.6.1.1.2.2 Suppose \( c_7 \equiv 0 \). Thus (A2) is automatic and \( t_{2,0} \) is a free variable. Taking \( t_{2,0} \equiv 1 \), \( t_{0,1} = 0 \), \( t_{1,0} = 0 \), and \( s_{2,1} = 0 \), the matrix \( x \) becomes

\[
v_4 = \begin{bmatrix} 0 & 0 & c_2(c_6 - 1) \\ 0 & 0 & 3(1 - c_6) \\ 1 & 0 & 0 \end{bmatrix} \quad \text{and} \quad 3v_4 = \begin{bmatrix} 0 & 0 & 3c_2(c_6 - 1) \\ 0 & 0 & 0 \\ 3 & 0 & 0 \end{bmatrix}.
\]

Take \( r_1 = 1, r_2 = 0, r_3 = c_2(c_6 - 1) \), then \( r_1 + c_2r_2 + r_3 = 1 + 0 + c_2(c_6 - 1) \). This is 0 precisely when \( c_2(c_6 - 1) \equiv -1 \), which occurs if and only if \( (c_2, c_6) \) is either \( (1, 0) \) or \( (2, 2) \).

Suppose \( (c_2, c_6) \) is neither \( (1, 0) \) nor \( (2, 2) \). Then \( v_4 + W_2 \) is an element of order greater than 3 and \( \partial^{-1}W_2/W_2 \cong \mathbb{Z}_9 \times \mathbb{Z}_3 \). Therefore \( W_2 \) is terminal.

Suppose \( (c_2, c_6) \) is either \( (1, 0) \) or \( (2, 2) \). Then \( v_4 + W_2 \) is an element of order 3. Thus \( \partial^{-1}W_2/W_2 \cong \mathbb{Z}_3 \times \mathbb{Z}_3 \times \mathbb{Z}_3 \) and \( W_2 \) is nonterminal. As shown earlier, since \( W_2 \) is nonterminal we will fix \( c_8, c_9 \in \{0, 1, 2\} \) and obtain 9 subgroups \( W_3 \), all satisfying \( |W_3| = 3^9 \). By the Modified Terminal Lemma, we will see that each of these 9 subgroups is terminal.

Case 7.4.2.6.1.2 Suppose \( c_5 \neq 0 \). Thus \( c_5^2 \equiv 1 \) and (B2) becomes \( t_{0,2} \equiv [c_6 + c_4c_5(c_2 - c_7)]t_{1,1} \). Hence \( x \in \partial^{-1}W_2 \) if and only if

\[
(c_4c_5c_6 - c_7)t_{2,0} \equiv (c_4c_5 + 1)t_{1,1} \quad \text{(A2)}
\]

\[
t_{0,2} \equiv [c_6 + c_4c_5(c_2 - c_7)]t_{1,1} \quad \text{(B2)}
\]

\[
c_4(c_2 + c_6)t_{1,1} \equiv (c_2c_4c_6 - 1)t_{2,0} + (c_2 + c_4)t_{0,2} \quad \text{(B5)}.
\]
Recall we have $c_4, c_5 \in \{1, 2\}$. We now consider cases $c_4 = c_5$ and $c_5 \equiv -c_4$ separately.

**Case 7.4.2.6.1.2.1.2.1** Suppose $c_5 \equiv -c_4$. Thus $c_4c_5 \equiv -1$. (A2) becomes $(-c_6 - c_7)t_{2,0} \equiv 0$ which is equivalent to $(c_6 + c_7)t_{2,0} \equiv 0$. (B2) becomes $t_{0,2} \equiv (c_6 + c_7 - c_2)t_{1,1}$. We now consider the case $c_6 + c_7 \equiv 0$ and $c_6 + c_7 \not\equiv 0$.

**Case 7.4.2.6.1.2.1.1** Suppose $c_6 + c_7 \equiv 0$. Thus (A2) is automatic and (B2) becomes $t_{0,2} \equiv -c_2t_{1,1}$ and (B5) becomes $(c_2c_4c_6 - 1)t_{2,0} \equiv [c_4(c_2 + c_6 + c_2(c_2 + c_4))]t_{1,1}$. After simplifying, (B5) becomes $qt_{2,0} \equiv rt_{1,1}$ where $q = c_2c_4c_6 - 1$ and $r = 1 + c_4(c_6 - c_2)$. Note that condition $r \equiv 0$ holds if and only if $c_4(c_6 - c_2) \equiv -1$. We now determine ordered triples $(c_2, c_4, c_6)$ for which the condition $c_4(c_6 - c_2) \equiv -1$ holds. The necessary conditions for this are $c_2 \not\equiv c_6$ and $c_4 \not\equiv 0$. Recall that $c_2 \in \{1, 2\}$. We consider cases $c_2 = 1$ and $c_2 = 2$. Suppose $c_2 = 1$. Then consider $c_6 \in \{0, 2\}$. If $c_6 = 0$ then $c_6 - c_2 \equiv -1$ so take $c_4 = 1$. If $c_6 = 2$ then $c_6 - c_2 \equiv 1$ so take $c_4 = 2$. Suppose $c_2 = 2$. Then consider $c_6 \in \{0, 1\}$. If $c_6 = 0$ then $c_6 - c_2 \equiv 1$ so take $c_4 = 2$. If $c_6 = 1$ then $c_6 - c_2 \equiv -1$ so take $c_4 = 1$. In conclusion, we see that condition $r \equiv 0$ holds if and only if $(c_2, c_4, c_6)$ is one of the following: $(1, 1, 0), (1, 2, 2), (2, 2, 0), (2, 1, 1)$. We now consider cases $r \equiv 0$ and $r \not\equiv 0$.

**Case 7.4.2.6.1.2.1.1.1** Suppose $r \not\equiv 0$. Thus $r^2 \equiv 1$. (B5) becomes $t_{1,1} \equiv rqt_{2,0}$. Recall (B2) says $t_{0,2} \equiv -c_2t_{1,1}$. Use (B5) to substitute for $t_{1,1}$ in (B2), so (B2) becomes
\( t_{0,2} \equiv -c_2rq_{t_{2,0}} \). Hence \( t_{2,0} \) is the only new free variable and corresponds to
\[
v_4 = \begin{bmatrix}
0 & 0 & -c_2rq \\
0 & rq & 3(c_2c_4rq + 1 - c_2c_4c_6) \\
1 & 0 & 3c_2rq \\
\end{bmatrix}
\]
and
\[
3v_4 = \begin{bmatrix}
0 & 0 & -3c_2rq \\
0 & 3rq & 0 \\
3 & 0 & 0 \\
\end{bmatrix}.
\]

Take \( r_1 = 1, r_2 = rq, r_3 = -c_2rq \). Since \( r_1 + c_2r_2 + r_3 \equiv 1 + c_2rq - c_2rq \equiv 1 \neq 0 \), we see that \( v_4 + W_2 \) has order 9. Thus \( \partial^{-1}W_2 / W_2 \cong \mathbb{Z}_9 \times \mathbb{Z}_3 \) and \( W_2 \) is terminal.

**Case 7.4.2.6.1.2.1.1.2** Suppose \( r \equiv 0 \). Thus \( (B5) \) becomes \( qt_{2,0} \equiv 0 \). Recall we also have \( (B2) \) which says \( t_{0,2} \equiv -c_2t_{1,1} \). We consider the cases \( q \equiv 0 \) and \( q \not\equiv 0 \).

**Case 7.4.2.6.1.2.1.1.2.1** Suppose \( q \not\equiv 0 \). Then \( (B5) \) is equivalent to \( t_{2,0} \equiv 0 \). Hence \( t_{1,1} \) is the only new free variable. Note that since \( r \equiv 0 \) and \( q \not\equiv 0 \), we see by earlier work that \((c_2, c_4, c_6)\) is one of the following: \((1, 1, 0), (2, 2, 0), (2, 1, 1)\). Taking \( t_{1,1} \equiv 1, t_{0,1} = 0, t_{1,0} = 0, \) and \( s_{2,1} = 0 \), the matrix \( x \) becomes
\[
v_4 = \begin{bmatrix}
0 & 0 & -c_2 \\
0 & 1 & 3(c_2c_4 + 1 - c_2c_4c_6) \\
0 & 0 & 3 \\
\end{bmatrix}
\]
and
\[
3v_4 = \begin{bmatrix}
0 & 0 & -3c_2 \\
0 & 3 & 0 \\
0 & 0 & 0 \\
\end{bmatrix}.
\]

Take \( r_1 = 0, r_2 = 1, r_3 = -c_2 \). Then \( r_1 + c_2r_2 + r_3 \equiv 0 + c_2 - c_2 \equiv 0 \) and \( v_4 + W_2 \) has order 3. Therefore \( \partial^{-1}W_2 / W_2 \cong \mathbb{Z}_3 \times \mathbb{Z}_3 \times \mathbb{Z}_3 \). As shown in earlier cases, since \( W_2 \) is nonterminal we will fix \( c_8, c_9 \in \{0, 1, 2\} \) and obtain 9 subgroups \( W_3 \) for each of the 3 nonterminal subgroups \( W_2 \), all of which satisfying \( |W_3| = 3^9 \). By the Modified Terminal Lemma, we will see that each of these 27 subgroups is terminal.
Case 7.4.2.6.1.2.1.2.2 Suppose $q \equiv 0$. Thus (B5) is automatic. We only have (B2) which says $t_{0,2} \equiv -c_2 t_{1,1}$. Hence $t_{2,0}$ and $t_{1,1}$ are both free variables. Note that since $r \equiv 0$ and $q \equiv 0$, we see by earlier work that this forces $(c_2, c_4, c_6) = (1, 2, 2)$.

Taking $t_{2,0} \equiv 1$, $t_{0,1} = 0$, $t_{1,0} = 0$, $s_{2,1} = 0$, and $t_{1,1} = 0$, the matrix $x$ becomes

$$v_4 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix}.$$  

Taking $t_{1,1} \equiv 1$, $t_{0,1} = 0$, $t_{1,0} = 0$, $s_{2,1} = 0$, and $t_{2,0} = 0$, the matrix $x$ becomes

$$v_5 = \begin{bmatrix} 0 & 0 & -1 \\ 0 & 1 & -3 \\ 0 & 0 & 3 \end{bmatrix} \text{ and } 3v_5 = \begin{bmatrix} 0 & 0 & -3 \\ 0 & 3 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$  

By Lemma 10.4.1, we see that $v_4 + W_2$ has order 9 and $v_5 + W_2$ has order 3.

Hence $\partial^{-1}W_2/W_2 \cong \mathbb{Z}_9 \times \mathbb{Z}_3 \times \mathbb{Z}_3$ and $W_2$ is nonterminal. As shown in earlier work, since $W_2$ is nonterminal we will fix values $c_8, c_9 \in \{0, 1, 2\}$ and obtain 9 subgroups $W_3$, all satisfying $|W_3| = 3^9$. By the Modified Terminal Lemma, we will see that each of these 9 subgroups is terminal.

Case 7.4.2.6.1.2.2 Suppose $c_5 = c_4$. Hence $x \in \partial^{-1}W_2$ if and only if

$$t_{1,1} \equiv (c_7 - c_6)t_{2,0} \quad \text{(A2)}$$

$$t_{0,2} \equiv (c_6 - c_7 + c_2)t_{1,1} \quad \text{(B2)}$$

$$c_4(c_2 + c_6)t_{1,1} \equiv (c_2 c_4 c_6 - 1)t_{2,0} + (c_2 + c_4)t_{0,2} \quad \text{(B5)}.$$  

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Case 7.4.2.6.1.2.2.1 Suppose \(c_6 \equiv c_7\). Thus (A2) is equivalent to \(t_{1,1} \equiv 0\). Since \(t_{1,1} \equiv 0\), (B2) is equivalent to \(t_{0,2} \equiv 0\). Since \(t_{1,1} \equiv 0\) and \(t_{0,2} \equiv 0\), (B5) is equivalent to \((c_2c_4c_6 - 1)t_{2,0} \equiv 0\).

If \(c_2c_4c_6 \neq 1\) then (B5) is equivalent to \(t_{2,0} \equiv 0\), and so we get \(\partial^{-1}W_2 = \partial^{-1}W_1\) and \(W_2\) is terminal.

Now suppose \(c_2c_4c_6 \neq 1\) thus (B5) is automatic and \(t_{2,0}\) is a free variable and corresponds to

\[
v_4 = \begin{bmatrix}
0 & 0 & 0 \\
0 & 0 & 0 \\
1 & 0 & 0
\end{bmatrix}.
\]

By Lemma 10.4.1, we see that \(v_4 + W_2\) has order 9. Hence \(\partial^{-1}W_2/W_2 \cong \mathbb{Z}_9 \times \mathbb{Z}_3\) and \(W_2\) is terminal.

Case 7.4.2.6.1.2.2.2 Suppose \(c_6 \neq c_7\). Thus \((c_6 - c_7)^2 \equiv 1\). Use (A2) to substitute for \(t_{1,1}\) in (B2), so that (B2) becomes \(t_{0,2} \equiv [(c_6 - c_7) + c_2](-1)(c_6 - c_7)t_{2,0}\), which simplifies to \(t_{0,2} \equiv (-1)(c_6 - c_7)^2 + c_2(c_6 - c_7)t_{2,0}\). Since \((c_6 - c_7)^2 \equiv 1\), this becomes \(t_{0,2} \equiv (-1)[1 + c_2(c_6 - c_7)]t_{2,0}\), which becomes \(t_{0,2} \equiv [c_2(c_7 - c_6) - 1]t_{2,0}\). Now we have (A2) which says \(t_{1,1} \equiv (c_7 - c_6)t_{2,0}\) and we have (B2) which says \(t_{0,2} \equiv [c_2(c_7 - c_6) - 1]t_{2,0}\).

We can use these last two congruences to substitute for \(t_{1,1}\) and \(t_{0,2}\) in (B5), so that (B5) becomes \(qt_{2,0} \equiv 0\) where \(q = -c_2c_4c_6 + c_4c_6c_7 - c_4c_6^2 + 1 - c_7 + c_6 + c_2 + c_4\).

If \(q \neq 0\), then (B5) is equivalent to \(t_{2,0} \equiv 0\), and so (A2) and (B2) are equivalent to \(t_{1,1} \equiv 0\) and \(t_{0,2} \equiv 0\). Thus \(\partial^{-1}W_2 = \partial^{-1}W_1\) and \(W_2\) is terminal.
Now suppose $q \equiv 0$. Thus (B5) is automatic and so $t_{2,0}$ is the unique new free variable. $t_{2,0}$ corresponds to

$$v_4 = \begin{bmatrix} 0 & 0 & c_2(c_7 - c_6) - 1 \\ 0 & c_7 - c_6 & 3(c_2c_4c_7 + c_2c_4c_6 + 1) \\ 1 & 0 & 3c_2(c_7 - c_6) \end{bmatrix}$$

and $3v_4 = \begin{bmatrix} 0 & 0 & 3[c_2(c_7 - c_6) - 1] \\ 0 & 3(c_7 - c_6) & 0 \\ 3 & 0 & 0 \end{bmatrix}$.

Take $r_1 = 1, r_2 = c_7 - c_6, r_3 = c_2(c_7 - c_6) - 1$. Hence $r_1 = c_2r_2 + r_3 \equiv 1 + c_2(c_7 - c_6) + c_2(c_7 - c_6) - 1 \equiv c_2(c_6 - c_7)$. Now our assumption $c_6 \neq c_7$ means $c_6 - c_7 \neq 0$.

We are also assuming $c_2 \neq 0$. We deduce that $r_1 + c_2r_2 + r_3 \neq 0$. Hence $v_4 + W_2$ has order 9. Therefore $\partial^{-1}W_2/W_2 \cong \mathbb{Z}_9 \times \mathbb{Z}_3$ and $W_2$ is terminal.

**Case 7.4.2.6.2** Suppose $c_4 \equiv 0$. Thus $c_5 \neq 0$ and $c_5 \in \{1, 2\}$. (A1) becomes $t_{1,1} \equiv c_6 t_{2,0}$. (B1) becomes $t_{0,2} \equiv c_6 t_{1,1}$. We substitute for $t_{1,1}$ using (A1) so that (B1) becomes $t_{0,2} \equiv c_6^2 t_{2,0}$. (A2) becomes $a_4 \equiv -c_5 t_{1,1} - c_5 c_7 t_{2,0}$. (B2) becomes $b_4 \equiv c_5 s_{2,2} - c_5 c_7 t_{1,1}$. Use (A4) to substitute for $s_{2,2}$ in (B2) so that (B2) becomes $b_4 \equiv c_5(c_2 - c_7)t_{1,1}$. Use (A1) to substitute for $t_{1,1}$ in (B2) so that (B2) becomes $b_4 \equiv c_5c_6(c_2 - c_7)t_{2,0}$. Use (A2) to substitute for $a_4$ in (A5) so that (A5) becomes $s_{1,2} \equiv -c_2 s_{2,1} + [1 - c_2 c_5(c_6 + c_7)]t_{2,0}$. Now use (B1), (A2), (B2) to substitute for $t_{0,2}, a_4, b_4$ in (B5) so (B5) becomes $[1 - c_2 c_5(c_6 + c_7)]t_{2,0} \equiv c_2 c_6^2 t_{2,0} + c_5 c_6(c_2 - c_7)t_{2,0}$. Multiply (B5) by $c_2$ so that (B5) becomes $[c_2 - c_5(c_6 + c_7)]t_{2,0} \equiv c_2^2 t_{2,0} + c_5 c_6(1 - c_2 c_7)t_{2,0}$.
which becomes \( q t_{2,0} \equiv 0 \) where \( q = c_2 - c_5 c_6 - c_5 c_7 - c_6^2 - c_5 c_6 + c_2 c_5 c_6 c_7 \) so \( q \equiv c_2 - c_6^2 + c_5 c_6 - c_5 c_7 + c_2 c_5 c_6 c_7 \).

Hence \( x \in \partial^{-1} W_2 \) if and only if

\[
\begin{align*}
t_{1,1} &\equiv c_6 t_{2,0} \quad (A1) \\
s_{2,2} &\equiv c_2 t_{1,1} \quad (A4) \\
s_{1,2} &\equiv -c_2 s_{2,1} + [1 - c_2 c_5 (c_6 + c_7)] t_{2,0} \quad (A5) \\
t_{0,2} &\equiv c_6^2 t_{2,0} \quad (B1) \\
qt_{2,0} &\equiv 0 \quad (B5).
\end{align*}
\]

Note that if \( q \not\equiv 0 \) then \( t_{2,0} \equiv 0 \) and so \( t_{1,1} \equiv t_{0,2} \equiv 0 \). Hence \( \partial^{-1} W_2 = \partial^{-1} W_1 \) and \( W_2 \) is terminal.

Suppose \( q \equiv 0 \). Thus \( t_{2,0} \) is the unique new free variable for which we obtain

\[
v_4 = \begin{bmatrix} 0 & 0 & c_6^2 \\ 0 & c_6 & 3(1 - c_2 c_4 (c_6 + c_7)) \\ 1 & 0 & 3c_2 c_6 \end{bmatrix}
\quad \text{and} \quad
3v_4 = \begin{bmatrix} 0 & 0 & 3c_6^2 \\ 0 & 3c_6 & 0 \\ 3 & 0 & 0 \end{bmatrix}.
\]

Take \( r_1 = 1, r_2 = c_6, r_3 = c_6^2 \). Thus \( r_1 + c_2 r_2 + r_3 \equiv 1 + c_2 c_6 + c_6^2 \). Let \( Q = 1 + c_2 c_6 + c_6^2 \).

Recall \( c_2 \in \{1, 2\} \). Is \( c_2 = 1 \), then \( Q \equiv 1 + c_6 + c_6^2 \equiv c_6^2 - 2c_6 + 1 \equiv (c_6 - 1)^2 \). If \( c_2 = 2 \) then \( Q \equiv 1 - c_6 + c_6^2 \equiv 1 + 2c_6 + c_6^2 \equiv (c_6 + 1)^2 \). So in the case \( c_2 = 1 \) we get \( Q \equiv 0 \) if and only if \( c_6 \equiv 1 \). In the case \( c_2 = 2 \) we get \( Q \equiv 0 \) if and only if \( c_6 = 2 \). Hence we get \( Q \equiv 0 \) if and only if \( c_6 \equiv c_2 \).

Note the pair of conditions \( q \equiv 0 \) and \( c_6 \equiv c_2 \) hold if and only if \( c_2 = c_6 = 2 \) and \( c_5 = 1 \) and \( c_7 \in \{0, 1, 2\} \). For each of the three subgroups \( W_2 \) for which \( q \equiv 0 \) and
$c_2 \equiv c_6$, we have $\partial^{-1}W_2/W_2 \cong \mathbb{Z}_3 \times \mathbb{Z}_3 \times \mathbb{Z}_3$ while $\partial^{-1}W_1/W_2 \cong \mathbb{Z}_3 \times \mathbb{Z}_3$. Therefore, as shown earlier, we have exactly 9 subgroups $W_3$, all of which have $|W_3/W_2| = 3$ and all of which are terminal.

### 10.4.3 Case 7.4.3

We consider the 3-dimensional subspaces $W_2/W_1$. Let $d_1, d_2, d_3, d_4, e_1, e_2, e_3, e_4, f_1, f_2, f_3, f_4$ be unspecified variables. Let $m_4 = d_1y_2 + d_2v_1 + d_3v_2 + d_4v_3$, $m_5 = e_1y_2 + e_2v_1 + e_3v_2 + e_4v_3$, and $m_6 = f_1y_2 + f_2v_1 + f_3v_2 + f_4v_3$. In all the cases we consider the values of $e_1 = 0, f_1 = 0, f_2 = 0$ therefore we may exclude them from our expressions of $m_5$ and $m_6$. A formal expression for $m_4$ is

$$m_4 = d_1 \begin{bmatrix} 0 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 0 \end{bmatrix} + d_2 \begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} + d_3 \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} + d_4 \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & -3c_2 \\ 0 & 3 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & d_3 & 0 \\ d_2 & 3d_1 & -3c_2d_4 \\ 0 & 3d_4 & 0 \end{bmatrix}.$$

A formal expression for $m_5$ is

$$m_5 = e_2 \begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} + e_3 \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} + e_4 \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & -3c_2 \\ 0 & 3 & 0 \end{bmatrix} = \begin{bmatrix} 0 & e_3 & 0 \\ e_2 & 0 & -3c_2e_4 \\ 0 & 3e_4 & 0 \end{bmatrix}.$$
A formal expression for \( m_6 \) is

\[
m_6 = f_3 \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} + f_4 \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & -3c_2 \\ 0 & 3 & 0 \end{bmatrix} = \begin{bmatrix} 0 & f_3 & 0 \\ 0 & 0 & -3c_2 f_4 \\ 0 & 3 f_4 & 0 \end{bmatrix}.
\]

Let \( W_2 = \langle W_1, m_4, m_5, m_6 \rangle \in \mathcal{L}_2 \). We now calculate the pullback \( \partial^{-1} W_2 \).

The subgroup \( W_2 \) is contained in the pattern subgroup

\[
\begin{bmatrix} 2 & 2 & 1 \\ 2 & 1 & 1 \\ 1 & 1 & 0 \end{bmatrix}.
\]

Thus the pullback \( \partial^{-1} W_2 \) is contained in the pattern subgroup

\[
\begin{bmatrix} 2 & 2 & 2 \\ 2 & 2 & 1 \\ 2 & 1 & 1 \end{bmatrix} \cap \begin{bmatrix} 2 & 2 & 2 \\ 2 & 2 & 1 \\ 2 & 1 & 1 \end{bmatrix} = \begin{bmatrix} 2 & 2 & 2 \\ 2 & 2 & 1 \\ 2 & 1 & 1 \end{bmatrix}.
\]

Let

\[
x = \begin{bmatrix} 3s_{0,0} + t_{0,0} & 3s_{0,1} + t_{0,1} & 3s_{0,2} + t_{0,2} \\ 3s_{1,0} + t_{1,0} & 3s_{1,1} + t_{1,1} & 3s_{1,2} \\ 3s_{2,0} + t_{2,0} & 3s_{2,1} & 3s_{2,2} \end{bmatrix} \in \begin{bmatrix} 2 & 2 & 2 \\ 2 & 2 & 1 \\ 2 & 1 & 1 \end{bmatrix}.
\]

Thus

\[
\partial_1 x = \begin{bmatrix} 3s_{1,0} + t_{1,0} & 3s_{1,1} + t_{1,1} & 3s_{1,2} \\ 3s_{2,0} + t_{2,0} & 3s_{2,1} & 3s_{2,2} \\ -3(t_{1,0} + t_{2,0}) & -3t_{1,1} & 0 \end{bmatrix}
\]

and

\[
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\]
\[ \partial_2 x = \begin{bmatrix} 3s_{0,1} + t_{0,1} & 3s_{0,2} + t_{0,2} & -3(t_{0,1} + t_{0,2}) \\ 3s_{1,1} + t_{1,1} & 3s_{1,2} & -3t_{1,1} \\ 3s_{2,1} & 3s_{2,2} & 0 \end{bmatrix}. \]

The variables that are in play are those appearing in the matrix

\[
\begin{bmatrix}
0 & t_{0,1} & t_{0,2} \\
t_{1,0} & t_{1,1} & 3s_{1,2} \\
t_{2,0} & 3s_{2,1} & 3s_{2,2}
\end{bmatrix}.
\]

We wish to identify values \( a_1, a_2, a_3, a_4, a_5, a_6 \in \mathbb{Z}_9 \) such that \( \partial_1 x \equiv a_1 m_1 + a_2 m_2 + a_3 m_3 + a_4 m_4 + a_5 m_5 + a_6 m_6 \pmod{I} \). A formal expression for \( a_1 m_1 + a_2 m_2 + a_3 m_3 + a_4 m_4 + a_5 m_5 + a_6 m_6 \) is

\[
\begin{bmatrix}
1 & 0 & -3 \\
0 & 0 & 0 \\
0 & 0 & 0
\end{bmatrix} a_1 + \begin{bmatrix}
c_2 & 0 & -3c_2 \\
0 & 3 & 0 \\
-3c_2 & 0 & 0
\end{bmatrix} a_2 + \begin{bmatrix}
1 & 0 & 0 \\
0 & 0 & 0 \\
-3 & 0 & 0
\end{bmatrix} a_3 + \begin{bmatrix}
0 & d_3 & 0 \\
d_2 & 3d_1 & -3c_2d_4 \\
0 & 3d_4 & 0
\end{bmatrix} a_4 + \begin{bmatrix}
0 & e_3 & 0 \\
e_2 & 0 & -3c_2e_4 \\
0 & 3e_4 & 0
\end{bmatrix} a_5 + \begin{bmatrix}
0 & f_3 & 0 \\
f_2 & 0 & -3c_2f_4 \\
0 & 3f_4 & 0
\end{bmatrix} a_6.
\]

Comparing \((0,1)\)-entries, we get \( t_{1,1} \equiv a_4 d_3 + a_5 e_3 + a_6 f_3 \).
Comparing $(2, 1)$-entries, we get $-t_{1,1} \equiv a_4d_4 + a_5e_4 + a_6f_4$.

Comparing $(1, 0)$-entries, we get $t_{2,0} \equiv a_4d_2 + a_5e_2$.

Comparing $(1, 2)$-entries, we get $s_{2,2} \equiv -c_2(a_4d_4 + a_5e_4 + a_6f_4)$. Substituting $a_4d_4 + a_5e_4 + a_6f_4 \equiv -t_{1,1}$ we obtain $s_{2,2} \equiv c_2t_{1,1}$.

Comparing $(1, 1)$-entries, we get $a_2 \equiv s_{2,1} - a_4d_1$.

Comparing $(0, 2)$-entries, we get $s_{1,2} \equiv -(a_1 + a_2c_2)$. Substituting $a_2$ we obtain $a_1 \equiv c_2a_4d_1 - c_2s_{2,1} - s_{1,2}$.

Comparing $(2, 0)$-entries, we get $c_2a_2 + a_3 \equiv t_{1,0} + t_{2,0}$. Substituting $a_2$ we obtain $a_3 \equiv t_{1,0} + t_{2,0} - c_2s_{2,1} + c_2a_4d_1$.

Comparing $(0, 0)$-entries, we get $a_1 + a_2c_2 + a_3 \equiv t_{1,0}$. Substituting $a_1, a_2, a_3$, we obtain $s_{1,2} \equiv c_2a_4d_1 + t_{2,0} - c_2s_{2,1}$.

We see that $\partial_1 x \in W_2$ if and only if

\[
\begin{align*}
t_{1,1} & \equiv a_4d_3 + a_5e_3 + a_6f_3 \quad (A1) \\
-t_{1,1} & \equiv a_4d_4 + a_5e_4 + a_6f_4 \quad (A2) \\
t_{2,0} & \equiv a_4d_2 + a_5e_2 \quad (A3) \\
s_{2,2} & \equiv c_2t_{1,1} \quad (A4) \\
s_{1,2} & \equiv c_2a_4d_1 + t_{2,0} - c_2s_{2,1} \quad (A5).
\end{align*}
\]

We wish to identify values $b_1, b_2, b_3, b_4, b_5, b_6 \in \mathbb{Z}_9$ such that $\partial_2 x \equiv b_1m_1 + b_2m_2 + b_3m_3 + b_4m_4 + b_5m_5 + b_6m_6 \pmod{I}$. A formal expression for $b_1m_1 + b_2m_2 + b_3m_3 =$
\( b_4m_4 + b_5m_5 + b_6m_6 \) is

\[
b_1 \begin{bmatrix} 1 & 0 & -3 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} + b_2 \begin{bmatrix} c_2 & 0 & -3c_2 \\ 0 & 3 & 0 \\ -3c_2 & 0 & 0 \end{bmatrix} + b_3 \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ -3 & 0 & 0 \end{bmatrix} + b_4 \begin{bmatrix} 0 & d_3 & 0 \\ d_2 & 3d_4 & -3c_2d_4 \\ 0 & 3d_4 & 0 \end{bmatrix} + b_5 \begin{bmatrix} 0 & e_3 & 0 \\ e_2 & 0 & -3c_2e_4 \\ 0 & 3e_4 & 0 \end{bmatrix} + b_6 \begin{bmatrix} 0 & f_3 & 0 \\ 0 & 0 & -3c_2f_4 \\ 0 & 3f_4 & 0 \end{bmatrix}
\]

\[
= \begin{bmatrix} b_1 + b_2c_2 + b_3 & b_4d_3 + b_5e_3 + b_6f_3 & -3(b_1 + b_2c_2) \\ b_4d_2 + b_5e_2 & 3(b_2 + b_4d_1) & -3c_2(b_4d_4 + b_5e_4 + b_6f_4) \\ -3(c_2b_2 + b_3) & 3(b_4d_4 + b_5e_4 + b_6f_4) & 0 \end{bmatrix}
\]

Comparing \((0,1)\)-entries, we get \( t_{0,2} \equiv b_4d_3 + b_5e_3 + b_6f_3 \).

Comparing \((2,1)\)-entries, we get \( t_{s_{2,2}} \equiv b_4d_4 + b_5e_4 + b_6f_4 \).

Comparing \((1,0)\)-entries, we get \( t_{1,1} \equiv b_4d_2 + b_5e_2 \).

Comparing \((1,2)\)-entries, we get \( t_{1,1} \equiv c_2(b_4d_4 + b_5e_4 + b_6f_4) \). Substituting \( b_4d_4 + b_5e_4 + b_6f_4 \equiv s_{2,2} \) we obtain \( t_{1,1} \equiv c_2s_{2,2} \).

Comparing \((1,1)\)-entries, we get \( b_2 \equiv s_{1,2} - b_4d_1 \).

Comparing \((0,2)\)-entries, we get \( b_1 + b_2c_2 \equiv t_{0,1} + t_{0,2} \). Substituting \( b_2 \) we obtain \( b_1 \equiv t_{0,1} + t_{0,2} - c_2s_{1,2} + c_2b_4d_1 \).

Comparing \((2,0)\)-entries, we get \(-c_2b_2 - b_3 \equiv s_{2,1} \). Substituting \( b_2 \) we obtain \( b_3 \equiv -s_{2,1} - c_2s_{1,2} + c_2b_4d_1 \).
Comparing \( (0, 0) \)-entries, we get \( b_1 + b_2 c_2 + b_3 \equiv t_{0,1} \). Substituting \( b_1, b_2, b_3 \), we obtain 
\[ s_{1,2} \equiv -c_2 s_{2,1} + b_4 d_1 + c_2 t_{0,2}. \]

We see that \( \partial_2 x \in W_2 \) if and only if
\[
\begin{align*}
t_{0,2} &\equiv b_4 d_3 + b_5 e_3 + b_6 f_3 \quad \text{(B1)} \\
s_{2,2} &\equiv b_4 d_4 + b_5 e_4 + b_6 f_4 \quad \text{(B2)} \\
t_{1,1} &\equiv b_4 d_2 + b_5 e_2 \quad \text{(B3)} \\
t_{1,1} &\equiv c_2 s_{2,2} \quad \text{(B4)} \\
s_{1,2} &\equiv -c_2 s_{2,1} + b_4 d_1 + c_2 t_{0,2} \quad \text{(B5)}.
\end{align*}
\]

Case 7.4.3.1

Let \( m_4 = v_1, m_5 = v_2 \) and \( m_6 = v_3 \). Thus
\[
m_4 = \begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \quad \text{and} \quad m_5 = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \quad \text{and} \quad m_6 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}.
\]

Let \( W_2 = \langle W_1, m_4, m_5, m_6 \rangle \in L_2 \). The number of subgroups \( W_2 \) of this type is 1. Since \( m_4, m_5, m_6 \notin W_1 \) and \( 3m_4, 3m_5, 3m_6 \in W_1 \) we have \( |W_2/W_1| = 3^3 \). Since \( |W_1| = 3^6 \) it follows that \( |W_2| = 3^9 \). Note that \( \partial^{-1}W_1/W_2 \cong \mathbb{Z}_3 \) and \( \text{rank}(\partial^{-1}W_1/W_2) = 1 \). We now calculate the pullback of \( \partial^{-1}W_2 \). We observed in case 7.4.2 that \( \partial^{-1}W_2 \) is contained in the pattern subgroup
\[
\begin{bmatrix} 2 & 2 & 2 \\ 2 & 2 & 1 \\ 2 & 1 & 1 \end{bmatrix}.
\]
Let 
\[ x = \begin{bmatrix} 3s_{0,0} + t_{0,0} & 3s_{0,1} + t_{0,1} & 3s_{0,2} + t_{0,2} \\ 3s_{1,0} + t_{1,0} & 3s_{1,1} + t_{1,1} & s_{1,2} \\ 3s_{2,0} + t_{2,0} & 3s_{2,1} & s_{2,2} \end{bmatrix} \in \begin{bmatrix} 2 & 2 & 2 \\ 2 & 2 & 1 \\ 2 & 1 & 1 \end{bmatrix}. \]

The variables that are in play are those appearing in the matrix
\[
\begin{bmatrix} 0 & t_{0,1} & t_{0,2} \\ t_{1,0} & t_{1,1} & 3s_{1,2} \\ t_{2,0} & 3s_{2,1} & 3s_{2,2} \end{bmatrix}.
\]

In the notation of Case 7.4.2, we are taking \( d_1 = 0, \ d_2 = 1, \ d_3 = 0, \ d_4 = 0, \ e_2 = 0, \ e_3 = 1, \ e_4 = 0, \ f_3 = 0, \) and \( f_4 = 1. \) We wish to identify values \( a_1, a_2, a_3, a_4, a_5, a_6 \in \mathbb{Z}_9 \) such that \( \partial_1 x \equiv a_1 m_1 + a_2 m_2 + a_3 m_3 + a_4 m_4 + a_5 m_5 + a_6 m_6 \pmod{I}. \)

We see that \( \partial_1 x \in W_2 \) if and only if
\[
\begin{align*}
t_{1,1} & \equiv a_5 \quad (A1) \\
-t_{1,1} & \equiv a_6 \quad (A2) \\
t_{2,0} & \equiv a_4 \quad (A3) \\
s_{2,2} & \equiv c_2 t_{1,1} \quad (A4) \\
s_{1,2} & \equiv t_{2,0} - c_2 s_{2,1} \quad (A5). 
\end{align*}
\]
We wish to identify values \( b_1, b_2, b_3, b_4, b_5, b_6 \in \mathbb{Z}_9 \) such that \( \partial_1 x \equiv b_1 m_1 + b_2 m_2 + b_3 m_3 + b_4 m_4 + b_5 m_5 + b_6 m_6 \pmod{I} \). We see that \( \partial_2 x \in W_2 \) if and only if

\[
\begin{align*}
t_{0,2} &\equiv b_5 \quad \text{(B1)} \\
s_{2,2} &\equiv b_6 \quad \text{(B2)} \\
t_{1,1} &\equiv b_4 \quad \text{(B3)} \\
t_{1,1} &\equiv c_2 s_{2,2} \quad \text{(B4)} \\
s_{1,2} &\equiv -c_2 s_{2,1} + c_2 t_{0,2} \quad \text{(B5)}.
\end{align*}
\]

Multiplying (B4) by \( c_2 \) we obtain \( s_{2,2} \equiv c_2 t_{1,1} \) which is redundant with (A4). Combining (A5) and (B5) we obtain \( t_{0,2} \equiv c_2 t_{2,0} \) which we will denote as our new (B5).

Hence \( x \in \partial^{-1} W_2 \) if and only if

\[
\begin{align*}
s_{2,2} &\equiv c_2 t_{1,1} \quad \text{(A4)} \\
s_{1,2} &\equiv -c_2 s_{2,1} + t_{2,0} \quad \text{(A5)} \\
t_{0,2} &\equiv c_2 t_{2,0} \quad \text{(B5)}.
\end{align*}
\]

We regard \( t_{1,0}, t_{0,1}, s_{2,1}, t_{2,0}, t_{1,1} \) as free variables. Taking \( t_{1,0} \equiv 1, t_{0,1} = 0, s_{2,1} = 0, t_{2,0} = 0, \) and \( t_{1,1} = 0 \), the matrix \( x \) becomes

\[
v_1 = \begin{bmatrix}
0 & 0 & 0 \\
1 & 0 & 0 \\
0 & 0 & 0
\end{bmatrix}.
\]

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Taking \( t_{0,1} \equiv 1, t_{1,0} = 0, s_{2,1} = 0, t_{2,0} = 0, \) and \( t_{1,1} = 0, \) the matrix \( x \) becomes

\[
v_2 = \begin{bmatrix}
0 & 1 & 0 \\
0 & 0 & 0 \\
0 & 0 & 0
\end{bmatrix}
\]

Taking \( s_{2,1} \equiv 1, t_{0,1} = 0, t_{1,0} = 0, t_{2,0} = 0, \) and \( t_{1,1} = 0, \) the matrix \( x \) becomes

\[
v_3 = \begin{bmatrix}
0 & 0 & 0 \\
0 & 0 & -3c_2 \\
0 & 3 & 0
\end{bmatrix}
\]

Taking \( t_{2,0} \equiv 1, t_{0,1} = 0, t_{1,0} = 0, s_{2,1} = 0, \) and \( t_{1,1} = 0, \) the matrix \( x \) becomes

\[
v_4 = \begin{bmatrix}
0 & 0 & c_2 \\
0 & 0 & 3 \\
1 & 0 & 0
\end{bmatrix}
\]

Taking \( t_{1,1} \equiv 1, t_{0,1} = 0, t_{1,0} = 0, s_{2,1} = 0, \) and \( t_{2,0} = 0, \) the matrix \( x \) becomes

\[
v_5 = \begin{bmatrix}
0 & 0 & 0 \\
0 & 1 & 0 \\
0 & 0 & 3c_2
\end{bmatrix}
\]

We see that \( \partial^{-1}W_2 = \langle \partial^{-1}W_1, v_4, v_5 \rangle. \) Hence \(|\partial^{-1}W_2/\partial^{-1}W_1| = 3^2\). Recall that \(|\partial^{-1}W_1| = 3^{10}\) hence \(|\partial^{-1}W_2| = 3^{12}\). Since \(|W_2| = 3^9\), then \(|\partial^{-1}W_2/W_2| = 3^3\).

We obtain \( \partial^{-1}W_2/W_2 \cong \mathbb{Z}_9 \times \mathbb{Z}_3 \) or \( \mathbb{Z}_3 \times \mathbb{Z}_3 \times \mathbb{Z}_3 \). Recall \( c_2 \in \{1, 2\} \). We want to determine the order of \( v_5 + W_2 \) of \( \partial^{-1}W_2/W_2 \). We see \( 3v_5 = y_2 \not\in W_1 \) so by Lemma 10.4.2 \( 3v_5 \not\in W_1 \) and \( 3v_5 + W_2 \) has order 9. Since \( \partial^{-1}W_2/W_2 \cong \mathbb{Z}_9 \times \mathbb{Z}_3 \).
We know $\Omega_1(\partial^{-1}W_2/W_2) \cong \mathbb{Z}_3 \times \mathbb{Z}_3$ and $\Omega_1(\partial^{-1}W_2/W_2) \cong \partial^{-1}W_1/W_1$. Hence every $W_3/W_2$ has order 3 and is thus contained in $\Omega_1(\partial^{-1}W_2/W_2) \cong \mathbb{Z}_3 \times \mathbb{Z}_3$.

We want to determine the order of $v_4 + W_2$ of $\partial^{-1}W_2/W_2$. Note

$$3v_4 = \begin{bmatrix} 0 & 0 & 3c_2 \\ 0 & 0 & 0 \\ 3 & 0 & 0 \end{bmatrix}.$$ 

Taking $r_1 = 1, r_2 = 0, \text{ and } r_3 = c_2$ then $r_1 + c_2r_2 + r_3 \equiv 1 + c_2$. Recall that $c_2 \neq 0$.

We see $r_1 + c_2r_2 + r_3 \equiv 0$ if and only if $c_2 \equiv 2$. Thus if $c_2 \equiv 2$, then by Lemma 10.4.2 $3v_4 \in W_2$ and $v_4 + W_2$ has order 3. If $c_2 = 1$ then by Lemma 10.4.2 $3v_4 \not\in W_2$ and $v_4 + W_2$ has order 9. Since $\text{rank}(\partial^{-1}W_2/W_2) = 2$ and $\text{rank}(\partial^{-1}W_1/W_2) = 1$, then $\text{rank}(\partial^{-1}W_1/W_2) < \text{rank}(\partial^{-1}W_2/W_2)$. Hence $W_2$ is nonterminal and $W_2 \in \hat{L}_2$. We consider the cases $c_2 = 1$ and $c_2 = 2$ separately.

**Case 7.4.3.1.1** Suppose $c_2 = 1$. Thus $v_4 + W_2, v_5 + W_2$ both have order 9 and so $\partial^{-1}W_2/W_2 \cong \mathbb{Z}_9 \times \mathbb{Z}_3$. We want to determine the order of $v_4 + v_5 + W_2$. Note

$$3(v_4 + v_5) = \begin{bmatrix} 0 & 0 & 3 \\ 0 & 3 & 0 \\ 3 & 0 & 0 \end{bmatrix} = m_1 + m_2 + m_3.$$ 

Taking $r_1 = 1, r_2 = 1, \text{ and } r_3 = 1$ then $r_1 + c_2r_2 + c_3 \equiv 0$. Hence by Lemma 10.4.2 $3(v_4 + v_5) \in W_2$ and $v_4 + v_5 + W_2$ is an element of order 3 in $\partial^{-1}W_2/W_2$. We see $y_2 + W_2, v_4 + v_5 + W_2$ is a basis for $\Omega_1(\partial^{-1}W_2/W_2)$ while $y_2 + W_2$ is a basis for
\(\partial^{-1}W_1/W_2\). Let \(c_4 \in \{0, 1, 2\}\). Let \(m_7 = c_4y_2 + (v_4 + v_5)\). Thus
\[
m_7 = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 + 3c_4 & 3 \\ 1 & 0 & 3 \end{bmatrix}.
\]

Let \(\partial^{-1}W_2/W_3 \sim Z_3 \times Z_3\) and \(\text{rank}(\partial^{-1}W_2/W_3) = 2\). By the Modified Terminal Lemma, we conclude that \(\partial^{-1}W_3 = \partial^{-1}W_2\). Thus \(\text{rank}(\partial^{-1}W_3/W_3) = \text{rank}(\partial^{-1}W_2/W_3)\), \(W_3\) is terminal, and \(W_3 \not\in \hat{L}_3\).

**Case 7.4.3.1.2** Suppose \(c_2 = 2\). Thus \(v_4 + W_2\) has order 4 and \(v_5 + W_2\) has order 9. Hence \(y_2 + W_2, v_4 + W_2\) is a basis for the vector space \(\Omega_1(\partial^{-1}W_2/W_2)\) and that \(y_2 + W_2\) is a basis for \(\partial^{-1}W_1/W_2\). Let \(c_4 \in \{0, 1, 2\}\). Let \(m_7 = c_4y_2 + v_4\). Thus
\[
m_7 = \begin{bmatrix} 0 & 0 & 2 \\ 0 & 3c_4 & 3 \\ 1 & 0 \end{bmatrix}.
\]

Let \(\partial^{-1}W_2/W_3 \sim Z_3 \times Z_3\) and \(\text{rank}(\partial^{-1}W_2/W_3) = 2\). By the Modified Terminal Lemma, we conclude that \(\partial^{-1}W_3 = \partial^{-1}W_2\). Thus \(\text{rank}(\partial^{-1}W_3/W_3) = \text{rank}(\partial^{-1}W_2/W_3)\), \(W_3\) is terminal, and \(W_3 \not\in \hat{L}_3\).
Case 7.4.3.2

We fix arbitrary value $c_4 \in \{1, 2\}$. There are 2 ways to choose the value $c_4$. Let

$$m_4 = y_2 + c_4 v_1, m_5 = v_2, \text{ and } m_6 = v_3.$$ Thus

$$m_4 = \begin{bmatrix} 0 & 0 & 0 \\ c_4 & 3 & 0 \\ 0 & 0 & 0 \end{bmatrix} \quad \text{and} \quad m_5 = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \quad \text{and} \quad m_6 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 3 & 0 \end{bmatrix}.$$ 

Let $W_2 = \langle W_1, m_4, m_5, m_6 \rangle \in \mathcal{L}_2$. The number of subgroups $W_2$ of this type is 2. Since $m_4, m_5, m_6 \notin W_1$ and $3m_4, 3m_5, 3m_6 \in W_1$ we have $|W_2/W_1| = 3^3$. Since $|W_1| = 3^6$ it follows that $|W_2| = 3^9$. Note that $\partial^{-1}W_1/W_2 \cong \mathbb{Z}_3$ and $\ker(\partial^{-1}W_1/W_2) = 1$. We now calculate the pullback of $\partial^{-1}W_2$. We observed in case 7.4.2 that $\partial^{-1}W_2$ is contained in the pattern subgroup

$$\begin{bmatrix} 2 & 2 & 2 \\ 2 & 2 & 1 \\ 2 & 1 & 1 \end{bmatrix}.$$ 

Let

$$x = \begin{bmatrix} 3s_{0,0} + t_{0,0} & 3s_{0,1} + t_{0,1} & 3s_{0,2} + t_{0,2} \\ 3s_{1,0} + t_{1,0} & 3s_{1,1} + t_{1,1} & 3s_{1,2} \\ 3s_{2,0} + t_{2,0} & 3s_{2,1} & 3s_{2,2} \end{bmatrix} \in \begin{bmatrix} 2 & 2 & 2 \\ 2 & 2 & 1 \\ 2 & 1 & 1 \end{bmatrix}.$$ 

The variables that are in play are those appearing in the matrix

$$\begin{bmatrix} 0 & t_{0,1} & t_{0,2} \\ t_{1,0} & t_{1,1} & 3s_{1,2} \\ t_{2,0} & 3s_{2,1} & 3s_{2,2} \end{bmatrix}.$$
In the notation of Case 7.4.2, we are taking $d_1 = 1$, $d_2 = c_4$, $d_3 = 0$, $d_4 = 0$, $e_2 = 0$, $e_3 = 1$, $e_4 = 0$, $f_3 = 0$, and $f_4 = 1$. We wish to identify values $a_1, a_2, a_3, a_4, a_5, a_6 \in \mathbb{Z}_9$ such that $\partial_1 x \equiv a_1 m_1 + a_2 m_2 + a_3 m_3 + a_4 m_4 + a_5 m_5 + a_6 m_6 \pmod{I}$. We see that $\partial_1 x \in W_2$ if and only if

$$t_{1,1} \equiv a_5 \quad (A1)$$
$$-t_{1,1} \equiv a_6 \quad (A2)$$
$$t_{2,0} \equiv a_4 c_4 \quad (A3)$$
$$s_{2,2} \equiv c_2 t_{1,1} \quad (A4)$$
$$s_{1,2} \equiv c_2 a_4 + t_{2,0} - c_2 s_{2,1} \quad (A5).$$

We wish to identify values $b_1, b_2, b_3, b_4, b_5, b_6 \in \mathbb{Z}_9$ such that $\partial_1 x \equiv b_1 m_1 + b_2 m_2 + b_3 m_3 + b_4 m_4 + b_5 m_5 + b_6 m_6 \pmod{I}$. We see that $\partial_2 x \in W_2$ if and only if

$$t_{0,2} \equiv b_5 \quad (B1)$$
$$s_{2,2} \equiv b_6 \quad (B2)$$
$$t_{1,1} \equiv b_4 c_4 \quad (B3)$$
$$t_{1,1} \equiv c_2 s_{2,2} \quad (B4)$$
$$s_{1,2} \equiv -c_2 s_{2,1} + b_4 + c_2 t_{0,2} \quad (B5).$$

Multiplying (B4) by $c_2$ we obtain $s_{2,2} \equiv c_2 t_{1,1}$ which is redundant with (A4). Since $c_4 \in \{1, 2\}$, then $c_4^2 \equiv 1$. We can rewrite (A3) and (B3) as $a_4 \equiv c_4 t_{2,0}$ and $b_4 \equiv c_4 t_{1,1}$. Combining (A5) and (B5) we obtain $c_2 a_4 + t_{2,0} \equiv c_4 t_{1,1} + c_2 t_{0,2}$ which
we will denote as our new (B5). Substituting $a_4$ and $b_4$ into (B5) we obtain $t_{0,2} \equiv (c_4 + c_2)t_{2,0} - c_2c_4t_{1,1}$. (A5) becomes $s_{1,2} \equiv (c_2c_4 + 1)t_{2,0} - c_2s_{2,1}$.

Hence $x \in \partial^{-1}W_2$ if and only if

$$s_{2,2} \equiv c_2t_{1,1} \quad \text{(A4)}$$

$$s_{1,2} \equiv (c_2c_4 + 1)t_{2,0} - c_2s_{2,1} \quad \text{(A5)}$$

$$t_{0,2} \equiv (c_4 + c_2)t_{2,0} - c_2c_4t_{1,1} \quad \text{(B5)}.$$

We regard $t_{1,0}, t_{0,1}, s_{2,1}, t_{2,0}, t_{1,1}$ as free variables. Taking $t_{1,0} \equiv 1$, $t_{0,1} = 0$, $s_{2,1} = 0$, $t_{2,0} = 0$, and $t_{1,1} = 0$, the matrix $x$ becomes

$$v_1 = \begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

Taking $t_{0,1} \equiv 1$, $t_{1,0} = 0$, $s_{2,1} = 0$, $t_{2,0} = 0$, and $t_{1,1} = 0$, the matrix $x$ becomes

$$v_2 = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

Taking $s_{2,1} \equiv 1$, $t_{0,1} = 0$, $t_{1,0} = 0$, $t_{2,0} = 0$, and $t_{1,1} = 0$, the matrix $x$ becomes

$$v_3 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & -3c_2 \\ 0 & 3 & 0 \end{bmatrix}.$$
Taking \( t_{2,0} \equiv 1, t_{0,1} = 0, t_{1,0} = 0, s_{2,1} = 0, \text{ and } t_{1,1} = 0 \), the matrix \( x \) becomes

\[
v_4 = \begin{bmatrix}
0 & 0 & c_4 + c_2 \\
0 & 3(c_2 c_4 + 1) & 0 \\
1 & 0 & 0
\end{bmatrix}.
\]

Taking \( t_{1,1} \equiv 1, t_{0,1} = 0, t_{1,0} = 0, s_{2,1} = 0, \text{ and } t_{2,0} = 0 \), the matrix \( x \) becomes

\[
v_5 = \begin{bmatrix}
0 & 0 & -c_2 c_4 \\
0 & 1 & 0 \\
0 & 0 & 3c_2
\end{bmatrix}.
\]

We see that \( \partial^{-1}W_2 = < \partial^{-1}W_1, v_4, v_5 >. \) Hence \( |\partial^{-1}W_2/\partial^{-1}W_1| = 3^2 \). Recall that \( |\partial^{-1}W_1| = 3^{10} \) hence \( |\partial^{-1}W_2| = 3^{12} \). Since \( |W_2| = 3^9 \), then \( |\partial^{-1}W_2/W_2| = 3^3 \). We obtain \( \partial^{-1}W_2/W_2 \cong \mathbb{Z}_9 \times \mathbb{Z}_3 \) or \( \mathbb{Z}_3 \times \mathbb{Z}_3 \times \mathbb{Z}_3 \).

Recall \( c_2, c_4 \in \{1, 2\} \). We want to determine the order of element \( v_4 + W_2 \) of \( \partial^{-1}W_2/W_2 \). This element has order 3 if and only if \( 3v_4 \in W_2 \). Note

\[
3v_4 = \begin{bmatrix}
0 & 3(c_2 + c_4) & 0 \\
0 & 0 & 0 \\
3 & 0 & 0
\end{bmatrix}.
\]

Taking \( r_1 = 1, r_2 = 0, \) and \( r_3 = c_2 + c_4 \). Then \( r_1 + c_2 r_2 + r_3 = 1 + c_2 + c_4 \). If \( (c_2, c_4) = (1, 1) \) then \( r_1 + c_2 r_2 + r_3 \equiv 0 \) so by Lemma 10.4.2 \( 3v_4 \in W_2 \) and \( v_4 + W_2 \) has order 3. If \( (c_2, c_4) \) is either \( (1, 2) \) or \( (2, 1) \) or \( (2, 2) \) then \( r_1 + c_2 r_2 + r_3 \not\equiv 0 \) and so by Lemma 10.4.2 \( 3v_4 \not\in W_2 \) and \( v_4 + W_2 \) has order 9.
We want to determine the order of element \( v_5 + W_2 \) of \( \partial^{-1}W_2/W_2 \). This element has order 3 if and only if \( 3v_5 \in W_2 \). Note
\[
3v_5 = \begin{bmatrix}
0 & 0 & -3c_2c_4 \\
0 & 3 & 0 \\
0 & 0 & 0
\end{bmatrix}.
\]
Taking \( r_1 = 0, r_2 = 1, \) and \( r_3 = -c_2c_4, \) then \( r_1 + c_2r_2 + r_3 \equiv 0 + c_2 - c_2c_4 = c_2(1 - c_4). \)
Recalling that \( c_2 \not\equiv 0, \) we see that \( r_1 + c_2r_2 + r_3 \equiv 0 \) if and only if \( c_4 \equiv 1. \) Thus, if \( c_4 = 1 \) then by Lemma 10.4.2 \( 3v_5 \in W_2 \) and \( v_5 + W_2 \) has order 3. If \( c_4 = 2, \) then by Lemma 10.4.2 \( 3v_5 \not\in W_2 \) and \( v_5 + W_2 \) has order 9.

It is convenient to consider the following three cases separately. The first case is \( c_4 = 2. \) The second case is \((c_2, c_4) = (2, 1).\) The third case is \((c_2, c_4) = (1, 1).\)

**Case 7.4.3.2.1** Suppose \( c_4 = 2. \) Thus \( v_4 + W_2, v_5 + W_2 \) both have order 9 and so \( \partial^{-1}W_2/W_2 \cong \mathbb{Z}_9 \times \mathbb{Z}_3. \) Note
\[
3v_4 = \begin{bmatrix}
0 & 0 & 3(c_2 - 1) \\
0 & 0 & 0 \\
3 & 0 & 0
\end{bmatrix}
\quad \text{and} \quad
3v_5 = \begin{bmatrix}
0 & 0 & 3c_2 \\
0 & 3 & 0 \\
0 & 0 & 0
\end{bmatrix}.
\]
We show that element \( v_4 + v_5 + W_2 \) has order 3. To see this, note
\[
3(v_4 + v_5) = \begin{bmatrix}
0 & 0 & 3(-c_2 - 1) \\
0 & 3 & 0 \\
3 & 0 & 0
\end{bmatrix}.
\]
Taking \( r_1 = 1, r_2 = 1, \) and \( r_3 = -c_2 - 1 \) then \( r_1 + c_2 r_2 + r_3 = 1 + c_2 (1) + (-c_2 - 1) \equiv 0. \) Hence \( 3(v_4 + v_5) \in W_2, \) as desired. Note \( \mathcal{O}_1(\partial^{-1} W_2/W_2) = \partial^{-1} W_1/W_1. \) Hence every \( W_3/W_2 \) has order 3 and is thus contained in \( \Omega_1(\partial^{-1} W_2/W_2) \cong \mathbb{Z}_3 \times \mathbb{Z}_3. \) Vector space \( \Omega_1(\partial^{-1} W_2/W_2) \) has basis \( y_2 + W_2, v_4 + v_5 + W_2 \) while its subspace \( \partial^{-1} W_1/W_2 \) has basis \( y_2 + W_2. \) Let \( c_5 \in \{0, 1, 2\}. \) Let \( m_7 = c_5 y_2 + v_4 v_5. \) Note that
\[
v_4 + v_5 = \begin{bmatrix} 0 & 0 & 2 - c_2 \\ 0 & 1 & 3(1 - c_2) \\ 1 & 0 & 0 \end{bmatrix}
\] and thus \( m_7 = \begin{bmatrix} 0 & 0 & 2 - c_2 \\ 0 & 1 + 3c_5 & 3(1 - c_2) \\ 1 & 0 & 0 \end{bmatrix}. \)

Let \( W_3 = < W_2, m_7 >. \) There are 3 subgroups of this type. Since \( m_7 \not\in W_2 \) and \( 3m_7 \in W_2 \) then \( |W_3/W_2| = 3. \) Since \( |W_2| = 3^9 \) then \( |W_3| = 3^{10}. \) Then \( \partial^{-1} W_2/W_3 \cong \mathbb{Z}_3 \times \mathbb{Z}_3 \) and \( \text{rank}(\partial^{-1} W_2/W_3) = 2. \) By the Modified Terminal Lemma we conclude that \( \partial^{-1} W_3 = \partial^{-1} W_2. \) Thus \( \text{rank}(\partial^{-1} W_3/W_3) = \text{rank}(\partial^{-1} W_2/W_3), \) \( W_3 \) is terminal, and \( W_3 \not\in \hat{\mathcal{L}}_3. \)

**Case 7.4.3.2.2** Suppose \((c_2, c_4) = (2, 1). \) Thus \( v_4 + W_2 \) has order 9 while \( v_5 + W_2 \) has order 3. Also,
\[
v_4 = \begin{bmatrix} 0 & 0 & 3 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix}, \quad 3v_4 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 3 & 0 & 0 \end{bmatrix}, \quad v_5 = \begin{bmatrix} 0 & 0 & -2 \\ 0 & 1 & 0 \\ 0 & 0 & -3 \end{bmatrix}, \quad 3v_5 = \begin{bmatrix} 0 & 0 & 3 \\ 0 & 3 & 0 \\ 0 & 0 & 0 \end{bmatrix}.
\]

Note \( \partial^{-1} W_2/W_2 \cong \mathbb{Z}_9 \times \mathbb{Z}_3 \) and \( \mathcal{O}_1(\partial^{-1} W_2/W_2) = \partial^{-1} W_1/W_2. \) Note \( \Omega_1(\partial^{-1} W_2/W_2) \) is isomorphic to \( \mathbb{Z}_3 \times \mathbb{Z}_3 \) and has basis \( y_2 + W_2, v_5 + W_2 \) while its subspace \( \partial^{-1} W_1/W_2 \) has basis \( y_2 + W_2. \) Each \( W_3/W_2 \) has order 3 and is thus contained in \( \Omega_1(\partial^{-1} W_2/W_2). \)
Let $c_5 \in \{0, 1, 2\}$. Let $m_5 = c_5 y_2 + v_5$. Thus

$$m_5 = \begin{bmatrix} 0 & 0 & -2 \\ 0 & 1 + 3c_5 & 0 \\ 0 & 0 & -3 \end{bmatrix}.$$ 

Let $W_3 = \langle W_2, m_7 \rangle$. There are 3 subgroups of this type. Since $m_7 \not\in W_2$ and $3m_7 \in W_2$ then $|W_3/W_2| = 3$. Since $|W_2| = 3^9$ then $|W_3| = 3^{10}$. Then $\partial^{-1}W_2/W_3 \cong \mathbb{Z}_3 \times \mathbb{Z}_3$ and $\text{rank}(\partial^{-1}W_2/W_3) = 2$. By the Modified Terminal Lemma we conclude that $\partial^{-1}W_3 = \partial^{-1}W_2$. Thus $\text{rank}(\partial^{-1}W_3/W_3) = \text{rank}(\partial^{-1}W_2/W_3)$, $W_3$ is terminal, and $W_3 \not\in \hat{\mathcal{L}}_3$.

**Case 7.4.3.2.3** Suppose $(c_2, c_4) = (1, 1)$. Then $v_4 + W_2, v_5 + W_2$ both have order 3. Recall $|\partial^{-1}W_2/W_2| = 3^3$. We know $\partial^{-1}W_2 = \langle \partial^{-1}W_1, v_4, v_5 \rangle$ and $\partial^{-1}W_1 = \langle W_2, y_2 \rangle$. Hence the group $\partial^{-1}W_2/W_2$ of order $3^3$ is generated by its three elements $y_2 + W_2, v_4 + W_2, v_5 + W_2$ of order 3. Hence $\partial^{-1}W_2/W_2$ is isomorphic to $\mathbb{Z}_3 \times \mathbb{Z}_3 \times \mathbb{Z}_3$ and has basis $y_2 + W_2, v_4 + W_2, v_5 + W_2$ while its subspace $\partial^{-1}W_1/W_2$ has basis $y_2 + W_2$. Since $\text{rank}(\partial^{-1}W_2/W_2) = 3$ and $\text{rank}(\partial^{-1}W_1/W_2) = 1$, then $\text{rank}(\partial^{-1}W_1/W_2) < \text{rank}(\partial^{-1}W_2/W_2)$. Hence $W_2$ is nonterminal and $W_2 \in \hat{\mathcal{L}}_2$. Thus we have some $W_3/W_2$ isomorphic to $\mathbb{Z}_3$ and some $W_3/W_2$ isomorphic $\mathbb{Z}_3 \times \mathbb{Z}_3$. Therefore we have the following six cases.

In Case 7.4.3.2.3.1 we consider the 1-dimensional subspaces $W_3/W_2$. There are three possible forms for the matrix $m$. The first form is

$$m = [0, 0, 1]$$
(1 possibility), which is considered in Case 7.4.3.2.3.1.1. The second form is

\[ m = [0, 1, c_6] \quad \text{for } c_6 \in \{0, 1, 2\} \]

(3 possibilities), which is considered in Case 7.4.3.2.3.1.2. The third form is

\[ m = [1, c_6, c_7] \quad \text{for } c_6, c_7 \in \{0, 1, 2\}, \ (c_6, c_7) \neq (0, 0) \]

(8 possibilities), which is considered in Case 7.4.3.2.3.1.3.

In Case 7.4.3.2.3.2 we consider the 2-dimensional subspaces \( W_3/W_2 \). There are three possible forms for the matrix \( m \). The first form is

\[
m = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}
\]

(1 possibility), which is considered in Case 7.4.3.2.3.2.1. The second form is

\[
m = \begin{bmatrix} 1 & c_6 & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad \text{for } c_6 \in \{1, 2\}
\]

(2 possibilities), which is considered in Case 7.4.3.2.3.2.2. The third form is

\[
m = \begin{bmatrix} 1 & 0 & c_6 \\ 0 & 1 & c_7 \end{bmatrix} \quad \text{for } c_6 \in \{1, 2\} \quad \text{and} \quad c_7 \in \{0, 1, 2\}
\]

(6 possibilities), which is considered in Case 7.4.3.2.3.2.3.

**Case 7.4.3.2.3.1.1** Let \( m_7 = v_5 \). Thus

\[
m_7 = \begin{bmatrix} 0 & 0 & -2 \\ 0 & 1 & 0 \\ 0 & 0 & -3 \end{bmatrix}
\]
Let $W_3 = \langle W_2, m_7 \rangle \in \mathcal{L}_3$. The number of subgroups $W_2$ of this type is 1. Since $m_7 \notin W_2$ and $3m_7 \in W_2$ we have $|W_3/W_2| = 3$. Since $|W_2| = 3^9$ it follows that $|W_3| = 3^{10}$. By the Modified Terminal Lemma we conclude that $\partial^{-1}W_3 = \partial^{-1}W_2$. Thus $\text{rank}(\partial^{-1}W_3/W_3) = \text{rank}(\partial^{-1}W_2/W_3)$, $W_3$ is terminal, and $W_3 \notin \mathcal{L}_3$.

**Case 7.4.3.2.3.1.2** We fix arbitrary value $c_6 \in \{0, 1, 2\}$. Let $m_7 = v_4 + c_6v_5$. Thus

\[
m_7 = \begin{bmatrix}
0 & 0 & 3 - 2c_6 \\
0 & c_6 & 0 \\
1 & 0 & -3c_6
\end{bmatrix}.
\]

Let $W_3 = \langle W_2, m_7 \rangle \in \mathcal{L}_3$. The number of subgroups $W_2$ of this type is 3. Since $m_7 \notin W_2$ and $3m_7 \in W_2$ we have $|W_3/W_2| = 3$. Since $|W_2| = 3^9$ it follows that $|W_3| = 3^{10}$. By the Modified Terminal Lemma we conclude that $\partial^{-1}W_3 = \partial^{-1}W_2$. Thus $\text{rank}(\partial^{-1}W_3/W_3) = \text{rank}(\partial^{-1}W_2/W_3)$, $W_3$ is terminal, and $W_3 \notin \mathcal{L}_3$.

**Case 7.4.3.2.3.1.3** We fix arbitrary values $c_6, c_7 \in \{0, 1, 2\}$ such that $(c_6, c_7) \neq (0, 0)$.

Let $m_7 = y_2 + c_6v_4 + c_7v_5$. Thus

\[
m_7 = \begin{bmatrix}
0 & 0 & 3c_6 - 2c_7 \\
0 & 3 + c_7 & 0 \\
c_6 & 0 & -3c_7
\end{bmatrix}.
\]

Let $W_3 = \langle W_2, m_7 \rangle \in \mathcal{L}_3$. The number of subgroups $W_2$ of this type is 8. Since $m_7 \notin W_2$ and $3m_7 \in W_2$ we have $|W_3/W_2| = 3$. Since $|W_2| = 3^9$ it then follows that $|W_3| = 3^{10}$. Therefore by the Modified Terminal Lemma we conclude
that $\partial^{-1}W_3 = \partial^{-1}W_2$. Thus $\text{rank}(\partial^{-1}W_3/W_3) = \text{rank}(\partial^{-1}W_2/W_3)$, $W_3$ is terminal, and $W_3 \not\in \hat{\mathcal{L}}_3$.

**Case 7.4.3.2.3.2.1** Let $m_7 = v_4$ and $m_8 = v_5$. Thus

$$m_7 = \begin{bmatrix} 0 & 0 & 3 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix} \quad \text{and} \quad m_8 = \begin{bmatrix} 0 & 0 & -2 \\ 0 & 1 & 0 \\ 0 & 0 & -3 \end{bmatrix}.$$  

Let $W_3 =< W_2, m_7, m_8 > \in \mathcal{L}_3$. The number of subgroups $W_2$ of this type is 1. Since $m_7 \not\in W_2$, $m_8 \not\in W_2$, $3m_7 \in W_2$, and $3m_8 \in W_2$ we have $|W_3/W_2| = 3^2$. Since $|W_2| = 3^9$ it follows that $|W_3| = 3^{11}$. By the Modified Terminal Lemma we conclude that $\partial^{-1}W_3 = \partial^{-1}W_2$. Thus $\text{rank}(\partial^{-1}W_3/W_3) = \text{rank}(\partial^{-1}W_2/W_3)$, $W_3$ is terminal, and $W_3 \not\in \hat{\mathcal{L}}_3$.

**Case 7.4.3.2.3.2.2** We fix arbitrary value $c_6 \in \{1, 2\}$. Let $m_7 = y_2 + c_6 v_4$ and $m_8 = v_5$. Thus

$$m_7 = \begin{bmatrix} 0 & 0 & 3c_6 \\ 0 & 3 & 0 \\ c_6 & 0 & 0 \end{bmatrix} \quad \text{and} \quad m_8 = \begin{bmatrix} 0 & 0 & -2 \\ 0 & 1 & 0 \\ 0 & 0 & -3 \end{bmatrix}.$$  

Let $W_3 =< W_2, m_7, m_8 > \in \mathcal{L}_3$. The number of subgroups $W_2$ of this type is 2. Since $m_7, m_8 \not\in W_2$ and $3m_7, 3m_8 \in W_2$ we have $|W_3/W_2| = 3^2$. Since $|W_2| = 3^9$ then $|W_3| = 3^{11}$. By the Modified Terminal Lemma we conclude that $\partial^{-1}W_3 = \partial^{-1}W_2$. Thus $\text{rank}(\partial^{-1}W_3/W_3) = \text{rank}(\partial^{-1}W_2/W_3)$, $W_3$ is terminal, and $W_3 \not\in \hat{\mathcal{L}}_3$.  

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Case 7.4.3.2.3 We fix arbitrary values $c_6 \in \{1, 2\}$ and $c_7 \in \{0, 1, 2\}$. Let

$m_7 = y_2 + c_6 v_4$ and $m_8 = y_2 + c_7 v_5$. Thus

$$m_7 = \begin{bmatrix} 0 & 0 & 3c_6 \\ 0 & 3 & 0 \\ c_6 & 0 & 0 \end{bmatrix} \quad \text{and} \quad m_8 = \begin{bmatrix} 0 & 0 & -2c_7 \\ 0 & 3 + c_7 & 0 \\ 0 & 0 & -3c_7 \end{bmatrix}.$$  

Let $W_3 = \langle W_2, m_7, m_8 \rangle \in L_3$. The number of subgroups $W_2$ of this type is 6. Since $m_7 \notin W_2$, $m_8 \notin W_2$, $3m_7 \in W_2$, and $3m_8 \in W_2$ we have $|W_3/W_2| = 3^2$. Since $|W_2| = 3^9$ it follows that $|W_3| = 3^{11}$. By the Modified Terminal Lemma we conclude that $\partial^{-1}W_3 = \partial^{-1}W_2$. Thus $\text{rank}(\partial^{-1}W_3/W_3) = \text{rank}(\partial^{-1}W_2/W_3)$, $W_3$ is terminal, and $W_3 \notin \hat{L}_3$.

Case 7.4.3.3

We fix arbitrary values $c_4 \in \{1, 2\}$ and $c_5 \in \{0, 1, 2\}$. There are 6 ways to choose the values $c_4, c_5$. Let $m_4 = y_2 + c_4 v_2$, $m_5 = v_1 + c_5 v_2$, and $m_6 = v_3$. Thus

$$m_4 = \begin{bmatrix} 0 & c_4 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 0 \end{bmatrix} \quad \text{and} \quad m_5 = \begin{bmatrix} 0 & c_5 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \quad \text{and} \quad m_6 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & -3c_2 \\ 0 & 3 & 0 \end{bmatrix}.$$  

Let $W_2 = \langle W_1, m_4, m_5, m_6 \rangle \in L_2$. The number of subgroups $W_2$ of this type is 6. Since $m_4, m_5, m_6 \notin W_1$ and $3m_4, 3m_5, 3m_6 \in W_1$ we have $|W_2/W_1| = 3^3$. Since $|W_1| = 3^6$ it follows that $|W_2| = 3^9$. Note that $\partial^{-1}W_1/W_2 \cong \mathbb{Z}_3$ and $\text{rank}(\partial^{-1}W_1/W_2) = 1$. We now calculate the pullback of $\partial^{-1}W_2$. We observed in case
7.4.2 that $\partial^{-1}W_2$ is contained in the pattern subgroup

$$\begin{bmatrix} 2 & 2 & 2 \\ 2 & 2 & 1 \\ 2 & 1 & 1 \end{bmatrix}.$$

Let

$$x = \begin{bmatrix} 3s_{0,0} + t_{0,0} & 3s_{0,1} + t_{0,1} & 3s_{0,2} + t_{0,2} \\ 3s_{1,0} + t_{1,0} & 3s_{1,1} + t_{1,1} & 3s_{1,2} \\ 3s_{2,0} + t_{2,0} & 3s_{2,1} & 3s_{2,2} \end{bmatrix} \in \begin{bmatrix} 2 & 2 & 2 \\ 2 & 2 & 1 \end{bmatrix}.$$

The variables that are in play are those appearing in the matrix

$$\begin{bmatrix} 0 & t_{0,1} & t_{0,2} \\ t_{1,0} & t_{1,1} & 3s_{1,2} \\ t_{2,0} & 3s_{2,1} & 3s_{2,2} \end{bmatrix}.$$

In the notation of Case 7.4.2, we are taking $d_1 = 1$, $d_2 = 0$, $d_3 = c_4$, $d_4 = 0$, $e_2 = 1$, $e_3 = c_5$, $e_4 = 0$, $f_3 = 0$, and $f_4 = 1$. We wish to identify values $a_1, a_2, a_3, a_4, a_5, a_6 \in \mathbb{Z}_9$ such that $\partial_1 x \equiv a_1 m_1 + a_2 m_2 + a_3 m_3 + a_4 m_4 + a_5 m_5 + a_6 m_6 \pmod{I}$. We see that $\partial_1 x \in W_2$ if and only if

$$t_{1,1} \equiv a_4 c_4 + a_5 c_5 \quad (A1)$$

$$-t_{1,1} \equiv a_6 \quad (A2)$$

$$t_{2,0} \equiv a_5 \quad (A3)$$

$$s_{2,2} \equiv c_2 t_{1,1} \quad (A4)$$

$$s_{1,2} \equiv c_2 a_4 + t_{2,0} - c_2 s_{2,1} \quad (A5).$$
We wish to identify values \(b_1, b_2, b_3, b_4, b_5, b_6 \in \mathbb{Z}_9\) such that \(\partial_1 x \equiv b_1 m_1 + b_2 m_2 + b_3 m_3 + b_4 m_4 + b_5 m_5 + b_6 m_6 \pmod{I}\). We see that \(\partial_2 x \in W_2\) if and only if

\[
\begin{align*}
t_{0,2} &\equiv b_4 c_4 + b_5 c_5 & (B1) \\
s_{2,2} &\equiv b_6 & (B2) \\
t_{1,1} &\equiv b_5 & (B3) \\
t_{1,1} &\equiv c_2 s_{2,2} & (B4) \\
s_{1,2} &\equiv -c_2 s_{2,1} + b_4 + c_2 t_{0,2} & (B5).
\end{align*}
\]

Multiplying (B4) by \(c_2\) we obtain \(s_{2,2} \equiv c_2 t_{1,1}\) which is redundant with (A4). Substituting \(a_5\) into (A1) we obtain \(a_4 \equiv c_4 t_{1,1} - c_4 c_5 t_{2,0}\). Substituting \(b_5\) into (B1) we obtain \(b_4 \equiv c_4 t_{0,2} - c_4 c_5 t_{1,1}\). Combining (A5) and (B5) we obtain \(c_2 a_4 + t_{2,0} \equiv b_4 + c_2 t_{0,2}\) which we will denote as our new (B5). Substituting \(a_4\) and \(b_4\) into (B5) we obtain

\[
(c_2 c_4 + c_4 c_5) t_{1,1} + (1 - c_2 c_4 c_5 t_{2,0}) \equiv (c_4 + c_2) t_{0,2}.
\]

(A5) becomes

\[
s_{1,2} \equiv c_2 c_4 t_{1,1} + (-c_2 c_4 c_5 + 1) t_{2,0} - c_2 s_{2,1}.
\]

Hence \(x \in \partial^{-1}W_2\) if and only if

\[
\begin{align*}
s_{2,2} &\equiv c_2 t_{1,1} & (A4) \\
s_{1,2} &\equiv c_2 c_4 t_{1,1} + (-c_2 c_4 c_5 + 1) t_{2,0} - c_2 s_{2,1} & (A5) \\
(c_2 c_4 + c_4 c_5) t_{1,1} + (1 - c_2 c_4 c_5) t_{2,0} &\equiv (c_4 + c_2) t_{0,2} & (B5).
\end{align*}
\]

It is convenient to consider the cases \(c_2 = c_4\) and \(c_2 \neq c_4\) separately.
Case 7.4.3.3.1 Suppose $c_2 = c_4$. Thus $c_2 + c_4 = 2c_2 \equiv -c_2 \not\equiv 0$. (B5) becomes

$$c_2(c_2 + c_5)t_{1,1} + (1 - c_5)t_{2,0} \equiv -c_2t_{0,2}.$$ 

Multiply by $-c_2$ to get $t_{0,2} \equiv -(c_2 + c_5)t_{1,1} + c_2(c_5 - 1)t_{2,0}$. Recall $c_2^2 \equiv 1$. Hence $x \in \partial^{-1}W_2$ if and only if

$$s_{2,2} \equiv c_2 t_{1,1} \quad \text{(A4)}$$

$$s_{1,2} \equiv -c_2 s_{2,2} + t_{1,1} + (1 - c_5)t_{2,0} \quad \text{(A5)}$$

$$t_{0,2} \equiv -(c_2 + c_5)t_{1,1} + c_2(c_5 - 1)t_{2,0} \quad \text{(B5)}.$$ 

We regard $t_{1,1}$ and $t_{2,0}$ as our new free variables. Taking $t_{1,1} \equiv 1$ and $t_{2,0} = 0$, the matrix $x$ becomes

$$v_4 = \begin{bmatrix} 0 & 0 & -(c_2 + c_5) \\ 0 & 1 & 3 \\ 0 & 0 & 3c_2 \end{bmatrix}.$$ 

Taking $t_{2,0} \equiv 1$ and $t_{1,1} = 0$, the matrix $x$ becomes

$$v_5 = \begin{bmatrix} 0 & 0 & c_2(c_5 - 1) \\ 0 & 0 & 3(1 - c_5) \\ 1 & 0 & 0 \end{bmatrix}.$$ 

Recall $c_2 \in \{1, 2\}, c_5 \in \{0, 1, 2\}$. We want to determine the order of $v_4 + W_2$. This elements has order 3 if and only if $3v_4 \in W_2$. Note

$$3v_4 = \begin{bmatrix} 0 & 0 & 3(-c_2 - c_5) \\ 0 & 3 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$ 

Taking $r_1 = 0, r_2 = 1, r_3 = -c_2 - c_5$. Thus $r_1 + c_2 r_2 + r_3 = 0 + c_2 + (-c_2 - c_5) = -c_5$. Thus $r_1 + c_2 r_2 + r_3 \equiv 0$ if and only if $c_5 = 0$. If $c_5 = 0$ then by Lemma 10.4.2 $3v_4 \in W_2$.
and \(v_4 + W_2\) has order 3. If \(c_5 \in \{1, 2\}\) then by Lemma 10.4.2 \(3v_4 \not\in W_2\) and \(v_4 + W_2\) has order 9.

We want to determine the order of \(v_5 + W_2\). This element has order 3 if and only if \(3v_5 \in W_2\). Note

\[
3v_5 = \begin{bmatrix}
0 & 0 & 3c_2(c_5 - 1) \\
0 & 0 & 0 \\
0 & 0 & 0
\end{bmatrix}.
\]

Take \(r_1 = 1, r_2 = 0, \) and \(r_3 = c_2(c_5 - 1)\). Thus \(r_1 + c_2r_2 + r_3 = c_2c_5 - c_2 + 1\). We see that \(r_1 + c_2r_2 + r_3 \equiv 0\) if and only if \((c_2, c_5)\) is either \((1, 0)\) or \((2, 2)\). If \((c_2, c_5)\) is either \((1, 0)\) or \((2, 2)\) then by Lemma 10.4.2 \(3v_5 \in W_2\) and \(v_5 + W_2\) has order 3. If \((c_2, c_5)\) is neither \((1, 0)\) or \((2, 2)\) then by Lemma 10.4.2 \(3v_5 + W_2\) has order 9. It is convenient to consider the following four cases separately. The first case is when \((c_2, c_5) = (1, 0)\).

The second case is when \((c_2, c_5) = (2, 2)\). The third case is when \((c_2, c_5) = (2, 0)\). The fourth case is when \((c_2, c_5) = (1, 1)\) or \((c_2, c_5) = (1, 2)\), or \((c_2, c_5) = (2, 1)\).

**Case 7.4.3.3.1.1** Suppose \((c_2, c_5) = (1, 0)\). Then \(v_4 + W_2\) and \(v_5 + W_2\) both have order 3. Hence the group \(\partial^{-1}W_2/W_2\) is generated by its three elements \(y_2 + W_2, v_4 + W_2, v_5 + W_2\) of order 3. Hence \(\partial^{-1}W_2/W_2 \cong \mathbb{Z}_3 \times \mathbb{Z}_3 \times \mathbb{Z}_3\) has basis \(y_2 + W_2, v_4 + W_2, v_5 + W_2\) while its subspace \(\partial^{-1}W_1/W_2\) has basis \(y_2 + W_2\). Since \(\text{rank}(\partial^{-1}W_2/W_2) = 3\) and \(\text{rank}(\partial^{-1}W_1/W_2) = 1\), then \(\text{rank}(\partial^{-1}W_1/W_2) < \text{rank}(\partial^{-1}W_2/W_2)\). Hence \(W_2\) is nonterminal and \(W_2 \in \hat{\mathcal{L}}_2\). Thus we have some \(W_3/W_2\) isomorphic to \(\mathbb{Z}_3\) and some \(W_3/W_2\) isomorphic \(\mathbb{Z}_3 \times \mathbb{Z}_3\).
In Case 7.4.3.3.1.1 we consider the 1-dimensional subspaces $W_3/W_2$. There are three possible forms for the matrix $m$. The first form is

$$m = [0, 0, 1]$$

(1 possibility), which is considered in Case 7.4.3.3.1.1.1. The second form is

$$m = [0, 1, c_6] \quad \text{for} \quad c_6 \in \{0, 1, 2\}$$

(3 possibilities), which is considered in Case 7.4.3.3.1.1.2. The third form is

$$m = [1, c_6, c_7] \quad \text{for} \quad c_6, c_7 \in \{0, 1, 2\}, \ (c_6, c_7) \neq (0, 0)$$

(8 possibilities), which is considered in Case 7.4.3.3.1.1.3.

In Case 7.4.3.3.1.2 we consider the 2-dimensional subspaces $W_3/W_2$. There are three possible forms for the matrix $m$. The first form is

$$m = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

(1 possibility), which is considered in Case 7.4.3.3.1.2.1. The second form is

$$m = \begin{bmatrix} 1 & c_6 & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad \text{for} \quad c_6 \in \{1, 2\}$$

(2 possibilities), which is considered in Case 7.4.3.3.1.2.2. The third form is

$$m = \begin{bmatrix} 1 & 0 & c_6 \\ 0 & 1 & c_7 \end{bmatrix} \quad \text{for} \quad c_6 \in \{1, 2\} \quad \text{and} \quad c_7 \in \{0, 1, 2\}$$

(6 possibilities), which is considered in Case 7.4.3.3.1.2.3.
Case 7.4.3.3.1.1.1 Let $m_7 = v_5$. Thus

$$m_7 = \begin{bmatrix}
0 & 0 & c_2(c_5 - 1) \\
0 & 0 & 3(1 - c_5) \\
1 & 0 & 0
\end{bmatrix}. $$

Let $W_3 = < W_2, m_7 > \in \mathcal{L}_3$. The number of subgroups $W_2$ of this type is 1.

Since $m_7 \notin W_2$ and $3m_7 \in W_2$ we have $|W_3/W_2| = 3$. Since $|W_2| = 3^9$ it follows that $|W_3| = 3^{10}$. By the Modified Terminal Lemma we conclude that $\partial^{-1}W_3 = \partial^{-1}W_2$.

Thus $\text{rank}(\partial^{-1}W_3/W_3) = \text{rank}(\partial^{-1}W_2/W_3)$, $W_3$ is terminal, and $W_3 \notin \mathcal{L}_3$.

Case 7.4.3.3.1.1.1.2 We fix arbitrary value $c_6 \in \{0, 1, 2\}$. Let $m_7 = v_4 + c_6 v_5$. Thus

$$m_7 = \begin{bmatrix}
0 & 0 & c_2 c_6 (c_5 - 1) - (c_2 + c_5) \\
0 & 1 & 3(c_6 - c_5 c_6 + 1) \\
c_6 & 0 & 3c_2
\end{bmatrix}. $$

Let $W_3 = < W_2, m_7 > \in \mathcal{L}_3$. The number of subgroups $W_2$ of this type is 3.

Since $m_7 \notin W_2$ and $3m_7 \in W_2$ we have $|W_3/W_2| = 3$. Since $|W_2| = 3^9$ it follows that $|W_3| = 3^{10}$. By the Modified Terminal Lemma we conclude that $\partial^{-1}W_3 = \partial^{-1}W_2$.

Thus $\text{rank}(\partial^{-1}W_3/W_3) = \text{rank}(\partial^{-1}W_2/W_3)$, $W_3$ is terminal, and $W_3 \notin \mathcal{L}_3$.

Case 7.4.3.3.1.1.1.3 We fix arbitrary values $c_6, c_7 \in \{0, 1, 2\}$ such that $(c_6, c_7) \neq (0, 0)$. Let $m_7 = y_2 + c_6 v_4 + c_7 v_5$. 

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Thus

\[
m_7 = \begin{bmatrix}
0 & 0 & -c_6(c_2 + c_5) - c_2c_7c_5 - 1 \\
0 & 3 + c_6 & 3c_6 + 3c_7(1 - c_5) \\
c_7 & 0 & 3c_2c_6
\end{bmatrix}.
\]

Let \( W_3 = \langle W_2, m_7 \rangle \in \mathcal{L}_3 \). The number of subgroups \( W_2 \) of this type is 8. Since \( m_7 \notin W_2 \) and \( 3m_7 \in W_2 \) we have \( |W_3/W_2| = 3 \). Since \( |W_2| = 3^9 \) it follows that \( |W_3| = 3^{10} \). By the Modified Terminal Lemma we conclude that \( \partial^{-1}W_3 = \partial^{-1}W_2 \).

Thus \( \text{rank}(\partial^{-1}W_3/W_3) = \text{rank}(\partial^{-1}W_2/W_3) \), \( W_3 \) is terminal, and \( W_3 \notin \hat{\mathcal{L}}_3 \).

**Case 7.4.3.3.1.1.2.1** Let \( m_7 = v_4 \) and \( m_8 = v_5 \). Thus

\[
m_7 = \begin{bmatrix}
0 & 0 & -(c_2 + c_5) \\
0 & 1 & 3 \\
0 & 0 & 3c_2
\end{bmatrix} \quad \text{and} \quad m_8 = \begin{bmatrix}
0 & 0 & c_2(c_5 - 1) \\
0 & 0 & 3(1 - c_5) \\
1 & 0 & 0
\end{bmatrix}.
\]

Let \( W_3 = \langle W_2, m_7, m_8 \rangle \in \mathcal{L}_3 \). The number of subgroups \( W_2 \) of this type is 1. Since \( m_7 \notin W_2 \), \( m_8 \notin W_2 \), \( 3m_7 \in W_2 \), and \( 3m_8 \in W_2 \) we have \( |W_3/W_2| = 3^2 \). Since \( |W_2| = 3^9 \) it follows that \( |W_3| = 3^{11} \). By the Modified Terminal Lemma we conclude that \( \partial^{-1}W_3 = \partial^{-1}W_2 \). Thus \( \text{rank}(\partial^{-1}W_3/W_3) = \text{rank}(\partial^{-1}W_2/W_3) \), \( W_3 \) is terminal, and \( W_3 \notin \hat{\mathcal{L}}_3 \).

**Case 7.4.3.3.1.1.2.2** We fix arbitrary value \( c_6 \in \{1, 2\} \). Let \( m_7 = y_2 + c_6v_4 \) and \( m_8 = v_5 \).
Thus

\[ m_7 = \begin{bmatrix}
0 & 0 & -c_6(c_2 + c_5) \\
0 & 3 + c_6 & 3c_6 \\
0 & 0 & 3c_2c_6
\end{bmatrix} \quad \text{and} \quad m_8 = \begin{bmatrix}
0 & 0 & 2(c_5 - 1) \\
0 & 0 & 3(1 - c_5) \\
1 & 0 & 0
\end{bmatrix}. \]

Let \( W_3 = \langle W_2, m_7, m_8 \rangle \in \mathcal{L}_3 \). The number of subgroups \( W_2 \) of this type is 2. Since \( m_7 \notin W_2, m_8 \notin W_2, 3m_7 \in W_2, \) and \( 3m_8 \in W_2 \) we have \( |W_3/W_2| = 3^2 \). Since \( |W_2| = 3^9 \) it follows that \( |W_3| = 3^{11} \). By the Modified Terminal Lemma we conclude that \( \partial^{-1}W_3 = \partial^{-1}W_2 \). Thus \( \text{rank}(\partial^{-1}W_3/W_3) = \text{rank}(\partial^{-1}W_2/W_3) \), \( W_3 \) is terminal, and \( W_3 \notin \mathcal{L}_3 \).

**Case 7.4.3.3.1.1.2.3** We fix arbitrary values \( c_6 \in \{1, 2\} \) and \( c_7 \in \{0, 1, 2\} \). Let \( m_7 = y_2 + c_6v_4 \) and \( m_8 = y_2 + c_7v_5 \). Thus

\[ m_7 = \begin{bmatrix}
0 & 0 & -c_6(c_2 + c_5) \\
0 & 3 + c_6 & 3c_6 \\
0 & 0 & 3c_2c_6
\end{bmatrix} \quad \text{and} \quad m_8 = \begin{bmatrix}
0 & 0 & 2c_7(c_5 - 1) \\
0 & 0 & 3c_7(1 - c_5) \\
c_7 & 0 & 0
\end{bmatrix}. \]

Let \( W_3 = \langle W_2, m_7, m_8 \rangle \in \mathcal{L}_3 \). The number of subgroups \( W_2 \) of this type is 6. Since \( m_7 \notin W_2, m_8 \notin W_2, 3m_7 \in W_2, \) and \( 3m_8 \in W_2 \) we have \( |W_3/W_2| = 3^2 \). Since \( |W_2| = 3^9 \) it follows that \( |W_3| = 3^{11} \). By the Modified Terminal Lemma we conclude that \( \partial^{-1}W_3 = \partial^{-1}W_2 \). Thus \( \text{rank}(\partial^{-1}W_3/W_3) = \text{rank}(\partial^{-1}W_2/W_3) \), \( W_3 \) is terminal, and \( W_3 \notin \mathcal{L}_3 \).
Case 7.4.3.3.1.2 Suppose \((c_2, c_5) = (2, 2)\). Then \(v_4 + W_2\) has order 9 while \(v_5 + W_2\) has order 3. Since \(|\partial^{-1}W_2/W_2| = 3^3\) we get \(\partial^{-1}W_2/W_2 \cong Z_9 \times Z_3\). Note \(\Omega_1(\partial^{-1}W_2/W_2) \cong Z_3 \times Z_3\) has basis \(y_2 + W_2, v_5 + W_2\) while its subspace \(\partial^{-1}W_1/W_2\) has basis \(y_2 + W_2\).

Since \((c_2, c_5) = (2, 2)\) we have

\[
v_5 = \begin{pmatrix}
0 & 0 & 2 \\
0 & 0 & -3 \\
1 & 0 & 0
\end{pmatrix}.
\]

Each of \(W_3/W_2\) has order 3 and is contained in \(\Omega_1(\partial^{-1}W_2/W_2)\). Let \(c_6 \in \{0, 1, 2\}\).

Let \(m_7 = c_6y_2 + v_5\). Thus

\[
m_7 = \begin{pmatrix}
0 & 0 & 2 \\
0 & 3c_6 & -3 \\
1 & 0 & 0
\end{pmatrix}.
\]

Let \(W_3 = \langle W_2, m_7 \rangle\). There are 3 subgroups of this type. Since \(m_7 \not\in W_2\) and \(3m_7 \in W_2\) then \(|W_3/W_2| = 3^3\) and \(|W_2| = 3^9\) then \(|W_3| = 3^{10}\). Then \(\partial^{-1}W_2/W_3 \cong Z_3 \times Z_3\) and \(\text{rank}(\partial^{-1}W_2/W_3) = 2\). By the Modified Terminal Lemma we conclude that \(\partial^{-1}W_3 = \partial^{-1}W_2\). Thus \(\text{rank}(\partial^{-1}W_3/W_3) = \text{rank}(\partial^{-1}W_2/W_3)\), \(W_3\) is terminal, and \(W_3 \not\in \hat{L}_3\).

Case 7.4.3.3.1.3 Suppose \((c_2, c_5) = (2, 0)\). Then \(v_4 + W_2\) has order 3 while \(v_5 + W_2\) has order 9. Since \(|\partial^{-1}W_2/W_2| = 3^3\) we get \(\partial^{-1}W_2/W_2 \cong Z_9 \times Z_3\). Note \(\Omega_1(\partial^{-1}W_2/W_2) \cong Z_3 \times Z_3\) has basis \(y_2 + W_2, v_4 + W_2\) while its subspace \(\partial^{-1}W_1/W_2\) has basis \(y_2 + W_2\).
Since \((c_2, c_5) = (2, 0)\) we have
\[
v_4 = \begin{bmatrix} 0 & 0 & -2 \\ 0 & 1 & 3 \\ 0 & 0 & -3 \end{bmatrix}.
\]
Each of \(W_3/W_2\) has order 3 and is contained in \(\Omega_1(\partial^{-1}W_2/W_2)\). Let \(c_6 \in \{0, 1, 2\}\).

Let \(m_7 = c_6y_2 + v_4\). Thus
\[
m_7 = \begin{bmatrix} 0 & 0 & -2 \\ 0 & 1 + 3c_6 & 3 \\ 0 & 0 & -3 \end{bmatrix}.
\]
Let \(W_3 = < W_2, m_7 >\). There are 3 subgroups of this type. Since \(m_7 \not\in W_2\) and \(3m_7 \in W_2\) then \(|W_3/W_2| = 3\). Since \(|W_2| = 3^9\) then \(|W_3| = 3^{10}\). Then \(\partial^{-1}W_2/W_3 \cong \mathbb{Z}_3 \times \mathbb{Z}_3\) and \(\text{rank}(\partial^{-1}W_2/W_3) = 2\). By the Modified Terminal Lemma we conclude that \(\partial^{-1}W_3 = \partial^{-1}W_2\). Thus \(\text{rank}(\partial^{-1}W_3/W_3) = \text{rank}(\partial^{-1}W_2/W_3)\), \(W_3\) is terminal, and \(W_3 \not\in \hat{\mathcal{L}}_3\).

**Case 7.4.3.3.1.4** Suppose \((c_2, c_5)\) is \((1, 1)\) or \((1,2)\), or \((2, 1)\). Then the elements \(v_4 + W_2\) and \(v_5 + W_2\) both have order 9. Since \(|\partial^{-1}W_2/W_2| = 3^3\) we get \(\partial^{-1}W_2/W_2 \cong \mathbb{Z}_9 \times \mathbb{Z}_3\).

We show that element \(v_4 + v_5 + W_2\) has order 3. For this it suffices to show \(3(v_4 + v_5) \in W_2\). Note
\[
3(v_4 + v_5) = \begin{bmatrix} 0 & 0 & 3(c_2 - c_5 + c_2c_5) \\ 0 & 3 & 0 \\ 3 & 0 & 0 \end{bmatrix}.
\]
Taking \( r_1 = 1, r_2 = 1 \), and \( r_3 = c_2 - c_5 + c_2c_5 \) then \( r_1 + c_2r_2 + r_3 = (1 - c_2)(1 - c_5) \). By assumption we know at least one of the \( c_2, c_5 \) is equal to 1, so we get \( r_1 + c_2r_2 + r_3 \equiv 0 \). Hence by Lemma 10.4.2 \( 3(v_4 + v_5) \in W_2 \) as desired. Now \( \Omega_1(\partial^{-1}W_2/W_2) \cong \mathbb{Z}_3 \times \mathbb{Z}_3 \) has basis \( y_2 + W_2, v_4 + v_5 + W_2 \) while its subspace \( \partial^{-1}W_1/W_2 \) has basis \( y_2 + W_2 \). Each \( W_3/W_3 \) has order 3 and is contained in \( \Omega_1(\partial^{-1}W_2/W_2) \). Note that

\[
v_4 + v_5 = \begin{bmatrix} 0 & 0 & c_2c_5 - 2c_2 - c_5 \\ 0 & 1 & 3(2 - c_5) \\ 1 & 0 & 3c_2 \end{bmatrix}.
\]

Let \( c_6 \in \{0, 1, 2\} \). Let \( m_7 = c_6y_2 + v_4 + v_5 \). Thus

\[
m_7 = \begin{bmatrix} 0 & 0 & c_2c_5 - 2c_2 - c_5 \\ 0 & 1 + 3c_6 & 3(2 - c_5) \\ 1 & 0 & 3c_2 \end{bmatrix}.
\]

Let \( W_3 = \langle W_2, m_7 \rangle \). There are 3 subgroups of this type. Since \( m_7 \not\in W_2 \) and \( 3m_7 \in W_2 \) then \(|W_3/W_2| = 3\). Since \(|W_2| = 3^9\) then \(|W_3| = 3^{10}\). Then \( \partial^{-1}W_2/W_3 \cong \mathbb{Z}_3 \times \mathbb{Z}_3 \) and \( \text{rank}(\partial^{-1}W_2/W_3) = 2 \). By the Modified Terminal Lemma we conclude that \( \partial^{-1}W_3 = \partial^{-1}W_2 \). Thus \( \text{rank}(\partial^{-1}W_3/W_3) = \text{rank}(\partial^{-1}W_2/W_3) \), \( W_3 \) is terminal, and \( W_3 \not\in \hat{L}_3 \).

**Case 7.4.3.3.2** Now we consider the case \( c_2 \neq c_4 \). Recall \( c_2^2 \equiv 1 \). (B5) becomes

\[-c_2(c_2 + c_5)t_{1,1} + (1 + c_2^2c_5)t_{2,0} \equiv 0, \] which says \(-(1 + c_2c_5)t_{1,1} + (1 + c_5)t_{2,0} \equiv 0\). Thus \((1 + c_5)t_{2,0} \equiv (1 + c_2c_5)t_{1,1}\).
Hence $x \in \partial^{-1}W_2$ if and only if

$$s_{2,2} \equiv c_2 t_{1,1} \quad (A4)$$

$$s_{1,2} \equiv -c_2 s_{2,1} - t_{1,1} + (1 + c_5)t_{2,0} \quad (A5)$$

$$(1 + c_5)t_{2,0} \equiv (1 + c_2 c_5)t_{1,1} \quad (B5).$$

Recall $c_2 \in \{1, 2\}$ and $c_5 \in \{0, 1, 2\}$. So there are six possibilities for the ordered pair $(c_2, c_5)$. We now consider four separate cases. The first case is when $(c_2, c_5) = (1, 2)$. The second case is when $(c_2, c_5) = (2, 2)$. The third case is when $(c_2, c_5) = (2, 1)$. The fourth case is when $(c_2, c_5)$ is either $(1, 1)$ or $(1, 0)$ or $(2, 0)$.

**Case 7.4.3.3.2.1** Suppose $(c_2, c_5) = (1, 2)$. Then (B5) holds automatically. Hence $x \in \partial^{-1}W_2$ if and only if

$$s_{2,2} \equiv t_{1,1} \quad (A4)$$

$$s_{1,2} \equiv -s_{2,1} - t_{1,1} \quad (A5).$$

We regard $t_{1,1}, t_{2,0},$ and $t_{0,2}$ as our new free variables. Taking $t_{2,0} \equiv 1, t_{1,1} = 0,$ and $t_{0,2} = 0$, the matrix $x$ becomes

$$v_4 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix}.$$
Taking $t_{1,1} \equiv 1$, $t_{2,0} = 0$, and $t_{0,2} = 0$, the matrix $x$ becomes

$$v_5 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & -3 \\ 0 & 0 & 3 \end{bmatrix}. $$

Taking $t_{0,2} \equiv 1$, $t_{1,1} = 0$, and $t_{2,0} = 0$, the matrix $x$ becomes

$$v_6 = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}. $$

Note that $|\partial^{-1}W_2/W_2| = 3^4$. We want to determine the orders of the elements $v_4 + W_2$, $v_5 + W_2$, and $v_6 + W_2$. Note $3v_4 = y_1 \notin W_2$ so $3v_4 \notin W_2$ by Lemma 10.4.2 so the element $v_4 + W_2$ has order 9. Note $3v_5 = y_2 \notin W_1$ so $3v_5 \notin W_2$ by Lemma 10.4.2 so $v_5 + W_2$ has order 9. Note $3v_6 = y_3 \notin W_3$ so $3v_6 \notin W_2$ by Lemma 10.4.2 so $v_6 + W_2$ has order 9. Since

$$3(v_4 - v_6) = \begin{bmatrix} 0 & 0 & -3 \\ 0 & 0 & 0 \\ 3 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 & -3 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} - \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ -3 & 0 & 0 \end{bmatrix} = m_1 - m_3 \in W_1 \subseteq W_2, $$

the element $v_4 - v_6 + W_2$ of $\partial^{-1}W_2/W_2$ has order 3. Since

$$3(v_5 - v_6) = \begin{bmatrix} 0 & 0 & -3 \\ 0 & 3 & 0 \\ 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 & -3 \\ 0 & 3 & 0 \\ -3 & 0 & 0 \end{bmatrix} - \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ -3 & 0 & 0 \end{bmatrix} = m_2 - m_3 \in W_1 \subseteq W_2, $$

the element $v_5 - v_6 + W_2$ of $\partial^{-1}W_2/W_2$ has order 3. The elements $y_2 + W_2$, $v_4 - v_6 + W_2$, $v_5 - v_6 + W_2$ of order 3 in $\partial^{-1}W_2/W_2$ are linearly independent and generate

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a subgroup isomorphic to $\mathbb{Z}_3 \times \mathbb{Z}_3 \times \mathbb{Z}_3$. Since $|\partial^{-1}W_2/W_2| = 3^4$, it follows that $\partial^{-1}W_2/W_2 \cong \mathbb{Z}_9 \times \mathbb{Z}_3 \times \mathbb{Z}_3$ and that $\Omega_1(\partial^{-1}W_2/W_2)$ has basis $y_2 + W_2, v_4 - v_6 + W_2, v_5 - v_6 + W_2$ while its subspace $\partial^{-1}W_1/W_2$ has basis $y_2 + W_2$. For each $W_3/W_2$, indeed $W_3$ is contained in $\Omega_1(\partial^{-1}W_2/W_2)$. Since $\text{rank}(\partial^{-1}W_2/W_2) = 3$ and $\text{rank}(\partial^{-1}W_1/W_2) = 1$, then $\text{rank}(\partial^{-1}W_1/W_2) < \text{rank}(\partial^{-1}W_2/W_2)$. Hence $W_2$ is nonterminal and $W_2 \in \hat{\mathcal{L}}_2$.

Thus there are some $W_3/W_2$ isomorphic to $\mathbb{Z}_3$ and some isomorphic to $\mathbb{Z}_3 \times \mathbb{Z}_3$.

In Case 7.4.3.3.2.1.1 we consider the 1-dimensional subspaces $W_3/W_2$. There are three possible forms for the matrix $m$. The first form is

$$m = [0, 0, 1]$$

(1 possibility), which is considered in Case 7.4.3.3.2.1.1. The second form is

$$m = [0, 1, c_6] \quad \text{for } c_6 \in \{0, 1, 2\}$$

(3 possibilities), which is considered in Case 7.4.3.3.2.1.2. The third form is

$$m = [1, c_6, c_7] \quad \text{for } c_6, c_7 \in \{0, 1, 2\}, \ (c_6, c_7) \neq (0, 0)$$

(8 possibilities), which is considered in Case 7.4.3.3.2.1.3.

In Case 7.4.3.3.2.1.2 we consider the 2-dimensional subspaces $W_3/W_2$. There are three possible forms for the matrix $m$. The first form is

$$m = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

(1 possibility), which is considered in Case 7.4.3.3.2.2.1. The second form is

$$m = \begin{bmatrix} 1 & c_6 & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad \text{for } c_6 \in \{1, 2\}$$

(8 possibilities), which is considered in Case 7.4.3.3.2.2.2.
(2 possibilities), which is considered in Case 7.4.3.2.2.2. The third form is

\[
m = \begin{bmatrix} 1 & 0 & c_6 \\ 0 & 1 & c_7 \end{bmatrix} \text{ for } c_6 \in \{1, 2\} \text{ and } c_7 \in \{0, 1, 2\}
\]

(6 possibilities), which is considered in Case 7.4.3.2.2.3.

**Case 7.4.3.2.1.1** Let \( m_7 = v_4 - v_6 \). Thus

\[
m_7 = \begin{bmatrix} 0 & 0 & -1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix}.
\]

Let \( W_3 = \langle W_2, m_7 \rangle \in \mathcal{L}_3 \). The number of subgroups \( W_2 \) of this type is 1. Since \( m_7 \notin W_2 \) and \( 3m_7 \in W_2 \) we have \( |W_3/W_2| = 3 \). Since \( |W_2| = 3^9 \) it follows that \( |W_3| = 3^{10} \). By the Modified Terminal Lemma we conclude that \( \partial^{-1}W_3 = \partial^{-1}W_2 \).

Thus \( \text{rank}(\partial^{-1}W_3/W_3) = \text{rank}(\partial^{-1}W_2/W_3) \), \( W_3 \) is terminal, and \( W_3 \notin \hat{\mathcal{L}}_3 \).

**Case 7.4.3.2.1.1.2** We fix arbitrary value \( c_6 \in \{0, 1, 2\} \). Let \( m_7 = v_4 - v_6 + c_6(v_5 - v_6) \).

Thus

\[
m_7 = \begin{bmatrix} 0 & 0 & -c_6 \\ 0 & c_6 & -3c_6 \\ 1 & 0 & 3c_6 \end{bmatrix}.
\]

Let \( W_3 = \langle W_2, m_7 \rangle \in \mathcal{L}_3 \). The number of subgroups \( W_2 \) of this type is 3. Since \( m_7 \notin W_2 \) and \( 3m_7 \in W_2 \) we have \( |W_3/W_2| = 3 \). Since \( |W_2| = 3^9 \) it follows that
\[ |W_3| = 3^{10}. \] By the Modified Terminal Lemma we conclude that \( \partial^{-1}W_3 = \partial^{-1}W_2. \)

Thus \( \text{rank}(\partial^{-1}W_3/W_3) = \text{rank}(\partial^{-1}W_2/W_3), \) \( W_3 \) is terminal, and \( W_3 \notin \hat{L}_3. \)

**Case 7.4.3.3.2.1.1.3** We fix arbitrary values \( c_6, c_7 \in \{0, 1, 2\} \) such that \( (c_6, c_7) \neq (0, 0). \) Let \( m_7 = y_2 + c_6(v_4 - v_6) + c_7(v_5 - v_6). \) Thus

\[
m_7 = \begin{bmatrix}
0 & 0 & -(c_6 + c_7) \\
0 & 3 + c_7 & -3c_7 \\
c_6 & 0 & 3c_7
\end{bmatrix}.
\]

Let \( W_3 = <W_2, m_7> \in L_3. \) The number of subgroups \( W_2 \) of this type is 8.

Since \( m_7 \notin W_2 \) and \( 3m_7 \in W_2 \) we have \( |W_3/W_2| = 3. \) Since \( |W_2| = 3^9 \) it follows that \( |W_3| = 3^{10}. \) By the Modified Terminal Lemma we conclude that \( \partial^{-1}W_3 = \partial^{-1}W_2. \)

Thus \( \text{rank}(\partial^{-1}W_3/W_3) = \text{rank}(\partial^{-1}W_2/W_3), \) \( W_3 \) is terminal, and \( W_3 \notin \hat{L}_3. \)

**Case 7.4.3.3.2.1.2.1** Let \( m_7 = v_4 - v_6 \) and \( m_8 = v_5 - v_6. \) Thus

\[
m_7 = \begin{bmatrix}
0 & 0 & -1 \\
0 & 0 & 0 \\
1 & 0 & 0
\end{bmatrix} \quad \text{and} \quad m_8 = \begin{bmatrix}
0 & 0 & -1 \\
0 & 1 & -3 \\
0 & 0 & 3
\end{bmatrix}.
\]

Let \( W_3 = <W_2, m_7, m_8> \in L_3. \) The number of subgroups \( W_2 \) of this type is 1. Since \( m_7 \notin W_2, m_8 \notin W_2, 3m_7 \in W_2, \) and \( 3m_8 \in W_2 \) we have \( |W_3/W_2| = 3^2. \) Since \( |W_2| = 3^9 \) it follows that \( |W_3| = 3^{11}. \) By the Modified Terminal Lemma we conclude that \( \partial^{-1}W_3 = \partial^{-1}W_2. \) Thus \( \text{rank}(\partial^{-1}W_3/W_3) = \text{rank}(\partial^{-1}W_2/W_3), \) \( W_3 \) is terminal, and \( W_3 \notin \hat{L}_3. \)

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Case 7.4.3.3.2.1.2.2 We fix arbitrary value $c_6 \in \{1, 2\}$. Let $m_7 = y_2 + c_6(v_4 - v_6)$ and $m_8 = v_5$. Thus

$$m_7 = \begin{bmatrix} 0 & 0 & -c_6 \\ 0 & 3 & 0 \\ c_6 & 0 & 0 \end{bmatrix} \quad \text{and} \quad m_8 = \begin{bmatrix} 0 & 0 & -1 \\ 0 & 1 & -3 \\ 0 & 0 & 3 \end{bmatrix}.$$ 

Let $W_3 = \langle W_2, m_7, m_8 \rangle \in \mathcal{L}_3$. The number of subgroups $W_2$ of this type is 2. Since $m_7 \notin W_2$, $m_8 \notin W_2$, $3m_7 \in W_2$, and $3m_8 \in W_2$ we have $|W_3/W_2| = 3^2$. Since $|W_2| = 3^3$ it follows that $|W_3| = 3^{11}$. By the Modified Terminal Lemma we conclude that $\partial^{-1}W_3 = \partial^{-1}W_2$. Thus $	ext{rank}(\partial^{-1}W_3/W_3) = \text{rank}(\partial^{-1}W_2/W_3)$, $W_3$ is terminal, and $W_3 \notin \hat{\mathcal{L}}_3$.

Case 7.4.3.3.2.1.2.3 We fix arbitrary values $c_6 \in \{1, 2\}$ and $c_7 \in \{0, 1, 2\}$. Let $m_7 = y_2 + c_6(v_4 - v_6)$ and $m_8 = y_2 + c_7(v_5 - v_6)$. Thus

$$m_7 = \begin{bmatrix} 0 & 0 & -c_6 \\ 0 & 3 & 0 \\ c_6 & 0 & 0 \end{bmatrix} \quad \text{and} \quad m_8 = \begin{bmatrix} 0 & 0 & -c_7 \\ 0 & 3 + c_7 & -3c_7 \\ 0 & 0 & 3c_7 \end{bmatrix}.$$ 

Let $W_3 = \langle W_2, m_7, m_8 \rangle \in \mathcal{L}_3$. The number of subgroups $W_2$ of this type is 6. Since $m_7 \notin W_2$, $m_8 \notin W_2$, $3m_7 \in W_2$, and $3m_8 \in W_2$ we have $|W_3/W_2| = 3^2$. Since $|W_2| = 3^3$ it follows that $|W_3| = 3^{11}$. By the Modified Terminal Lemma we conclude that $\partial^{-1}W_3 = \partial^{-1}W_2$. Thus $	ext{rank}(\partial^{-1}W_3/W_3) = \text{rank}(\partial^{-1}W_2/W_3)$, $W_3$ is terminal, and $W_3 \notin \hat{\mathcal{L}}_3$. 

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Case 7.4.3.3.2.2 Suppose \((c_2, c_5) = (2, 2)\). Then (B5) becomes \(-t_{1,1} \equiv 0\) which forces \(t_{1,1} \equiv 0\). Since \(t_{1,1} \equiv 0\) and \((c_2, c_5) = (2, 2)\) (A4) becomes \(s_{2,2} \equiv 0\) and (A5) becomes \(s_{1,2} \equiv s_{2,2}\). Hence \(x \in \partial^{-1}W_2\) if and only if

\[
\begin{align*}
  s_{2,2} &\equiv 0 \quad \text{(A4)} \\
  s_{1,2} &\equiv s_{2,2} \quad \text{(A5)} \\
  t_{1,1} &\equiv 0 \quad \text{(B5)}.
\end{align*}
\]

We regard \(t_{2,0}\) and \(t_{0,2}\) as our new free variables. Taking \(t_{2,0} \equiv 1\) and \(t_{0,2} = 0\), the matrix \(x\) becomes

\[
v_4 = \begin{bmatrix}
0 & 0 & 0 \\
0 & 0 & 0 \\
1 & 0 & 0
\end{bmatrix}.
\]

Taking \(t_{0,2} \equiv 1\) and \(t_{2,0} = 0\), the matrix \(x\) becomes

\[
v_5 = \begin{bmatrix}
0 & 0 & 1 \\
0 & 0 & 0 \\
0 & 0 & 0
\end{bmatrix}.
\]

Note \(3v_4 = y_1 \not\in W_1\) so \(3v_4 \not\in W_2\) by Lemma 10.4.2 so \(v_4 + W_2\) has order 9. Note \(3v_5 = y_3 \not\in W_1\) so \(3v_5 \not\in W_2\) by Lemma 10.4.2 so \(v_4 + W_2\) has order 9. Since \(|\partial^{-1}W_2/W_2| = 3^3\) we deduce \(\partial^{-1}W_2/W_2 \cong \mathbf{Z}_9 \times \mathbf{Z}_3\). Since

\[
3(v_4 - v_5) = \begin{bmatrix}
0 & 0 & -3 \\
0 & 0 & 0 \\
3 & 0 & 0
\end{bmatrix} = m_1 - m_3 \in W_1 \subseteq W_2,
\]

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the element $v_4 - v_5 + W_2$ of $\partial^{-1}W_2/W_2$ has order 3. We see that $\Omega_1(\partial^{-1}W_2/W_2) \cong \mathbb{Z}_3 \times \mathbb{Z}_3$ has basis $y_2 + W_2, v_4 - v_5 + W_2$ while its subspace $\partial^{-1}W_1/W_2$ has basis $y_2 + W_2$.

Let $c_6 \in \{0, 1, 2\}$. Let $m_7 = c_6 y_2 + v_4 - v_5$. Thus

$$m_7 = \begin{bmatrix} 0 & 0 & -1 \\ 0 & 3c_6 & 0 \\ 1 & 0 & 0 \end{bmatrix}.$$ 

Let $W_3 = \langle W_2, m_7 \rangle \in \mathcal{L}_3$. By the Modified Terminal Lemma we conclude that $\partial^{-1}W_3 = \partial^{-1}W_2$. Thus $\text{rank}(\partial^{-1}W_3/W_3) = \text{rank}(\partial^{-1}W_2/W_3)$, $W_3$ is modified terminal, and $W_3 \notin \hat{\mathcal{L}}_3$.

**Case 7.4.3.3.2.3** Suppose $(c_2, c_5) = (2, 1)$. Then (B5) becomes $-t_{2,0} \equiv 0$ which says $t_{2,0} \equiv 0$. Hence $x \in \partial^{-1}W_2$ if and only if

$$s_{2,2} \equiv -t_{1,1} \quad (A4)$$

$$s_{1,2} \equiv s_{2,1} - t_{1,1} \quad (A5)$$

$$t_{2,0} \equiv 0 \quad (B5).$$

We regard $t_{0,2}, t_{1,1}$ as our new free variables. Taking $t_{0,2} \equiv 1$ and $t_{1,1} = 0$, the matrix $x$ becomes

$$v_4 = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$
Taking $t_{1,1} \equiv 1$ and $t_{0,2} = 0$, the matrix $x$ becomes

$$v_5 = \begin{bmatrix}
0 & 0 & 0 \\
0 & 1 & -3 \\
0 & 0 & -3
\end{bmatrix}.$$ 

Note $3v_4 = y_3 \not\in W_1$ so by Lemma 10.4.2 $3v_4 \not\in W_2$ and the element $v_4 + W_2$ has order 9.

9. Note $3v_5 = y_2 \not\in W_1$ so by Lemma 10.4.2 $3v_5 \not\in W_2$ and the element $v_5 + W_2$ has order 9. Since $|\partial^{-1}W_2/W_2| = 3^3$ we deduce $\partial^{-1}W_2/W_2 \cong \mathbb{Z}_9 \times \mathbb{Z}_3$. Since

$$3(v_5 - v_4) = \begin{bmatrix}
0 & 0 & -3 \\
0 & 3 & 0 \\
0 & 0 & 0
\end{bmatrix} = m_2 - m_3 \in W_1 \subseteq W_2,$$

the element $v_5 - v_4 + W_2$ of $\partial^{-1}W_2/W_2$ has order 3. We see that $\Omega_1(\partial^{-1}W_2/W_2) \cong \mathbb{Z}_3 \times \mathbb{Z}_3$ has basis $y_2 + W_2, v_5 - v_4 + W_2$ while its subspace $\partial^{-1}W_1/W_2$ has basis $y_2 + W_2$.

Let $c_6 \in \{0, 1, 2\}$ Let $m_7 = c_6 y_2 + v_5 - v_4$. Thus

$$m_7 = \begin{bmatrix}
0 & 0 & -1 \\
0 & 1 + 3c_6 & -3 \\
0 & 0 & -3
\end{bmatrix}.$$

Let $W_3 = \langle W_2, m_7 \rangle \in \mathcal{L}_3$. By the Modified Terminal Lemma we conclude that $\partial^{-1}W_3 = \partial^{-1}W_2$. Thus $\text{rank}(\partial^{-1}W_3/W_3) = \text{rank}(\partial^{-1}W_2/W_3)$, $W_3$ is terminal, and $W_3 \not\in \hat{\mathcal{L}}_3$.

**Case 7.4.3.3.2.4** Suppose $(c_2, c_5)$ is either $(1, 1)$ or $(1, 0)$ or $(2, 0)$. Thus $1 + c_5 \equiv 1 + c_2c_5 \not\equiv 0$ and (B5) becomes $t_{2,0} \equiv t_{1,1}$. 

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Hence $x \in \partial^{-1}W_2$ if and only if

\[ s_{2,2} \equiv c_2 t_{2,0} \quad \text{(A4)} \]

\[ s_{1,2} \equiv -c_2 s_{2,1} + c_5 t_{2,0} \quad \text{(A5)} \]

\[ t_{1,1} \equiv t_{2,0} \quad \text{(B5)}. \]

We regard $t_{2,0}, t_{0,2}$ as our new free variables. Taking $t_{2,0} \equiv 1$ and $t_{0,2} = 0$, the matrix $x$ becomes

\[
 v_4 = \begin{bmatrix}
 0 & 0 & 0 \\
 0 & 1 & 3c_5 \\
 1 & 0 & 3c_2
\end{bmatrix}.
\]

Taking $t_{0,2} \equiv 1$ and $t_{2,0} = 0$, the matrix $x$ becomes

\[
 v_5 = \begin{bmatrix}
 0 & 0 & 1 \\
 0 & 0 & 0 \\
 0 & 0 & 0
\end{bmatrix}.
\]

Note $3v_5 = y_3 \notin W_1$ so by Lemma 10.4.2 $3v_5 \notin W_2$ and so the element $v_5 + W_2$ has order 9. Since $|\partial^{-1}W_2/W_2| = 3^3$ we deduce $\partial^{-1}W_2/W_2 \cong \mathbb{Z}_9 \times \mathbb{Z}_3$. We determine the order of element $v_4 + W_2$. Note

\[
 3v_4 = \begin{bmatrix}
 0 & 0 & 0 \\
 0 & 3 & 0 \\
 3 & 0 & 0
\end{bmatrix}.
\]

Taking $r_1 = 1, r_2 = 1$, and $r_3 = 0$ then $r_1 + c_2 r_2 + r_3 = 1 + c_2$. We know $3v_4 \in W_2$ if and only if $r_1 + c_2 r_2 + r_3 \equiv 0$ which holds if and only if $c_2 = 2$. Hence element
$v_4 + W_2$ has order 9 if $(c_2, c_5)$ is either $(1, 1)$ or $(1, 0)$ and order 3 if $(c_2, c_5) = (2, 0)$. Now we consider subcases. The first subcase is when $(c_2, c_5)$ is either $(1, 1)$ or $(1, 0)$. The second subcase is when $(c_2, c_5) = (2, 0)$.

Case 7.4.3.2.4.1 Suppose $(c_2, c_5)$ is either $(1, 1)$ or $(1, 0)$. Thus $v_4 + W_2$ and $v_5 + W_2$ are both elements of order 9. Since $c_2 = 1$ we have

\[
v_4 = \begin{bmatrix}
0 & 0 & 0 \\
0 & 1 & 3c_5 \\
1 & 0 & 3
\end{bmatrix}
\quad\text{and } m_2 = \begin{bmatrix}
1 & 0 & -3 \\
0 & 3 & 0 \\
-3 & 0 & 0
\end{bmatrix}.
\]

Note $3(v_4 + v_5) = \begin{bmatrix}
0 & 0 & 3 \\
0 & 3 & 0 \\
3 & 0 & 0
\end{bmatrix} \cong \begin{bmatrix}
1 & 0 & -3 \\
1 & 0 & -1 \\
1 & 0 & 0
\end{bmatrix} + \begin{bmatrix}
0 & 0 & 0 \\
0 & 3 & 0 \\
-3 & 0 & 0
\end{bmatrix} + \begin{bmatrix}
0 & 0 & 0 \\
-3 & 0 & 0 \\
0 & 0 & 0
\end{bmatrix}
= m_1 + m_2 + m_3 \in W_1 \subseteq W_2$. Thus element $v_4 + v_5 + W_2$ has order 3. Thus $\Omega_1(\partial^{-1}W_2/W_2) \cong \mathbb{Z}_3 \times \mathbb{Z}_3$ has basis $y_2 + W_2, v_4 + v_5 + W_2$ while its subspace $\partial^{-1}W_1W_2$ has basis $y_2 + W_2$. Let $c_6 \in \{0, 1, 2\}$ Let $m_7 = c_6 y_2 + v_4 + v_5$. Thus

\[
m_7 = \begin{bmatrix}
0 & 0 & 1 \\
0 & 1 + 3c_6 & 3c_5 \\
1 & 0 & 3
\end{bmatrix}.
\]

Let $W_3 = \langle W_2, m_7 \rangle \in \mathcal{L}_3$. By the Modified Terminal Lemma we conclude that $\partial^{-1}W_3 = \partial^{-1}W_2$. Thus $\text{rank}(\partial^{-1}W_3/W_3) = \text{rank}(\partial^{-1}W_2/W_3)$, $W_3$ is terminal, and $W_3 \not\in \hat{\mathcal{L}}_3$. 

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Case 7.4.3.3.2.4.2 Suppose \((c_2, c_5) = (2, 0)\). Then

\[
v_4 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & 0 & -3 \end{bmatrix}
\]

and element \(v_4 + W_2\) has order 3. Thus \(\Omega_1 = (\partial^{-1}W_2/W_2) \cong \mathbb{Z}_3 \times \mathbb{Z}_3\) has basis \(y_2 + W_2, v_4 + W_2\) while its subspace \(\partial^{-1}W_1/W_2\) has basis \(y_2 + W_2\). Let \(c_6 \in \{0, 1, 2\}\). Let \(m_7 = c_6 y_2 + v_4\). Thus

\[
m_7 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 + 3c_6 & 0 \\ 1 & 0 & -3 \end{bmatrix}
\]

Let \(W_3 = \langle W_2, m_7 \rangle \in \mathcal{L}_3\). By the Modified Terminal Lemma we conclude that \(\partial^{-1}W_3 = \partial^{-1}W_2\). Thus \(\text{rank}(\partial^{-1}W_3/W_3) = \text{rank}(\partial^{-1}W_2/W_3)\), \(W_3\) is terminal, and \(W_3 \notin \hat{\mathcal{L}}_3\).

Case 7.4.3.4

We fix arbitrary values \(c_4 \in \{1, 2\}\) and \(c_5, c_6 \in \{0, 1, 2\}\). There are 18 ways to choose the values \(c_4, c_5, c_6\). Let \(m_4 = y_2 + c_4 v_3\), \(m_5 = v_1 + c_5 v_3\), and \(m_6 = v_2 + c_6 v_3\).

Thus

\[
m_4 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 3 & -3c_2c_4 \\ 0 & 3c_4 & 0 \end{bmatrix}, \quad m_5 = \begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & -3c_2c_5 \\ 0 & 3c_5 & 0 \end{bmatrix}, \quad \text{and} \quad m_6 = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & -3c_2c_6 \\ 0 & 3c_6 & 0 \end{bmatrix}
\]

Let \(W_2 = \langle W_1, m_4, m_5, m_6 \rangle \in \mathcal{L}_2\). The number of subgroups \(W_2\) of this type is 18. Since \(m_4, m_5, m_6 \notin W_1\) and \(3m_4, 3m_5, 3m_6 \in W_1\) we have \(|W_2/W_1| = 534\).
$3^3$. Since $|W_1| = 3^6$ it follows that $|W_2| = 3^9$. Note that $\partial^{-1}W_1/W_2 \cong \mathbb{Z}_3$ and $\text{rank}(\partial^{-1}W_1/W_2) = 1$. We now calculate the pullback of $\partial^{-1}W_2$. We observed in case 7.4.2 that $\partial^{-1}W_2$ is contained in the pattern subgroup

$$\begin{bmatrix} 2 & 2 & 2 \\ 2 & 2 & 1 \\ 2 & 1 & 1 \end{bmatrix}.$$ 

Let

$$x = \begin{bmatrix} 3s_{0,0} + t_{0,0} & 3s_{0,1} + t_{0,1} & 3s_{0,2} + t_{0,2} \\ 3s_{1,0} + t_{1,0} & 3s_{1,1} + t_{1,1} & 3s_{1,2} \\ 3s_{2,0} + t_{2,0} & 3s_{2,1} & 3s_{2,2} \end{bmatrix} \in \begin{bmatrix} 2 & 2 & 2 \\ 2 & 2 & 1 \\ 2 & 1 & 1 \end{bmatrix}.$$ 

The variables that are in play are those appearing in the matrix

$$\begin{bmatrix} 0 & t_{0,1} & t_{0,2} \\ t_{1,0} & t_{1,1} & 3s_{1,2} \\ t_{2,0} & 3s_{2,1} & 3s_{2,2} \end{bmatrix}.$$ 

In the notation of Case 7.4.2, we are taking $d_1 = 1$, $d_2 = 0$, $d_3 = 0$, $d_4 = c_4$, $e_2 = 1$, $e_3 = 0$, $e_4 = c_5$, $f_3 = 1$, and $f_4 = c_6$. We wish to identify values $a_1, a_2, a_3, a_4, a_5, a_6 \in \mathbb{Z}_9$ such that $\partial_1 x \equiv a_1m_1 + a_2m_2 + a_3m_3 + a_4m_4 + a_5m_5 + a_6m_6 \pmod{I}$. 

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We see that $\partial_1 x \in W_2$ if and only if

$$t_{1,1} \equiv a_6 \quad (A1)$$

$$-t_{1,1} \equiv a_4 c_4 + a_5 c_5 + a_6 c_6 \quad (A2)$$

$$t_{2,0} \equiv a_5 \quad (A3)$$

$$s_{2,2} \equiv c_2 t_{1,1} \quad (A4)$$

$$s_{1,2} \equiv c_2 a_4 + t_{2,0} - c_2 s_{2,1} \quad (A5).$$

We wish to identify values $b_1, b_2, b_3, b_4, b_5, b_6 \in \mathbb{Z}_9$ such that $\partial_1 x \equiv b_1 m_1 + b_2 m_2 + b_3 m_3 + b_4 m_4 + b_5 m_5 + b_6 m_6 \pmod{I}$. We see that $\partial_2 x \in W_2$ if and only if

$$t_{0,2} \equiv b_6 \quad (B1)$$

$$s_{2,2} \equiv b_4 c_4 + b_5 c_5 + b_6 c_6 \quad (B2)$$

$$t_{1,1} \equiv b_5 \quad (B3)$$

$$t_{1,1} \equiv c_2 s_{2,2} \quad (B4)$$

$$s_{1,2} \equiv -c_2 s_{2,1} + b_4 + c_2 t_{0,2} \quad (B5).$$

Multiplying (B4) by $c_2$ we obtain $s_{2,2} \equiv c_2 t_{1,1}$ which is redundant with (A4). Substituting $a_5$ and $a_6$ into (A2) we obtain $a_4 \equiv -c_4 c_5 t_{2,0} - c_4 (c_6 + 1) t_{1,1}$. Substituting $b_5$ and $b_6$ into (B2) we obtain $s_{2,2} \equiv c_4 b_4 + c_5 t_{1,1} + c_6 t_{0,2}$. Substituting (A4) into (B2) we obtain $b_4 \equiv c_4 (c_2 - c_5) t_{1,1} - c_4 c_6 t_{0,2}$. Combining (A5) and (B5) we obtain

$$c_2 a_4 + t_{2,0} \equiv b_4 + c_2 t_{0,2}$$

which we will denote as our new (B5). Substituting $a_4$ and $b_4$ into (B5) we obtain

$$(c_4 c_6 - c_2) t_{0,2} \equiv (c_2 c_4 c_5 - 1) t_{2,0} + c_4 (c_2 c_6 - c_2 - c_5) t_{1,1}. \quad (A5)$$

(A5) becomes

$$s_{1,2} \equiv (1 - c_2 c_4 c_5) t_{2,0} - c_2 c_4 (c_6 + 1) t_{1,1} - c_2 s_{2,1}. \quad (536)$$
Hence $x \in \partial^{-1}W_2$ if and only if

\[ s_{2,2} \equiv c_2 t_{1,1} \quad \text{(A4)} \]

\[ s_{1,2} \equiv (1 - c_2 c_4 c_5) t_{2,0} - c_2 c_4 (c_6 + 1) t_{1,1} - c_2 s_{2,1} \quad \text{(A5)} \]

\[ (c_4 c_6 - c_2) t_{0,2} \equiv (c_2 c_4 c_5 - 1) t_{2,0} + c_4 (c_2 c_6 - c_2 - c_5) t_{1,1} \quad \text{(B5)}. \]

**Case 7.4.3.4.1** Suppose $c_4 c_6 \neq c_2$. Thus $c_4 c_6 - c_2 \neq 0$ and so $(c_4 c_6 - c_2)^2 \equiv 1$.

We multiply both sides of (B5) by $c_4 c_6 - c_2$ and we obtain $t_{0,2} \equiv q t_{2,0} + r t_{1,1}$ where

\[ q = (c_2 c_4 c_5 - 1)(c_4 c_6 - c_2) \quad \text{and} \quad r \equiv c_4 (c_2 c_6 - c_2 - c_5)(c_4 c_6 - c_2). \]

Hence $x \in \partial^{-1}W_2$ if and only if

\[ s_{2,2} \equiv c_2 t_{1,1} \quad \text{(A4)} \]

\[ s_{1,2} \equiv -c_2 s_{2,1} + (1 - c_2 c_4 c_5) t_{2,0} - c_2 c_4 (c_6 + 1) t_{1,1} \quad \text{(A5)} \]

\[ t_{0,2} \equiv q t_{2,0} + r t_{1,1} \quad \text{(B5)}. \]

We regard $t_{2,0}$ and $t_{1,1}$ as our new free variables. Taking $t_{2,0} \equiv 1$ and $t_{1,1} = 0$, the matrix $x$ becomes

\[ v_4 = \begin{bmatrix} 0 & 0 & q \\ 0 & 0 & 3(1 - c_2 c_4 c_5) \\ 1 & 0 & 0 \end{bmatrix}. \]

Taking $t_{1,1} \equiv 1$ and $t_{2,0} = 0$, the matrix $x$ becomes

\[ v_5 = \begin{bmatrix} 0 & 0 & r \\ 0 & 1 & -3c_2 c_4 (c_6 + 1) \\ 1 & 0 & 0 \end{bmatrix}. \]
Note

\[
3v_4 = \begin{bmatrix}
0 & 0 & 3q \\
0 & 0 & 0 \\
3 & 0 & 0
\end{bmatrix}
\quad \text{and} \quad
3v_5 = \begin{bmatrix}
0 & 0 & 3r \\
0 & 3 & 0 \\
3 & 0 & 0
\end{bmatrix}
\]

Thus \( \partial^{-1}W_2 = \langle \partial^{-1}W_0, v_1, v_2, v_3, v_4, v_5 \rangle = \langle \partial^{-1}W_1, v_4, v_5 \rangle \). Recall \( \partial^{-1}W_1/W_2 \cong \mathbb{Z}_3 \). We want to determine the order of the element \( v_4 + W_2 \). Take \( r_1 = 1, \ r_2 = 0, \) and \( r_3 = q \). Then \( r_1 + c_2 r_2 + r_3 \equiv 1 + q \). By Lemma 10.4.1 we see that element \( v_4 + W_2 \) has order 3 if and only if \( 1 + q \equiv 0 \). Since \( c_2, c_4 \in \{1, 2\} \) while \( c_5, c_6 \in \{0, 1, 2\} \), there are \( 2^2 \cdot 2^2 = 36 \) possibilities for the ordered quadruple \((c_2, c_4, c_5, c_6)\). Element \( v_4 + W_2 \) has order 3 precisely when \( q \equiv 2 \). It is tedious but straightforward to show that the condition \( q \equiv 2 \) holds if and only if \((c_2, c_4, c_5, c_6)\) is one of the following: \((1, 1, 0, 2), (1, 1, 2, 0), (1, 2, 0, 1), (1, 2, 1, 0), (2, 1, 0, 0), (2, 1, 1, 1), (2, 2, 0, 0), (2, 2, 2, 2)\). We want to determine the order of the element \( v_5 + W_2 \). Take \( r_1 = 0, r_2 = 1, \) and \( r_3 = r \). Then \( r_1 + c_2 r_2 + r_3 \equiv c_2 + r \). By Lemma 10.4.1 we see that element \( v_5 + W_2 \) has order 3 if and only if \( c_2 + r \equiv 0 \). Element \( v_5 + W_2 \) has order 3 precisely when \( r \equiv -c_2 \). It is tedious but straightforward to show that condition \( r \equiv -c_2 \) holds if and only if \((c_2, c_4, c_5, c_6)\) is one of the following: \((1, 1, 1, 0), (1, 1, 2, 2), (1, 2, 0, 0), (1, 2, 2, 1), (2, 1, 0, 0), (2, 1, 1, 1), (2, 2, 1, 2), (2, 2, 2, 0)\). We note that the only quadruples that appear in both lists are \((2, 1, 0, 0)\) and \((2, 1, 1, 1)\). Thus we conclude that the elements \( v_4 + W_2 \) and \( v_5 + W_2 \) both have order 3 if and only if \((c_2, c_4, c_5, c_6)\) is one of these two. Hence \( |\partial^{-1}W_2/W_2| = 3^3 \). Recall \( \partial^{-1}W_1/W_2 \cong \mathbb{Z}_3 \).
Case 7.4.3.4.1.1 Suppose $1 + q \neq 0$ and $c_2 + r \neq 0$. Note

$$3v_4 = \begin{bmatrix} 0 & 0 & 3q \\ 0 & 0 & 0 \\ 3 & 0 & 0 \end{bmatrix} \quad \text{and} \quad 3v_5 = \begin{bmatrix} 0 & 0 & 3r \\ 0 & 3 & 0 \\ 3 & 0 & 0 \end{bmatrix}.$$ 

By Lemma 10.4.1, we see that both $v_4 + W_2$ and $v_5 + W_2$ have order 9. Hence $\partial^{-1}W_2/W_2 \cong \mathbb{Z}_9 \times \mathbb{Z}_3$ and $W_2$ is nonterminal. As shown in earlier work, since $W_2$ is nonterminal we will fix the value $c_8 \in \{0, 1, 2\}$ and obtain 3 subgroups $W_3$, all satisfying $|W_3| = 3^{10}$. By the Modified Terminal Lemma, we will see that each of these 3 subgroups is terminal.

Case 7.4.3.4.1.2 Suppose $1 + q \neq 0$ and $c_2 + r \equiv 0$. Thus $v_4 + W_2$ has order 9 while $v_5 + W_2$ has order 3. Also,

$$v_4 = \begin{bmatrix} 0 & 0 & q \\ 0 & 0 & 3(1 - c_2c_4c_5) \\ 1 & 0 & 0 \end{bmatrix}, \quad 3v_4 = \begin{bmatrix} 0 & 0 & 3q \\ 0 & 0 & 0 \\ 3 & 0 & 0 \end{bmatrix},$$

$$v_5 = \begin{bmatrix} 0 & 0 & -c_2 \\ 0 & 1 & -3c_2c_4(c_6 + 1) \\ 1 & 0 & 0 \end{bmatrix}, \quad 3v_5 = \begin{bmatrix} 0 & 0 & -3c_2 \\ 0 & 3 & 0 \\ 3 & 0 & 0 \end{bmatrix}.$$ 

Note $\partial^{-1}W_2/W_2 \cong \mathbb{Z}_9 \times \mathbb{Z}_3$ and $\mathcal{B}_1(\partial^{-1}W_2/W_2) = \partial^{-1}W_1/W_2$. Note $\Omega_1(\partial^{-1}W_2/W_2)$ is isomorphic to $\mathbb{Z}_3 \times \mathbb{Z}_3$ and has basis $y_2 + W_2, v_5 + W_2$ while its subspace $\partial^{-1}W_1/W_2$ has basis $y_2 + W_2$. Each $W_3/W_2$ has order 3 and is thus contained in $\Omega_1(\partial^{-1}W_2/W_2)$. 

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Let \( c_7 \in \{0, 1, 2\} \). Let \( m_7 = c_7y_2 + v_5 \). Thus

\[
m_7 = \begin{bmatrix}
0 & 0 & -c_2 \\
0 & 3c_7 + 1 & -3c_2c_4(c_6 + 1) \\
1 & 0 & 0
\end{bmatrix}.
\]

Let \( W_3 = \langle W_2, m_7 \rangle \). There are 3 subgroups of this type. Since \( m_7 \not\in W_2 \) and \( 3m_7 \in W_2 \) then \( |W_3/W_2| = 3 \). Since \( |W_2| = 3^9 \) then \( |W_3| = 3^{10} \). Then \( \partial^{-1}W_2/W_3 \cong \mathbb{Z}_3 \times \mathbb{Z}_3 \) and \( \text{rank}(\partial^{-1}W_2/W_3) = 2 \). By the Modified Terminal Lemma we conclude that \( \partial^{-1}W_3 = \partial^{-1}W_2 \). Thus \( \text{rank}(\partial^{-1}W_3/W_3) = \text{rank}(\partial^{-1}W_2/W_3) \), \( W_3 \) is terminal, and \( W_3 \not\in \hat{\mathcal{L}}_3 \).

**Case 7.4.3.4.1.3** Suppose \( 1 + q \equiv 0 \) and \( c_2 + r \not\equiv 0 \). Thus \( v_5 + W_2 \) has order 9 while \( v_4 + W_2 \) has order 3. Also,

\[
v_4 = \begin{bmatrix}
0 & 0 & 2 \\
0 & 0 & 3(1 - c_2c_4c_5) \\
1 & 0 & 0
\end{bmatrix}, \quad 3v_4 = \begin{bmatrix}
0 & 0 & -3 \\
0 & 0 & 0 \\
3 & 0 & 0
\end{bmatrix},
\]

\[
v_5 = \begin{bmatrix}
0 & 0 & r \\
0 & 1 & -3c_2c_4(c_6 + 1) \\
1 & 0 & 0
\end{bmatrix}, \quad 3v_5 = \begin{bmatrix}
0 & 0 & 3r \\
0 & 3 & 0 \\
3 & 0 & 0
\end{bmatrix}.
\]

Note \( \partial^{-1}W_2/W_2 \cong \mathbb{Z}_9 \times \mathbb{Z}_3 \) and \( \mathcal{O}_1(\partial^{-1}W_2/W_2) = \partial^{-1}W_1/W_2. \) Note \( \Omega_1(\partial^{-1}W_2/W_2) \) is isomorphic to \( \mathbb{Z}_3 \times \mathbb{Z}_3 \) and has basis \( y_2 + W_2, v_4 + W_2 \) while its subspace \( \partial^{-1}W_1/W_2 \) has basis \( y_2 + W_2 \). Each \( W_3/W_2 \) has order 3 and is thus contained in \( \Omega_1(\partial^{-1}W_2/W_2) \).
Let $c_7 \in \{0, 1, 2\}$. Let $m_7 = c_7 y_2 + v_4$. Thus

$$m_7 = \begin{bmatrix}
0 & 0 & q \\
0 & 3c_7 & 3(1 - c_2 c_4 c_5) \\
1 & 0 & 0
\end{bmatrix}.$$ 

Let $W_3 = \langle W_2, m_7 \rangle$. There are 3 subgroups of this type. Since $m_7 \not\in W_2$ and $3m_7 \in W_2$ then $|W_3/W_2| = 3$. Since $|W_2| = 3^9$ then $|W_3| = 3^{10}$. Then $\partial^{-1}W_2/W_3 \cong \mathbb{Z}_3 \times \mathbb{Z}_3$ and $\text{rank}(\partial^{-1}W_2/W_3) = 2$. By the Modified Terminal Lemma we conclude that $\partial^{-1}W_3 = \partial^{-1}W_2$. Thus $\text{rank}(\partial^{-1}W_3/W_3) = \text{rank}(\partial^{-1}W_2/W_3)$, $W_3$ is terminal, and $W_3 \not\in \mathcal{L}_3$.

**Case 7.4.3.4.1.4** Suppose $q \equiv 2$ and $r \equiv -c_2$. Then $v_4 + W_2, v_5 + W_2$ both have order 3. Recall $|\partial^{-1}W_2/W_2| = 3^3$. We know $\partial^{-1}W_2 = \langle \partial^{-1}W_1, v_4, v_5 \rangle$ and $\partial^{-1}W_1 = \langle W_2, y_2 \rangle$. Hence the group $\partial^{-1}W_2/W_2$ of order $3^3$ is generated by its three elements $y_2 + W_2, v_4 + W_2, v_5 + W_2$ of order 3. Hence $\partial^{-1}W_2/W_2$ is isomorphic to $\mathbb{Z}_3 \times \mathbb{Z}_3 \times \mathbb{Z}_3$ and has basis $y_2 + W_2, v_4 + W_2, v_5 + W_2$ while its subspace $\partial^{-1}W_1/W_2$ has basis $y_2 + W_2$. Since $\text{rank}(\partial^{-1}W_2/W_2) = 3$ and $\text{rank}(\partial^{-1}W_1/W_2) = 1$, then $\text{rank}(\partial^{-1}W_1/W_2) < \text{rank}(\partial^{-1}W_2/W_2)$. Hence $W_2$ is nonterminal and $W_2 \in \mathcal{L}_2$.

Thus we have some $W_3/W_2$ isomorphic to $\mathbb{Z}_3$ and some $W_3/W_2$ isomorphic $\mathbb{Z}_3 \times \mathbb{Z}_3$.

Therefore we have the following six cases.

In Case 7.4.3.4.1.4.1 we consider the 1-dimensional subspaces $W_3/W_2$. There are three possible forms for the matrix $m$. The first form is

$$m = [0, 0, 1]$$

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(1 possibility), which is considered in Case 7.4.3.4.1.4.1.1. The second form is

\[ m = [0, 1, c_7] \quad \text{for } c_7 \in \{0, 1, 2\} \]

(3 possibilities), which is considered in Case 7.4.3.4.1.4.1.2. The third form is

\[ m = [1, c_7, c_8] \quad \text{for } c_7, c_8 \in \{0, 1, 2\}, \quad (c_7, c_8) \neq (0, 0) \]

(8 possibilities), which is considered in Case 7.4.3.4.1.4.1.3.

In Case 7.4.3.4.1.4.2 we consider the 2-dimensional subspaces \( W_3/W_2 \). There are three possible forms for the matrix \( m \). The first form is

\[
\begin{bmatrix}
0 & 1 & 0 \\
0 & 0 & 1
\end{bmatrix}
\]

(1 possibility), which is considered in Case 7.4.3.4.1.4.2.1. The second form is

\[
\begin{bmatrix}
1 & c_7 & 0 \\
0 & 0 & 1
\end{bmatrix}
\quad \text{for } c_7 \in \{1, 2\}
\]

(2 possibilities), which is considered in Case 7.4.3.4.1.4.2.2. The third form is

\[
\begin{bmatrix}
1 & 0 & c_7 \\
0 & 1 & c_8
\end{bmatrix}
\quad \text{for } c_7 \in \{1, 2\} \quad \text{and } c_8 \in \{0, 1, 2\}
\]

(6 possibilities), which is considered in Case 7.4.3.4.1.4.2.3.

**Case 7.4.3.4.1.4.1.1** Let \( m_7 = v_5 \). Thus

\[
m_7 = \begin{bmatrix}
0 & 0 & 1 \\
0 & 1 & 3(c_6 + 1) \\
1 & 0 & 0
\end{bmatrix}
\]
Let $W_3 = \langle W_2, m_7 \rangle \in \mathcal{L}_3$. The number of subgroups $W_2$ of this type is 1.

Since $m_7 \notin W_2$ and $3m_7 \in W_2$ we have $|W_3/W_2| = 3$. Since $|W_2| = 3^9$ it follows that $|W_3| = 3^{10}$. By the Modified Terminal Lemma we conclude that $\partial^{-1}W_3 = \partial^{-1}W_2$.

Thus $\text{rank}(\partial^{-1}W_3/W_3) = \text{rank}(\partial^{-1}W_2/W_3)$, $W_3$ is terminal, and $W_3 \notin \hat{\mathcal{L}}_3$.

**Case 7.4.3.4.1.4.1.2** We fix arbitrary value $c_7 \in \{0, 1, 2\}$. Let $m_7 = v_4 + c_7v_5$. Thus

$$m_7 = \begin{bmatrix} 0 & 0 & 2 + c_7 \\ 0 & c_7 & 3(1 + c_5 + c_6c_7 + c_7) \\ 1 + c_7 & 0 & 0 \end{bmatrix}.$$ 

Let $W_3 = \langle W_2, m_7 \rangle \in \mathcal{L}_3$. The number of subgroups $W_2$ of this type is 3.

Since $m_7 \notin W_2$ and $3m_7 \in W_2$ we have $|W_3/W_2| = 3$. Since $|W_2| = 3^9$ it follows that $|W_3| = 3^{10}$. By the Modified Terminal Lemma we conclude that $\partial^{-1}W_3 = \partial^{-1}W_2$.

Thus $\text{rank}(\partial^{-1}W_3/W_3) = \text{rank}(\partial^{-1}W_2/W_3)$, $W_3$ is terminal, and $W_3 \notin \hat{\mathcal{L}}_3$.

**Case 7.4.3.4.1.4.1.3** We fix arbitrary values $c_7, c_8 \in \{0, 1, 2\}$ such that $(c_7, c_8) \neq (0, 0)$. Let $m_7 = y_2 + c_7v_4 + c_8v_5$.

Thus

$$m_7 = \begin{bmatrix} 0 & 0 & 2c_7 + c_8 \\ 0 & 3 + c_8 & 3(c_7(1 + c_5) + c_8(c_6 + 1)) \\ c_7 + c_8 & 0 & 0 \end{bmatrix}.$$ 

Let $W_3 = \langle W_2, m_7 \rangle \in \mathcal{L}_3$. The number of subgroups $W_2$ of this type is 8. Since $m_7 \notin W_2$ and $3m_7 \in W_2$ we have $|W_3/W_2| = 3$. Since $|W_2| = 3^9$ it follows that
|W_3| = 3^{10}. By the Modified Terminal Lemma we conclude that \( \partial^{-1}W_3 = \partial^{-1}W_2 \).

Thus \( \text{rank}(\partial^{-1}W_3/W_3) = \text{rank}(\partial^{-1}W_2/W_3) \), \( W_3 \) is terminal, and \( W_3 \notin \hat{L}_3 \).

**Case 7.4.3.4.1.4.2.1** Let \( m_7 = v_4 \) and \( m_8 = v_5 \). Thus

\[
\begin{align*}
m_7 &= \begin{bmatrix} 0 & 0 & 2 \\ 0 & 0 & 3(1 + c_5) \\ 1 & 0 & 0 \end{bmatrix} \quad \text{and} \quad m_8 = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 3(c_6 + 1) \\ 1 & 0 & 0 \end{bmatrix}.
\end{align*}
\]

Let \( W_3 = \langle W_2, m_7, m_8 \rangle \in L_3 \). The number of subgroups \( W_2 \) of this type is 1. Since \( m_7 \notin W_2, m_8 \notin W_2, 3m_7 \in W_2, \) and \( 3m_8 \in W_2 \) we have \( |W_3/W_2| = 3^2 \). Since \( |W_2| = 3^9 \) it follows that \( |W_3| = 3^{11} \). By the Modified Terminal Lemma we conclude that \( \partial^{-1}W_3 = \partial^{-1}W_2 \). Thus \( \text{rank}(\partial^{-1}W_3/W_3) = \text{rank}(\partial^{-1}W_2/W_3) \), \( W_3 \) is terminal, and \( W_3 \notin \hat{L}_3 \).

**Case 7.4.3.4.1.4.2.2** We fix arbitrary value \( c_7 \in \{1, 2\} \). Let \( m_7 = y_2 + c_7 v_4 \) and \( m_8 = v_5 \).

Thus

\[
\begin{align*}
m_7 &= \begin{bmatrix} 0 & 0 & 2c_7 \\ 0 & 3 & 3c_7(1 + c_5) \\ c_7 & 0 & 0 \end{bmatrix} \quad \text{and} \quad m_8 = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 3(c_6 + 1) \\ 1 & 0 & 0 \end{bmatrix}.
\end{align*}
\]

Let \( W_3 = \langle W_2, m_7, m_8 \rangle \in L_3 \). The number of subgroups \( W_2 \) of this type is 2. Since \( m_7 \notin W_2, m_8 \notin W_2, 3m_7 \in W_2, \) and \( 3m_8 \in W_2 \) we have \( |W_3/W_2| = 3^2 \). Since \( |W_2| = 3^9 \) it follows that \( |W_3| = 3^{11} \). By the Modified Terminal Lemma we conclude
that $\partial^{-1}W_3 = \partial^{-1}W_2$. Thus $\text{rank}(\partial^{-1}W_3/W_3) = \text{rank}(\partial^{-1}W_2/W_3)$, $W_3$ is terminal, and $W_3 \notin \hat{L}_3$.

**Case 7.4.3.4.1.4.2.3** We fix arbitrary values $c_7 \in \{1, 2\}$ and $c_8 \in \{0, 1, 2\}$. Let $m_7 = y_2 + c_7v_4$ and $m_8 = y_2 + c_8v_5$. Thus

$$m_7 = \begin{bmatrix} 0 & 0 & c_7 \\ 0 & 3 + c_7 & 3c_7(c_6 + 1) \\ c_7 & 0 & 0 \end{bmatrix} \quad \text{and} \quad m_8 = \begin{bmatrix} 0 & 0 & 2 + c_8 \\ 0 & c_8 & 3(1 + c_5 + c_8(c_6 + 1)) \\ 1 + c_8 & 0 & 0 \end{bmatrix}.$$ 

Let $W_3 = < W_2, m_7, m_8 > \in L_3$. The number of subgroups $W_2$ of this type is 6. Since $m_7 \notin W_2$, $m_8 \notin W_2$, $3m_7 \in W_2$, and $3m_8 \in W_2$ we have $|W_3/W_2| = 3^2$. Since $|W_2| = 3^9$ it follows that $|W_3| = 3^{11}$. By the Modified Terminal Lemma we conclude that $\partial^{-1}W_3 = \partial^{-1}W_2$. Thus $\text{rank}(\partial^{-1}W_3/W_3) = \text{rank}(\partial^{-1}W_2/W_3)$, $W_3$ is terminal, and $W_3 \notin \hat{L}_3$.

**Case 7.4.3.4.2** Suppose $c_4c_6 \equiv c_2$. Then (B5) becomes $qt_{2,0} + rt_{1,1} \equiv 0$ where $q = c_5c_6 - 1$ and $r = 1 - c_4c_5 - c_6$. Since $c_2 \neq 0$, the condition $c_2 \equiv c_4c_6$ forces $c_6 \neq 0$. Thus $c_4, c_6 \in \{1, 2\}$ while $c_5 \in \{0, 1, 2\}$. So there are 12 possibilities for the ordered triple $(c_4, c_5, c_6)$.
Hence $x \in \partial^{-1}W_2$ if and only if

\[ s_{2,2} \equiv c_4 c_0 t_{1,1} \quad \text{(A4)} \]
\[ s_{1,2} \equiv (1 - c_4^2 c_5 c_6) t_{2,0} - c_4^2 c_6 (c_6 + 1) t_{1,1} - c_4 c_6 s_{2,1} \quad \text{(A5)} \]

\[ qt_{2,0} + rt_{1,1} \equiv 0 \quad \text{(B5)}. \]

We now consider the following four cases separately. The first case is $q \equiv 0$ and $r \equiv 0$. The second case is $q \equiv 0$ and $r \not\equiv 0$. The third case is $q \not\equiv 0$ and $r \equiv 0$. The fourth case is $q \not\equiv 0$ and $r \not\equiv 0$.

**Case 7.4.2.4.2.1** Suppose $q \equiv 0$ and $r \equiv 0$. Then $t_{0,2}, t_{1,1},$ and $t_{2,0}$ are all free variables. We show that $q \equiv r \equiv 0$ if and only if $(c_4, c_5, c_6) = (1, 2, 2)$. The condition $q \equiv 0$ is equivalent to $c_5 = c_6 \not\equiv 0$. Hence the condition $r \equiv 0$ becomes $1 - c_4 c_5 - c_5 \equiv 0$, which says $1 - c_5 (c_4 + 1) \equiv 0$, which says $c_5 (c_4 + 1) \equiv 1$. Because $c_5 \not\equiv 0$, the condition $c_5 (c_4 + 1) \equiv 1$ is equivalent to $c_5 \equiv c_4 + 1$. Since $c_5 \not\equiv 0$ while $c_4 \in \{1, 2\}$, the condition $c_5 \equiv c_4 + 1$ is equivalent to $(c_4, c_5) = (1, 2)$. Now using $c_6 = c_5$ we get $c_6 = 2$. We regard $t_{0,2}, t_{1,1},$ and $t_{2,0}$ as free variables. Taking $t_{0,2} \equiv 1$, $t_{1,1} = 0$, and $t_{2,0} = 0$, the matrix $x$ becomes

\[
v_4 = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}.
\]
Taking \( t_{1,1} \equiv 1, t_{0,2} = 0, \) and \( t_{2,0} = 0, \) the matrix \( x \) becomes

\[
v_5 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}.
\]

Taking \( t_{2,0} \equiv 1, t_{0,2} = 0, \) and \( t_{1,1} = 0, \) the matrix \( x \) becomes

\[
v_6 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix}.
\]

We see that \( |\partial^{-1}W_2/W_2| = 3^4. \) By Lemma 10.4.1, we see that \( v_4 + W_2, v_5 + W_2, v_6 + W_2 \) each have order 9.

Note

\[
v_4 - v_6 = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{bmatrix}, \quad 3(v_4 - v_6) = \begin{bmatrix} 0 & 0 & 3 \\ 0 & 0 & -3 \\ 0 & 0 & 0 \end{bmatrix},
\]

\[
v_6 - v_4 = \begin{bmatrix} 0 & 0 & -1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix}, \quad 3(v_6 - v_4) = \begin{bmatrix} 0 & 0 & -3 \\ 0 & 0 & 0 \\ 3 & 0 & 0 \end{bmatrix}.
\]

Therefore by Lemma 10.4.1, we see that \( v_4 - v_6 + W_2 \) and \( v_6 - v_4 + W_2 \) have order 3. Hence the group \( \partial^{-1}W_2/W_2 \) is generated by its three elements \( y_2 + W_2, v_4 - v_6 + W_2, v_6 - v_4 + W_2 \) of order 3. Hence \( \partial^{-1}W_2/W_2 \) is isomorphic to \( \mathbb{Z}_9 \times \mathbb{Z}_3 \times \mathbb{Z}_3 \) and has basis \( y_2 + W_2, v_4 - v_6 + W_2, v_6 - v_4 + W_2 \) while its subspace \( \partial^{-1}W_1/W_2 \) has basis
\[ y_2 + W_2. \] Since \( \text{rank}(\partial^{-1}W_2/W_2) = 3 \) and \( \text{rank}(\partial^{-1}W_1/W_2) = 1 \), then
\[
\text{rank}(\partial^{-1}W_1/W_2) < \text{rank}(\partial^{-1}W_2/W_2). \]
Hence \( W_2 \) is nonterminal and \( W_2 \in \hat{\mathcal{L}}_2 \). Thus we have some \( W_3/W_2 \) isomorphic to \( \mathbb{Z}_3 \) and some \( W_3/W_2 \) isomorphic \( \mathbb{Z}_3 \times \mathbb{Z}_3 \). Therefore we have the following six cases.

In Case 7.4.3.4.2.1.1 we consider the 1-dimensional subspaces \( W_3/W_2 \). There are three possible forms for the matrix \( m \). The first form is

\[
m = [0, 0, 1]
\]
(1 possibility), which is considered in Case 7.4.3.4.2.1.1.1. The second form is

\[
m = [0, 1, c_7] \quad \text{for} \ c_7 \in \{0, 1, 2\}
\]
(3 possibilities), which is considered in Case 7.4.3.4.2.1.1.2. The third form is

\[
m = [1, c_7, c_8] \quad \text{for} \ c_7, c_8 \in \{0, 1, 2\}, \ (c_7, c_8) \neq (0, 0)
\]
(8 possibilities), which is considered in Case 7.4.3.4.2.1.1.3.

In Case 7.4.3.4.2.1.2 we consider the 2-dimensional subspaces \( W_3/W_2 \). There are three possible forms for the matrix \( m \). The first form is

\[
m = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}
\]
(1 possibility), which is considered in Case 7.4.3.4.2.1.2.1. The second form is

\[
m = \begin{bmatrix} 1 & c_7 & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad \text{for} \ c_7 \in \{1, 2\}
\]
(2 possibilities), which is considered in Case 7.4.3.4.2.2. The third form is

\[ m = \begin{bmatrix} 1 & 0 & c_7 \\ 0 & 1 & c_8 \end{bmatrix} \quad \text{for } c_7 \in \{1, 2\} \quad \text{and } c_8 \in \{0, 1, 2\} \]

(6 possibilities), which is considered in Case 7.4.3.4.2.1.2.3.

**Case 7.4.3.4.2.1.1** Let \( m_7 = v_6 - v_4 \). Thus

\[ m_7 = \begin{bmatrix} 0 & 0 & -1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix} \]

Let \( W_3 = \langle W_2, m_7 \rangle \in \mathcal{L}_3 \). The number of subgroups \( W_2 \) of this type is 1. Since \( m_7 \not\in W_2 \) and \( 3m_7 \in W_2 \) we have \( |W_3/W_2| = 3 \). Since \( |W_2| = 3^9 \) it follows that \( |W_3| = 3^{10} \). By the Modified Terminal Lemma we conclude that \( \partial^{-1}W_3 = \partial^{-1}W_2 \).

Thus \( \text{rank}(\partial^{-1}W_3/W_3) = \text{rank}(\partial^{-1}W_2/W_3) \), \( W_3 \) is terminal, and \( W_3 \not\in \hat{\mathcal{L}}_3 \).

**Case 7.4.3.4.2.1.1.2** We fix arbitrary value \( c_7 \in \{0, 1, 2\} \). Let \( m_7 = v_4 - v_6 + c_7(v_6 - v_4) \).

Thus

\[ m_7 = \begin{bmatrix} 0 & 0 & 1 - c_7 \\ 0 & 0 & 0 \\ -1 + c_7 & 0 & 0 \end{bmatrix} \]

Let \( W_3 = \langle W_2, m_7 \rangle \in \mathcal{L}_3 \). The number of subgroups \( W_2 \) of this type is 3. Since \( m_7 \not\in W_2 \) and \( 3m_7 \in W_2 \) we have \( |W_3/W_2| = 3 \). Since \( |W_2| = 3^9 \) it follows that
\(|W_3| = 3^{10}\). By the Modified Terminal Lemma we conclude that \(\partial^{-1}W_3 = \partial^{-1}W_2\).

Thus \(\text{rank}(\partial^{-1}W_3/W_3) = \text{rank}(\partial^{-1}W_2/W_3)\), \(W_3\) is terminal, and \(W_3 \notin \hat{\mathcal{L}}_3\).

**Case 7.4.3.4.2.1.1.3** We fix arbitrary values \(c_7, c_8 \in \{0, 1, 2\}\) such that \((c_7, c_8) \neq (0, 0)\). Let \(m_7 = y_2 + c_7(v_4 - v_6) + c_8(v_6 - v_4)\). Thus

\[
m_7 = \begin{bmatrix}
0 & 0 & c_7 - c_8 \\
0 & 3 & 0 \\
-c_7 + c_8 & 0 & 0
\end{bmatrix}.
\]

Let \(W_3 = <W_2, m_7> \in \mathcal{L}_3\). The number of subgroups \(W_2\) of this type is 8. Since \(m_7 \notin W_2\) and \(3m_7 \in W_2\) we have \(|W_3/W_2| = 3\). Since \(|W_2| = 3^9\) it follows that \(|W_3| = 3^{10}\). By the Modified Terminal Lemma we conclude that \(\partial^{-1}W_3 = \partial^{-1}W_2\).

Thus \(\text{rank}(\partial^{-1}W_3/W_3) = \text{rank}(\partial^{-1}W_2/W_3)\), \(W_3\) is terminal, and \(W_3 \notin \hat{\mathcal{L}}_3\).

**Case 7.4.3.4.1.2.1.2.1** Let \(m_7 = v_4 - v_6\) and \(m_8 = v_6 - v_4\). Thus

\[
m_7 = \begin{bmatrix}
0 & 0 & 1 \\
0 & 0 & 0 \\
-1 & 0 & 0
\end{bmatrix} \quad \text{and} \quad m_8 = \begin{bmatrix}
0 & 0 & -1 \\
0 & 0 & 0 \\
1 & 0 & 0
\end{bmatrix}.
\]

Let \(W_3 = <W_2, m_7, m_8> \in \mathcal{L}_3\). The number of subgroups \(W_2\) of this type is 1. Since \(m_7 \notin W_2\), \(m_8 \notin W_2\), \(3m_7 \in W_2\), and \(3m_8 \in W_2\) we have \(|W_3/W_2| = 3^2\). Since \(|W_2| = 3^9\) it follows that \(|W_3| = 3^{11}\). By the Modified Terminal Lemma we conclude that \(\partial^{-1}W_3 = \partial^{-1}W_2\). Thus \(\text{rank}(\partial^{-1}W_3/W_3) = \text{rank}(\partial^{-1}W_2/W_3)\), \(W_3\) is terminal, and \(W_3 \notin \hat{\mathcal{L}}_3\).
**Case 7.4.3.4.2.1.2.2** We fix arbitrary value $c_7 \in \{1, 2\}$. Let $m_7 = y_2 + c_7(v_4 - v_6)$ and $m_8 = v_6 - v_4$. Thus

$$m_7 = \begin{bmatrix} 0 & 0 & c_7 \\ 0 & 3 & 0 \\ -c_7 & 0 & 0 \end{bmatrix} \quad \text{and} \quad m_8 = \begin{bmatrix} 0 & 0 & -1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix}.$$ 

Let $W_3 = \langle W_2, m_7, m_8 \rangle \in \mathcal{L}_3$. The number of subgroups $W_2$ of this type is 2. Since $m_7 \notin W_2$, $m_8 \notin W_2$, $3m_7 \in W_2$, and $3m_8 \in W_2$ we have $|W_3/W_2| = 3^2$. Since $|W_2| = 3^3$ it follows that $|W_3| = 3^{11}$. By the Modified Terminal Lemma we conclude that $\partial^{-1}W_3 = \partial^{-1}W_2$. Thus $\text{rank}(\partial^{-1}W_3/W_3) = \text{rank}(\partial^{-1}W_2/W_3)$, $W_3$ is terminal, and $W_3 \notin \hat{\mathcal{L}}_3$.

**Case 7.4.3.4.2.1.2.3** We fix arbitrary values $c_7 \in \{1, 2\}$ and $c_8 \in \{0, 1, 2\}$. Let $m_7 = y_2 + c_7(v_4 - v_6)$ and $m_8 = y_2 + c_8(v_6 - v_4)$. Thus

$$m_7 = \begin{bmatrix} 0 & 0 & c_7 \\ 0 & 3 & 0 \\ -c_7 & 0 & 0 \end{bmatrix} \quad \text{and} \quad m_8 = \begin{bmatrix} 0 & 0 & -c_8 \\ 0 & 3 & 0 \\ c_8 & 0 & 0 \end{bmatrix}.$$ 

Let $W_3 = \langle W_2, m_7, m_8 \rangle \in \mathcal{L}_3$. The number of subgroups $W_2$ of this type is 6. Since $m_7 \notin W_2$, $m_8 \notin W_2$, $3m_7 \in W_2$, and $3m_8 \in W_2$ we have $|W_3/W_2| = 3^2$. Since $|W_2| = 3^3$ it follows that $|W_3| = 3^{11}$. By the Modified Terminal Lemma we conclude that $\partial^{-1}W_3 = \partial^{-1}W_2$. Thus $\text{rank}(\partial^{-1}W_3/W_3) = \text{rank}(\partial^{-1}W_2/W_3)$, $W_3$ is terminal, and $W_3 \notin \hat{\mathcal{L}}_3$.

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Case 7.4.3.4.2.2 Suppose $q \equiv 0$ and $r \neq 0$. We show that these conditions hold if and only if $(c_4, c_5, c_6)$ is $(1,1,1)$ or $(2,1,1)$ or $(2,2,2)$. The condition $q \equiv 0$ is equivalent to $c_5 = c_6 \neq 0$. Hence the condition $r \neq 0$ becomes $1 - c_4 c_5 - c_5 \neq 0$, which says $1 - c_5(c_4 + 1) \neq 0$, which says $c_5(c_4) \neq 1$. Now recall $c_4,c_5 \in \{1,2\}$. The only way to have $c_5(c_4 + 1) \equiv 1$ is when $(c_4,c_5) = (1,2)$. Hence the condition $c_5(c_4 + 1) \neq 1$ occurs if and only if $(c_4,c_5)$ is either $(1,1)$ or $(2,1)$ or $(2,2)$. If $q \equiv 0$, and $r \neq 0$, then (B5) is equivalent to $t_{1,1} \equiv 0$. Therefore $t_{0,2}$ and $t_{2,0}$ are both free variables. Taking $t_{2,0} \equiv 1$ and $t_{0,2} = 0$, the matrix $x$ becomes

$$v_4 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix}.$$  

Taking $t_{0,2} \equiv 1$ and $t_{2,0} = 0$, the matrix $x$ becomes

$$v_5 = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$  

By Lemma 10.4.1, we see that both $v_4 + W_2$ and $v_5 + W_2$ have order 9. Note

$$v_4 - v_5 = \begin{bmatrix} 0 & 0 & -1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix}.$$  

Thus by Lemma 10.4.1, we see that $v_4 - v_5 + W_2$ has order 3. We see that $\partial^{-1}W_2/W_2 \cong \mathbb{Z}_9 \times \mathbb{Z}_3$. Note that $\Omega(\partial^{-1}W_2/W_2) = \partial^{-1}W_1/W_2$. $\Omega_{1}(\partial^{-1}W_2/W_2)$ is isomorphic to
$\mathbb{Z}_3 \times \mathbb{Z}_3$ and had basis $y_2 + W_2, v_4 - v_5 + W_2$, while its subspace $\partial^{-1}W_1/W_2$ has basis $y_2 + W_2$. Each $W_3/W_2$ has order 3 and it thus contained in $\Omega_1(\partial^{-1}W_2/W_2)$. Let $c_7 \in \{0, 1, 2\}$. Let $m_7 = c_7 y_2 + v_4 - v_5$. Thus

$$
\begin{bmatrix}
0 & 0 & -1 \\
0 & 3c_7 & 0 \\
1 & 0 & 0
\end{bmatrix}
$$

Let $W_3 = \langle W_2, m_7 \rangle$. There are 3 subgroups of this type. Since $m_7 \not\in W_2$ and $3m_7 \in W_2$ then $|W_3/W_2| = 3$. Since $|W_2| = 3^9$ then $|W_3| = 3^{10}$. Then $\partial^{-1}W_2/W_3 \cong \mathbb{Z}_3 \times \mathbb{Z}_3$ and rank $(\partial^{-1}W_2/W_3) = 2$. By the Modified Terminal Lemma, we conclude that $\partial^{-1}W_3 = \partial^{-1}W_2$. Thus rank$(\partial^{-1}W_3/W_3) =$ rank$(\partial^{-1}W_2/W_3)$, $W_3$ is terminal, and $W_3 \not\in \hat{L}_3$.

**Case 7.4.3.4.2.3** Suppose $q \not\equiv 0$ and $r \equiv 0$. Then (B5) is equivalent to $t_{2,0} \equiv 0$ and $t_{1,1}$ and $t_{0,2}$ are both free variables. We show that $q \not\equiv 0$ and $r \equiv 0$ if and only if $(c_4, c_5, c_6)$ is $(1, 0, 1)$ or $(2, 0, 1)$ or $(2, 1, 2)$. Recall $c_5 \in \{0, 1, 2\}$. We argue by cases. Suppose $c_5 = 0$. Then condition $q \not\equiv 0$ is automatic while condition $r \equiv 0$ is equivalent to $c_6 = 1$. Recalling that $c_4 \in \{1, 2\}$ we get $(c_4, c_5, c_6)$ equals $(1, 0, 1)$ or $(2, 0, 1)$. Now suppose $c_5 \in \{1, 2\}$. Since $c_6 \in \{1, 2\}$, the condition $q \not\equiv 0$ is equivalent to $c_6 \equiv -c_5$. Thus condition $r \equiv 0$ becomes $1 - c_4 c_5 + c_5 \equiv 0$, or $1 + c_5 (1 - c_4) \equiv 0$, or $c_5 (1 - c_4) \equiv 2$. Since $c_4 \in \{1, 2\}$, condition $c_5 (1 - c_4) \equiv 2$ happens if and only if $(c_4, c_5) = (2, 1)$. Recalling $c_6 \equiv -c_5$, we get $(c_4, c_5, c_6) = (2, 1, 2)$. Therefore we see that $q \not\equiv 0$ and $r \equiv 0$ holds if and only if $(c_4, c_5, c_6)$ equals $(1, 0, 1)$ or $(2, 0, 1)$ or $(2, 1, 2)$.
We regard $t_{0,2}$ and $t_{1,1}$ as free variables. Taking $t_{0,2} \equiv 1$ and $t_{1,1} = 0$, the matrix $x$ becomes

$$
v_4 = \begin{bmatrix}
0 & 0 & 1 \\
0 & 0 & 0 \\
0 & 0 & 0
\end{bmatrix}.
$$

Taking $t_{1,1} \equiv 1$ and $t_{0,2} = 0$, the matrix $x$ becomes

$$
v_5 = \begin{bmatrix}
0 & 0 & 0 \\
0 & 1 & -3 + c_6 \\
0 & 3c_4c_6
\end{bmatrix}.
$$

By Lemma 10.4.1, we see that $v_4 + W_2$ and $v_5 + W_2$ both have order 9. Note

$$
-c_2v_4 + v_5 = \begin{bmatrix}
0 & 0 & -c_2 \\
0 & 1 & -3 + c_6 \\
0 & 0 & 3c_4c_6
\end{bmatrix} \quad \text{and} \quad 3(-c_2v_4 + v_5) = \begin{bmatrix}
0 & 0 & -3c_2 \\
0 & 3 & 0 \\
0 & 0 & 0
\end{bmatrix}.
$$

By Lemma 10.4.1, we see that $-c_2v_4 + v_5 + W_2$ has order 3. Hence $\partial^{-1}W_2/W_2 \cong \mathbb{Z}_9 \times \mathbb{Z}_3$. Therefore rank(\partial^{-1}W_2/W_1) < rank(\partial^{-1}W_2/W_2) and $W_2$ is nonterminal. $\partial^{-1}W_2/W_2$ has basis $y_2 + W_2, -c_2v_4 + v_5 + W_2$ while its subspace $\partial^{-1}W_2/W_1$ has basis $y_2 + W_2$. We fix $c_7 \in \{0, 1, 2\}$. Let $m_7 = c_7y_2 + -c_2v_4 + v_5$. Thus

$$
m_7 = \begin{bmatrix}
0 & 0 & -c_2 \\
0 & 3c_7 + 1 & -3 + c_6 \\
0 & 0 & 3c_4c_6
\end{bmatrix}.
$$

Let $W_3 = < W_2, m_7 >$. There are 3 subgroups of this type. Since $m_7 \notin W_2$ and $3m_7 \in W_2$ then $|W_3/W_2| = 3$. Since $|W_2| = 3^9$ then $|W_3| = 3^{10}$. Then
\( \partial^{-1}W_2/W_3 \cong \mathbb{Z}_3 \times \mathbb{Z}_3 \) and rank \( \partial^{-1}W_2/W_3 \) = 2. By the Modified Terminal Lemma, we conclude that \( \partial^{-1}W_3 = \partial^{-1}W_2 \). Thus rank\( \partial^{-1}W_3/W_3 \) = rank\( \partial^{-1}W_2/W_3 \), \( W_3 \) is terminal, and \( W_3 \not\in \hat{\mathcal{L}}_3 \).

**Case 7.4.3.4.2.4** Suppose \( q \not\equiv 0 \) and \( r \not\equiv 0 \). Then (B5) becomes \( t_{1,1} \equiv -qrt_{2,0} \) and \( t_{0,2} \) and \( t_{2,0} \) are free variables. We show that \( q \not\equiv 0 \) and \( r \not\equiv 0 \) if and only if \((c_4, c_5, c_6)\) is equal to either \((1, 0, 2)\) or \((2, 0, 2)\) or \((1, 1, 2)\) or \((1, 2, 1)\) or \((2, 2, 1)\). Recall there are exactly 12 possibilities for \((c_4, c_5, c_6)\). The five ordered triples appearing here are precisely those that do not appear in the preceding three cases. We regard \( t_{0,2} \) and \( t_{2,0} \) as our new free variables. Taking \( t_{0,2} \equiv 1 \) and \( t_{2,0} = 0 \), the matrix \( x \) becomes

\[
v_4 = \begin{bmatrix}
0 & 0 & 1 \\
0 & 0 & 0 \\
0 & 0 & 0
\end{bmatrix}.
\]

Taking \( t_{2,0} \equiv 1 \) and \( t_{0,2} = 0 \), the matrix \( x \) becomes

\[
v_5 = \begin{bmatrix}
0 & 0 & 0 \\
0 & -qr & 3(c_6qr - c_5c_6) \\
0 & 0 & -3c_4c_6qr
\end{bmatrix}.
\]

Take \( r_1 = 1, r_2 = -qr, r_3 = 0 \). Then \( r_1 + c_2r_2 + r_3 \equiv 1 - qrc_2 \). \( 1 - qrc_2 \equiv 0 \) holds if and only if \((c_4, c_5, c_6)\) equals \((2, 0, 2)\) or \((1, 1, 2)\) or \((2, 2, 1)\). Then by Lemma 10.4.1, \( v_5 + W_2 \) has order 3 and \( v_4 + W_2 \) has order 9. We see \( \partial^{-1}W_2/W_2 \cong \mathbb{Z}_9 \times \mathbb{Z}_3 \) and has basis \( y_2 + W_2, v_5 + W_2 \) while its subspace \( \partial^{-1}W_2/W_1 \) has basis \( y_2 + W_2 \).
We fix \( c_7 \in \{0, 1, 2\} \). Let \( m_7 = c_7y_2 + v_5 \). Thus

\[
m_7 = \begin{bmatrix}
0 & 0 & 0 \\
0 & 3c_7 - qr & 3(c_6qr - c_5c_6) \\
0 & 0 & -3c_4c_6
\end{bmatrix}.
\]

Let \( W_3 = \langle W_2, m_7 \rangle \). There are 3 subgroups of this type for each of the three \( W_2 \) subgroups that satisfied \( r_1 + c_2r_2 + r_3 \equiv 0 \). Since \( m_7 \not\in W_2 \) and \( 3m_7 \in W_2 \) then \( |W_3/W_2| = 3 \). Since \( |W_2| = 3^9 \) then \( |W_3| = 3^{10} \). Then \( \partial^{-1}W_2/W_3 \cong \mathbb{Z}_3 \times \mathbb{Z}_3 \) and \( \text{rank}(\partial^{-1}W_2/W_3) = 2 \). By the Modified Terminal Lemma, we conclude that \( \partial^{-1}W_3 = \partial^{-1}W_2 \). Thus \( \text{rank}(\partial^{-1}W_3/W_3) = \text{rank}(\partial^{-1}W_2/W_3) \), \( W_3 \) is terminal, and \( W_3 \not\in \hat{L}_3 \).

We see that for \((c_4, c_5, c_6)\) equal to \((1, 0, 2)\) and \((1, 1, 2)\) that \( r_1 + c_2r_2 + r_3 \not\equiv 0 \) and therefore \( v_4 + W_2 \) and \( v_5 + W_2 \) have order 9. Note

\[
v_4 + v_5 = \begin{bmatrix}
0 & 0 & 1 \\
0 & -qr & 3(c_6qr - c_5c_6) \\
1 & 0 & -3c_4c_6
\end{bmatrix} \text{ and } 3(v_4 + v_5) = \begin{bmatrix}
0 & 0 & 3 \\
0 & -3qr & 0 \\
3 & 0 & 0
\end{bmatrix}.
\]

By Lemma 10.4.1, when \((c_4, c_5, c_6)\) equals \((1, 0, 2)\) or \((1, 1, 2)\) \( r_1 + c_2r_2 + r_3 \equiv 0 \). Hence \( v_4 + v_5 + W_2 \) has order 3. We see \( \partial^{-1}W_2/W_2 \cong \mathbb{Z}_9 \times \mathbb{Z}_3 \) and has basis \( y_2 + W_2, v_4 + v_5 + W_2 \) while its subspace \( \partial^{-1}W_2/W_1 \) has basis \( y_2 + W_2 \). We fix \( c_7 \in \{0, 1, 2\} \). Let \( m_7 = c_7y_2 + v_4 + v_5 \). Thus

\[
m_7 = \begin{bmatrix}
0 & 0 & 1 \\
0 & 3c_7 - qr & 3(c_6qr - c_5c_6) \\
1 & 0 & -3c_4c_6
\end{bmatrix}.
\]
Let \( W_3 = \langle W_2, m_7 \rangle \). There are 3 subgroups of this type for each of the 2 \( W_2 \) subgroups that satisfied \( r_1 + c_2 r_2 + r_3 \equiv 0 \). Since \( m_7 \not\in W_2 \) and \( 3m_7 \notin W_2 \) then \( |W_3/W_2| = 3 \). Since \( |W_2| = 3^9 \) then \( |W_3| = 3^{10} \), \( \partial^{-1} W_2/W_3 \cong \mathbb{Z}_3 \times \mathbb{Z}_3 \), and rank \( (\partial^{-1} W_2/W_3) = 2 \). By the Modified Terminal Lemma, we obtain \( \partial^{-1} W_3 = \partial^{-1} W_2 \).

Thus \( \text{rank}(\partial^{-1} W_3/W_3) = \text{rank}(\partial^{-1} W_2/W_3) \), \( W_3 \) is terminal, and \( W_3 \not\in \hat{L}_3 \).

In Case 7 we found 8 subgroups \( W_1 \in L_1 \), all of which satisfy \( |W_1| = 3^6 \) and all of which are nonterminal. We found that 6 of these nonterminal members of \( L_1 \) are each contained in 21 members of \( L_2 \) and 2 of these nonterminal members of \( L_2 \) are each contained in 183 members of \( L_2 \). Thus we found 492 subgroups \( W_2 \) of \( L_2 \), 150 of which satisfy \( |W_2| = 3^7 \), 288 of which satisfy \( |W_2| = 3^8 \), and 54 of which satisfy \( |W_2| = 3^9 \). Each of the 150 members of \( L_2 \) that satisfy \( |W_2| = 3^7 \) is terminal. Exactly 69 members of \( L_2 \) that satisfy \( |W_2| = 3^8 \) are nonterminal. Each of these 69 nonterminal members of \( L_2 \) is contained in 9 members of \( L_3 \). Exactly 48 members of \( L_2 \) that satisfy \( |W_2| = 3^9 \) are nonterminal and are each contained in 3 members of \( L_3 \). Exactly 6 members of \( L_2 \) that satisfy \( |W_2| = 3^9 \) are nonterminal and are each contained in 21 members of \( L_3 \). Thus we found 891 subgroups \( W_3 \in L_3 \), 621 of which satisfy \( |W_3| = 3^9 \), 216 of which satisfy \( |W_3| = 3^{10} \), and 54 of which satisfy \( |W_3| = 3^{11} \). Every member of \( L_3 \) is terminal. So in Case 7 we found a total of 8 + 492 + 891 = 1,391 subgroups.

In summary, the number of subgroups that we have found in each of the seven previous cases are 29, 62, 53, 508, 277, 568, and 1,391 respectively. Thus in total we have found 1 + 29 + 62 + 53 + 508 + 277 + 568 + 1,391 = 2,889 subgroups.
BIBLIOGRAPHY
