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Elementary Hyperreal Analysis

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Abstract

This text explores elementary analysis through the lens of non-standard analysis. The hyperreals will be proven to be implied by the existence of the reals via the axiom of choice. The notion of a hyperextension will be defined, and the so-called Transfer Principle will be proved. This principle establishes equivalence between results in real and hyperreal analysis. Sequences, subsequences, and limit suprema/infima will then be explored. Finally, integration will be considered.

Introduction

In the middle of the last century, Abraham Robinson showed that one could extend the real numbers to the hyperreal numbers, and that the methods, definitions, and theorems of the so-called non-standard analysis performed on the hyperreals are equivalent to the methods, definitions, and theorems of standard analysis. However, Robinson's work is very complicated, and it requires a degree of expertise to fully digest and understand. There has been some interest in this field, but many of the texts are either at the most basic level, or are designed with graduate students in mind - either the proofs are neglected, or they are inaccessible to an undergraduate student. This work is an attempt to partially bridge the gap in the actually existing texts.

Robinson's method allows for one to talk about limits and calculus in terms of infinities and infinitesimals. This is not something Robinson made up, and it is said that he designed this system to be a formalization of Leibniz's intuition [3]. There is evidence to suggest that in order to solve optimization problems, Fermat imagined a number x such that x was not equal to zero, but could be made to "vanish" [8]. This idea was used extensively by Liebniz, who interpreted the differential to be such an x. L'Hospital and Johann Bernoulli continued this tradition in the first published Calculus textbook. Even as late as Cauchy, infinitesimal methods were still employed in otherwise rigorous proofs [3].

The problem, however, was that it was never shown how such an infinitesimal could come into being. These mathematicians were relying solely on intuition, and even in the hyperreals there is no x such that x is non-zero, but x can be made to "vanish". In order to satiate the need for rigor, Bolzano and Weierstrass discovered the ϵ-δ definition of a limit.

It was in the 60s when Abraham Robinson gave new life to the infinitesimal. He had discovered that the reason no one until then was able to formalize the idea of an infinitesimal was because a satisfactory definition of it relied on the Axiom of Choice, an axiom not investigated until 1904 by Zermelo. In developing it, Robinson did not merely use this for Calculus on the real line, but developed ideas of topology, metric spaces, and multivariable calculus in *Non-Standard Analysis* [1]. This, coupled with the fact his book is very dense, and assumes very intimate familiarity with very advanced logic and model theory, makes the book inaccessible to many. There are other texts which I will be making use of. Robert Goldblatt's text is designed with graduate students in mind, making it on the more advanced side. Jerome Keisler has written two books, one for professors, and another designed for Calculus I students - meaning that the book will lack a level of rigor. James Henle and Eugene Kleinberg have presented a short text, that is only a tip of the iceberg. Their book, while being the most concise and informative out of those listed, is not so much designed to prepare a reader for analysis, but to show that the basic ideas of Calculus I and Calculus II can be swiftly proved using the hyperreals (at the end, more advanced ideas are briefly presented). Henle and Kleinberg also use the notion of "transfinite induction" in their proof of the existence of an Ultrafilter, an advanced topic (in this text, Zorn's lemma is used instead).

The following work will attempt to develop the most basic notions of elementary analysis, in a way mirroring the way that it is usually taught to undergraduates. That is, it will present a theory of real-valued sequences and integration. The idea is to develop standard and non-standard analysis simultaneously. This will be accomplished by presenting the non-standard definitions, using non-standard methods for the proofs, but immediately presenting the corresponding standard definition, in an attempt to show that they complement one another. In order to accomplish the develop of elementary analysis, I will be following the structure (and sometimes referencing) Kenneth Ross's *Elementary Analysis.*

First things first, it will need to be established that the hyperreals are something that actually exist, and this will require a bit of set theory. Basically, a hyperreal number will be defined as an equivalent class of sequences, where two sequences are in the same class if the number of terms they agree on is "large." A large set will be defined as a set that is in a particular ultrafilter, and thus, a definition of an ultrafilter and filter will need to be given. But it isn't enough to define an ultrafilter, one has to show that one exists. In order to do that, one must first define what a chain is, and state Zorn's Lemma.

Then, with largeness having been defined, one may safely (for the purposes of this work) ignore the machinery built to define largeness. With the concept of largeness, one is now free to develop the hyperreal number system. One may establish what it means for two hyperreals to be equal, less than, or greater than, one may show how to add, divide, subtract, and multiply them. Then one shows that it contains the real numbers (or, at least, a field isomorphic to them), infinites, and infinitesimals. After we have "hyperextended" the reals to the hyperreals, we similarly "hyperextend" functions of real numbers to functions of hyperreal numbers.

A number of useful theorems will then be proved. First, it will be shown that non-infinite hyperreals are “infinitely close” to exactly one standard (real) number. The standard function, which “rounds” a hyperreals to the closest real, can then be defined. It will be shown that hyperreals can be expressed in decimal notation. Finally, it will be shown that certain logical statements pertaining to the reals are true if and only if the “corresponding” statement pertaining to the hyperreals is true - this is the Transfer Principle, which is of tremendous importance to non-standard analysis.

With the hyperreals having been established, one can then move on to the actual content of *Elementary Analysis*. The first thing to be defined will be the non-standard definition of a limit, and then the definition of a divergent sequence. Then, the definition of a bounded sequence will be introduced, and it will be shown that convergent sequences are bounded. Basic theorems about adding sequences and multiplying sequences by constants will be established, as well as the limits of four special sequences. The idea of a monotonic sequence will be defined, as well as theorems about monotonic sequences and their limits. Finally, Cauchy sequences will be defined, and it will be demonstrated that all Cauchy sequences are convergent, and vice versa. Examples of how to use these definitions and the corresponding standard definition (as well as a proof of their equivalence) will be presented.

Then, subsequences will be explored. First, the definition of a subsequence will be introduced, as well as a theorem establishing non-standard conditions for the limits of subsequences. Thereafter, it will be demonstrated that every sequence has a monotonic subsequence. This segues nicely into the Bolzano-Weierstrass Theorem. Finally, it will be demonstrated how the notion of limit suprema and infima is rendered in non-standard analysis, something left unmentioned in many texts.

The last chapter is about integration. In introductory courses, and in the sciences, one is told that the integral is a limit of a sum - the “differential” part going to zero, and the number of summands going to infinity. Instead of formalizing this concept - the integral as a limit of sequences - Ross prefers instead to introduce the notion of Lower and Upper Darboux Sums. Instead, this paper explores and formalizes the idea of the Integral as a limit of Riemann sums, by looking at the limit of Riemann sums whose associated partition's mesh goes to zero.

This has the added difficulty that many texts, articles, and books on non-standard analysis do not present a definition of the integral that is equivalent to the standard Riemann integral. Obviously, while that integral may be interesting, it presents a problem if non-standard analysis is conceived of as an alternative view, rather than an alternative theory. The reason for the difference in definitions is that the non-standard texts simply look at uniform partitions when defining the integral. However, Ross (and any textbook that is both standard and rigorous) looks at any partitioning of an interval, uniform or non-uniform. For example, both sets of authors will partition the interval $[0, 1]$ into $\{0, \frac{1}{3}, \frac{2}{3}, 1\}$, only Ross would also consider $\{0, \frac{1}{7}, \frac{6}{7}, 1\}$.

So, to account for this, two, somewhat technical, lemmas will be proved. The first states: Let f be a bounded function on $[a,b]$ such that for every partition P_n such that $\lim_{n\to\infty} mesh(P_n) = 0$, if S_n and R_n are both associated with P_n, then $\lim_{n\to\infty} S_n - R_n = 0$. Then, for all Riemann sums T_n and B_n such that their (respectively) associated partitions Q_n and M_n satisfies $\lim_{n\to\infty} mesh(Q_n) = \lim_{n\to\infty} mesh(P_n) = 0$, then $\lim_{n\to\infty} T_n - B_n = 0$.

This lemma, and the Bolzano-Weierstrass Theorem, are then used to prove the second lemma, which states: Let f be a bounded function on $[a,b]$ such that for every partition P_n such that $\lim_{n\to\infty} mesh(P_n) = 0$, if S_n and R_n are both associated with P_n, then $\lim_{n\to\infty} S_n - R_n = 0$. Then f is Riemann integrable. Then, using this condition of integrability, the functions Ross shows to be integrable will be shown to be integrable.

Finally, the Fundamental Theorem of Calculus will be proved, in which the first part fundamental theorem of Calculus reduces to a telescoping sum.

Chapter 1: Construction

The Basic Idea One wants to come up with a set of numbers that contains the reals, numbers whose magnitudes are greater than every real (the infinites), and non-zero numbers whose magnitudes are less than every positive real number. To do that, one identifies the so-called hyperreals with an equivalence class of sequences of real numbers, which requires a notion of largeness. Largeness is not hard to define, but it requires a little bit of set theory, specifically Zorn's Lemma. Then, after the existence of the hyperreals is established, one wants to show that they can be compared, define a notion of "positive" and "negative" hyperreals, and show that functions can be "extended" to the hyperreals. Next one wants to show that every finite hyperreal is "infinitely close" to a real number, so that one can make sense of the phrase "rounded to the nearest real number." Finally, one proves the Transfer Principle, that delineates a collection of logical statements that are true in the reals if and only if they are true in the hyperreals.

Definition: Let S be a set of sets. Let C be a subset of S such that for all A, B $\in$ C, either A $\subset$ B or B $\subset$ A. Then C is called a chain in S.

Definition: Let S be a set of sets. Let M $\in$ S. M is said to be maximal if for all A $\in$ S, M $\subset$ A implies M = A.

Definition: Let S be a set of sets. Let A be a subset of S, and U be in S. U is said to be an upperbound on A, if $\forall$B $\in$ A, B $\subset$ U, that is, every set in S is a subset of U.

Zorn's Lemma: Let S be a set of sets with the property that every chain C has an upperbound U in S. Then S contains at least one maximal element

Zorn's Lemma is much more general to this, and applies to all partially ordered sets. In addition, it can be shown that Zorn's Lemma is equivalent to the axiom of choice.

Definition: Let S be a set. A filter of S, F, is a subset of the powerset of S such that

1. $S \in F$,
2. if $A, B \in F$, then $A \cap B \in F$,
3. if $A \subset B \subset S$ and $A \in F$ then $B \in F$. If $\emptyset \notin F$ then F is said to be a proper filter.

Example: A subset of the natural numbers is said to be cofinite if its complement is a finite set. The set, H, of cofinite subsets of the natural numbers is a proper filter. Proof: 1) $\mathbb{N} \in H$, 2) Let $A, B \in H$, then A^c and B^c are both finite, so $A^c \cup B^c$ is cofinite. But $A^c \cup B^c = A \cap B$. Thus $A \cap B \in H$, 3) Let $A \subset B$, if A is cofinite, B is. Finally, the complement of the empty set is the naturals, an infinite set.

Definition: Let S be a set and H be a set of subsets on S. F is a filter generated by H if F is a filter and H is a subset of F.

Definition: Let S be a set. An ultrafilter, UF, on S is a filter such that for all subsets $A \subset S$ either $A \in UF$ or $A^c \in UF$. An ultrafilter that does not contain the empty set is a proper ultrafilter.

Theorem: There is at least one proper principal ultrafilter on the naturals generated by the set of cofinite sets [5].

Proof. Let M be the set of all proper filters generated by the set of cofinite sets. Let C be a chain in M, and U be the union of all of its elements. First note that U must be a proper filter generated by the set of cofinite sets, so it is in M. Clearly, for all $a \in C$, $a \in C$, implies $a \subset U$, so U is actually an upperbound for C in M. Thus every chain has an upperbound in U, so it has a maximal element, call it UF. To prove UF is an ultrafilter, assume that it isn't. Then, there must be a set A or A^c that is not in UF. However, consider all filters generated by $UF \cup \{A\}$ they are generated by UF, so they are generated by the set of cofinite sets, so they are in M. But then, there is an element that is a superset of UF, contradiction. □

Definition: Fix a particular proper ultrafilter, UF, on the naturals. This particular ultrafilter is not constructed - the previous theorem shows there is an ultrafilter, just not what it is. For example, it is unknown if the evens or odds are contained in this particular ultrafilter - what is known, however, is that exactly one of them is contained in UF. If $A \in UF$, A is said to be large.

Theorem:

1. $\mathbb{N}$ is large
2. If A is large and B is a superset of A, B is large
3. The empty set is not large
4. If A and B are large, their intersection is large
5. Either A or A^c is large

Proof. 1, 2, 3, and 4 are true because a set is large if it is UF, and UF is a principle ultrafilter. 5 is true either as a set or its complement is in an ultrafilter, as it is maximal. □

Definition: An equivalence relation is a relation, $\sim$, such that 1. Reflexivity: $a \sim a$ 2. Symmetry $a \sim b$ implies $b \sim a$ 3. Transitivity: $a \sim b$ and $b \sim c$ implies $a \sim c$. An equivalence class of a under $\sim$ is the set $\{x|x \sim a\}$.

Definitions: A sequence is a function from $\mathbb{N}$ to $\mathbb{R}$. Commonly it will be denoted by $\{a_n\}_{n=1}^{\infty}$

Theorem: Let $\{a_n\}_{n=1}^{\infty}$ $\mathbb{H}_{\mathbb{L}}$ $\{b_n\}_{n=1}^{\infty}$ be defined to be true when the set $\{n|a_n = b_n\}$ is large.

Theorem: $\mathbb{H}_{\mathbb{L}}$ is an equivalence relation.

Proof. Let $\{a_n\}_{n=1}^{\infty}$ and $\{b_n\}_{n=1}^{\infty}$ both be sequences such that $\{a_n\}_{n=1}^{\infty}$ $\mathbb{H}_{\mathbb{L}}$ $\{b_n\}_{n=1}^{\infty}$ is true. Clearly, $\{a_n\}_{n=1}^{\infty}$ $\mathbb{H}_{\mathbb{L}}$ $\{a_n\}_{n=1}^{\infty}$ is true, so $\mathbb{H}_{\mathbb{L}}$ is reflexive. Similarly, $\{b_n\}_{n=1}^{\infty}$ $\mathbb{H}_{\mathbb{L}}$ $\{a_n\}_{n=1}^{\infty}$ is also obviously true, so $\mathbb{H}_{\mathbb{L}}$ is symmetric. Finally, let $\{b_n\}_{n=1}^{\infty}$ $\mathbb{H}_{\mathbb{L}}$ $\{c_n\}_{n=1}^{\infty}$ be true. Then the $\{n|b_n = c_n\}$ is large. Then, the set $\{n|a_n = b_n\} \cap \{n|c_n = b_n\}$ is large. However, as this set is the set of naturals such that $a_n = b_n$ and $c_n = b_n$, it must then be that it is the set of naturals such that $a_n = c_n$. Thus, $\{n|a_n = c_n\}$ is large, and $\{a_n\}_{n=1}^{\infty}$ $\mathbb{H}_{\mathbb{L}}$ $\{c_n\}_{n=1}^{\infty}$ is true. Thus, $\mathbb{H}_{\mathbb{L}}$ is transitive. □

Definition: The set of equivalence classes under $\mathbb{H}_{\mathbb{L}}$ is called the set of hyperreals, and is denoted as $\mathbb{R}^*$. An element of the hyperreals is called a hyperreal. One may also refer to a sequence contained in a hyperreal a "sequential representation" of that hyperreal.

Definition: Let a, b be hyperreals. Let a_n and b_n be contained in the equivalence class a and b, respectively. Define $a \leq b$ to be true if and only if the set $\{n|a_n \leq b_n\}$ is large. The other inequality relations are defined in a similar way.

Theorem: The inequality relations are well defined.

Proof. Let a, b be hyperreals. Let $\{a_n\}_{n=1}^{\infty}$, $\{x_n\}_{n=1}^{\infty}$ be in the equivalence class of a, and $\{b_n\}_{n=1}^{\infty}$, $\{y_n\}_{n=1}^{\infty}$ be in the equivalence class of b. Then $\{n|a_n = x_n\}$ and $\{n|b_n = y_n\}$ are large. Let the set $\{n|a_n \leq b_n\}$ be large. The intersection of $\{n|a_n = x_n\}$ and $\{n|a_n \leq b_n\}$, that is $\{n|x_n \leq b_n\}$ is large. The intersection of $\{n|x_n \leq b_n\}$ and $\{n|b_n = y_n\}$, $\{n|x_n \leq y_n\}$, is large. So, to see if all sequential representatives of two hyperreals satisfy an inequality, one must only chose one representation. □

For example, consider the sequence $\{0\}_{n=1}^{\infty}$, and call its equivalence class 0. One can then say a hyperreal, x, is negative if $x < 0$ and is positive if $0 < x$.

Definition: Let f be a function from $\mathbb{R}$ to $\mathbb{R}$, and let x be a hyperreal. The hyperextension of f at x is denoted as $f^*(x)$, and is defined in the following way: Let x_n be a sequential representation of x, then the extension of f at x, $f^*(x)$, is defined as the equivalence class containg $\{f(x_n)\}_{n=1}^{\infty}$ [3].

Theorem: The hyperextension of f at x is well-defined.

Proof. Let $\{x_n\}_{n=1}^{\infty}$ and $\{y_n\}_{n=1}^{\infty}$ be in the equivalence class x, then $\{n|x_n = y_n\}$ is large. Clearly, $\{n|f(x_n) = f(y_n)\}$ is a superset of that set, so it is also large. Thus $\{f(x_n)\}_{n=1}^{\infty}$ and $\{f(y_n)\}_{n=1}^{\infty}$ are in the same equivalence class. □

Extending this to functions of many variables requires the same argument. Thus, addition, multiplication, absolute values, and so on, are defined on the hyperreals.

Definitions: A hyperreal number, x, is said to be standard if, given a sequential representation, x_n, there exists a real number a such that the set $\{n|a = x_n\}$ is large. The real number corresponding to a standard hyperreal number, x, is the unique number a such that, given any sequential representation of x, $\{x_n\}_{n=1}^{\infty}$, the set $\{n|x_n = a\}$ is large. A hyperreal, x, is said to be infinitesimal if x is non-zero and $|x|$ is less than all positive standard numbers. A hyperreal, x, is said to be infinite if $|x|$ is greater than all standard numbers. A finite hyperreal is a hyperreal, x, that is either 0, or there exists two standard hyperreals, a and b, such that $a < |x| < b$.

Theorem: There exists infinitesmial and infinte hyperreals.

Proof. There exists infinitesmials. A few examples include the hyperreals a, represented by $\{\frac{1}{n}\}_{n=1}^{\infty}$, b, represented by $\{\frac{1}{n+1}\}_{n=1}^{\infty}$, and c, represented by $\{\frac{1}{2n}\}_{n=1}^{\infty}$. $\{\frac{1}{n}\}_{n=1}^{\infty}$ represents a positive hyperreal, since, every term in this sequence is greater than 0. It is an infinitesmial, since given any positive standard number, one can find a constant sequential representation (a sequence with constant terms representing that standard number), and eventually, the sequence

$\{\frac{1}{n}\}_{n=1}^{\infty}$ is less than any arbitrary positive real number. Thus it is less than the constant sequence on a cofinite set, thus on a large set, and a is less than any standard positive real number, which makes it an infinitesmial. The same argument works for b and c. However, note that $b < a$, $c < b$, and $c < a$. All these three are distinct positive infinitesmials. Negative infinitesmials can be found merely by multiplying a positive by a negative standard number.

There exists infinites. A few examples include the hyperreals x, represented by $\{n\}_{n=1}^{\infty}$, y, represented by $\{n+1\}_{n=1}^{\infty}$, and z, represented by $\{n^2\}_{n=1}^{\infty}$. $\{n\}_{n=1}^{\infty}$ represents a positive hyperreal, since, every term in this sequence is greater than 0. It is an infinite, since given any standard number, one can find a constant sequential representation, and eventually, the sequence $\{n\}_{n=1}^{\infty}$ is greater than any arbitrary real number. Thus it is greater than the constant sequence on a cofinite set, thus on a large set, and a is greater than any standard positive real number, which makes it an infinite. The same argument works for b and c. However, note that $x < y$ (in fact, y-x=1), $y < z$, and $x < z$. All these three are distinct positive infinites. Negative infinites can be found merely by multiplying a positive infinitesmial by a negative standard number.

□

Definitions: Define $\mathbb{N}^*$ to be the set of all hyperreals for which there is a sequential representation, $\{x_n\}_{n=1}^{\infty}$, such that the set of natural numbers, n, such that x_n is a natural number, is large. One may call the set $\mathbb{N}^*$ the "hypernaturals." One may define the set solely consisting of the hypernaturals and the additive inverse of the hypernaturals the "hyperintegers."

Definition The hyperextension of a set S, S*, is the set of all hyperreals for which there is a sequential representation, $\{x_n\}_{n=1}^{\infty}$, such that the set {n | $x_n \in$ S } is large [6].

Definition Two hyperreals, a and b are said to be infinitely close to one another if $a - b$ is an infinitesimal or zero, and it is denoted as $a \approx b$.

Theorem: Given any hyperreal a, at most one standard number is infinitely close to it.

Proof. Let b, c be standard, and $a \approx b$ and $a \approx c$. Then, $c-b = (c-a)+(a-b)$, and as $c-a$ and $b-a$ are either infinitesimal or zero, $c-b$ must be infinitesimal or zero. But, c, b are standard, so $c - b$ is standard, so it must be zero. □

Theorem: The set of standard numbers has the least-upper bound property - that is, given any subset of the standard numbers bounded from above by a standard number, has a standard least-upper bound (a supremum).

Proof. Let S be a subset of the standard numbers such that there is a standard a such that $\forall x \in$ S $x < a$ (that is, it is bounded by a standard number).

Let θ be a map from standard numbers to $\mathbb{R}$ defined by taking the standard number to the real number it corresponds to (that is, the a such that the set $\{n|a = x_n\}$ is large, where x_n is a sequential representation of a standard hyperreal).

So, it is easy to see that θ(S) has a least upperbound, t. Consider the sequence $\{t\}_{n=1}^{\infty}$, let it be a sequential representation of the hyperreal t^*. t^* is the supremum of S. First, t^* is an upperbound on S. Secondly, let q^* be a standard hyperreal that's a bound on S, but less than t, and let $\{q_n\}_{n=1}^{\infty}$ be a sequential representation of q, let it be a constant sequential representation, such that $\forall n \in \mathbb{N}$, $q_n = q$ (where p is a real number). q must be less than t, but an upperbound on S - a contradiction, as t is the supremum.

□

Theorem: Every finite hyperreal number is infinitely close to a standard number [7].

Proof. Let t be a finite number. Then, by definition, there is a standard number b such that $-b < t < b$. Construct the set K := $\{y|y$ is standard and $y < t\}$. Clearly this set contains $-b$ and is bounded from above by b, so it has a supremum, a. $|t-a|$ is non-infinite as $-b-a \leq t-a \leq b-a$, so it is either finite or infinitesmial. Assume that it is not infinitesmial, so there exists a (positive) standard c such that $c < |t-a|$. There are two cases to consider

Case 1: $a > t$ Then $|t-a| = a-t$. Thus, $c < a-t$ so, $a-c > t$. Yet $t < a-c < a$. But then $a-c$ is a standard number greater than t, so it is a standard upperbound on K less than the supremum of K, a contradiction.

Case 2: $t > a$ So, $|t-a| = t-a$. So $c < t-a$, so $a+c < t$. Thus $a-c \in$ K. As c is positive, $a < a+c$, but, as $a+c \in$ K, a is an upperbound on K, so $a+c \leq a$. Contradiction.

Thus, $|t-a|$ is an infinitesmial or zero. Thus $t \approx a$.

□

Definition: One can now define the standard function, st, as such:

$$st(x) = \begin{cases} a & x \text{ is non-infinite, where } a \text{ is standard and x} \approx a \\ \infty & x \text{ is infinite} \end{cases}$$

Theorem: Let a, a_1, a_2, ... a_n be non-infinite numbers, and x, x_1, x_2, ... x_n be standard numbers such that $a_i \approx x_i$, and $a \approx x$. Then

1. $st(a_1 + a_2 + ... + a_n) = x_1 + x_2 + ... + x_n$
2. $st(-a) = -x$
3. $a_1 < a_2$ implies $x_1 \leq x_2$
4. $a_1 \leq a_2$ implies $x_1 \leq x_2$

5. $st(a_1 * a_2 * ... * a_n) = x_1 * x_2 * ... * x_n$
6. If a is nonzero and finite, $st(\frac{1}{a}) = \frac{1}{x}$
7. If a_2 is nonzero and finite, $st(\frac{a_1}{a_2}) = \frac{x_1}{x_2}$

Proof.

1. Because a, a_1, a_2, ... a_n are (respectively) infinitely close to the real numbers x_1, x_2, ... x_n, $a_1 + a_2 + ... + a_n = x_1 + \varepsilon_1 + x_2 + \varepsilon_2 + ... + x_n + \varepsilon_n = x_1 + x_2 + ... + x_n + \varepsilon_1 + \varepsilon_2 + ... + \varepsilon_n$, where the ε's are infinitesimal. So, as the sum of infinitesimals is infinitesimal, $st(a_1 + a_2 + ... + a_n) = x_1 + x_2 + ... + x_n$.
2. Let $st(a) = x$, a real number, then $a = x + \varepsilon$. Consider $-a$. We have $-a = -x - \varepsilon$, so $-a$ is infinitely close to $-x$. So, $st(-a) = -x$.
3. Let $a_1 < a_2$ and be finite. Then, $x_1 + \varepsilon_1 < x_1 + \varepsilon_1$, where ε's are infinitesimal. There are two cases, either $x_1 < x_2$, or $x_1 = x_2$ and $\varepsilon_1 < \varepsilon_2$, in either case, we have $x_1 \leq x_2$
4. If $a_1 \leq a_2$, then either $a_1 = a_2$, in which case the theorem follows, or $a_1 < a_2$, in which case we apply (3).
5. We proceed with induction. Observe that $a_1 * a_2 = (x_1 + \varepsilon_1)(x_2 + \varepsilon_2) = x_1 x_2 + \varepsilon_1 * x_2 + \varepsilon_2 x_1 + \varepsilon_1 \varepsilon_2 \approx x_1 x_2$, which means $st(a_1 a_2) = x_1 x_2$. Now, assume that it is true for n, that is, $st(a_1 * a_2 * ... * a_n) = x_1 * x_2 * ... * x_n$. Now, consider $st(a_1 * a_2 * ... * a_n * a_{n+1})$, where a_{n+1} is a finite number. Then by the base case, $st(a_1 * a_2 * ... * a_n * a_{n+1}) = st(a_1 * a_2 * ... * a_n) * st(a_{n+1})$, which, by the assumption, equals $x_1 * x_2 * ... * x_n * st(a_{n+1})$. This completes the proof.
6. Knowing that $a \approx x$, then $a = x + \varepsilon$, where ε is infinitesimal. Consider the quantity $\frac{1}{x+\varepsilon}$. Note that $\frac{1}{x} - \frac{2}{x+\varepsilon} = \frac{-\varepsilon}{x*(x+\varepsilon)}$ is an infinitesimal number (as $\frac{1}{x*(x+\varepsilon)}$ is finite). Then $\frac{1}{x}$ and $\frac{1}{x+\varepsilon} = \frac{1}{a}$ are infinitely close to one another.
7. This is an immediate application of (5) and (6).

□

Decimal Expansion of the Hyperreals The following is found in Henle's book. First, one expresses any standard hyperinteger using decimal notation. This is achieved by taking any standard hyperreal, finding a constant sequential representation of it, and using the constant of that sequence to represent the standard hypernatural. For example, this has already been implicitly done with 0 - the hyperreal containing a constant sequence of 0's is represented as 0. Similarly, the hyperreal containing a constant sequence of 1's may be represented as "1", and so on.

Now, consider the function $d(x,n)$ from $\mathbb{R} \times \mathbb{N}$ to $\mathbb{N}$. This function takes in a real number, x, and returns the natural number occupying the nth point of the decimal. Then consider the integer function, $i(x)$, which returns the greatest integer less than or equal to x if x is positive, and the least integer greater than or equal to x if x is negative. One may extend the integer function from being defined on the reals to being defined on the hyperreals, and it would return a hyperinfinite. If it is a finite hyperreal, it would simply return the hyperreal containing a sequence which is constant on a large amount of terms, thus, it has been established how to represent $\lfloor x \rfloor^*$ in decimal notation for finite hyperreals. Similary, the function $d(x,n)$ can be extended. For a standard number, $d^*(x,N)$ will be zero for all hyperinfinite N's. For all hyperreals, and for all finite n's, $d^*(x,n)$ will be a finite hypernatural, thus it can be expressed in decimal notation. Thus, for a standard number, one can express all standard numbers in decimal notation as follows: x is expressed as $i(x).d^*(x,1).d^*(x,2)d^*(x,3)...$

Note that there are many ways of representing a real number using decimal expansion. This just means there are many ways of representing a hyperreal using decimal expansion.

The Transfer Principle Before proving the Transfer Principle, due to Łos [2], a few definitions need to be established.

The most basic (and important) parts of a mathematical statement is its "terms." Terms maybe be constants, variables, or some combination thereof - in fact, we may regard a function evaluated at a constant, or containing variables, to itself be a term.

Definition An n-place relational symbol, $R(t_1, t_2, t_3, ..., t_n)$, is a combination of the terms $t_1, t_2, t_3, ..., t_n$ and $<$, $>$, $=$, $\leq$, $\neq$, $\geq$, and $\in$ S where S is a set.

Definition The hyperextension of a real relational symbol, denoted $R^*(t_1, t_2, t_3, ..., t_n)$, where $t_1, t_2, t_3, ..., t_n$ are hyperreal terms, and the real relations $<$, $>$, etc are replaced by their hyperreal counterparts, and $\in S$ is replaced by $\in S^*$. One may call these relational symbols "corresponding".

Definition A formula of a language is

1. $R(t_1, t_2, t_3, ..., t_n)$, a relational symbol
2. F $\sim \wedge$ G ("not F and G"), where F and G are formulas (as any logical operator may be expressed as a combination of $\sim \wedge$'s)
3. $\exists x\ F(x)$, where x is a term, and $F(x)$ is a formula in which x appears (it is also known that using negation and existence quantifiers, one has the universal quantifier).

Definition Let G^* be a formula in R^*. Wherever one sees the hyperreals a, b, c,..., pick sequential representations $\{a_n\}_{n=1}^{\infty}$, $\{b_n\}_{n=1}^{\infty}$, $\{x_n\}_{n=1}^{\infty}$, and replace a, b, c, ..., with a_i, b_i, c_i,.... Similarly for functions. Then, replace all the hyperreal relational symbols with the corresponding real relational symbols. Call this new formula, in $\mathbb{R}$, G_i, or, the ith formula corresponding to G^* generated by $\{a_n\}_{n=1}^{\infty}$, $\{b_n\}_{n=1}^{\infty}$, $\{x_n\}_{n=1}^{\infty}$,...".

Lemma Let G^* be a formula in $\mathbb{R}^*$ involving the hyperreals t_1, t_2, t_3,.... Each of these has representation $\{y_{k,n}\}_{n=1}^{\infty}$ and $\{x_{k,n}\}_{n=1}^{\infty}$. Let G_i be the ith formula corresponding to G^* generated by $\{y_{k,n}\}_{n=1}^{\infty}$ and let T_i be the ith formula corresponding to G^* generated by $\{x_{k,n}\}_{n=1}^{\infty}$. The set $\{i|T_i$ is true $\}$ is large if and only if $\{i|G_i$ is true $\}$ is large.

Proof. Define S to be the set $\{n|y_{1,n} = x_{1,n}, y_{2,n} = x_{2,n}, y_{3,n} = x_{3,n},..\}$. Clearly, S must be large. And obviously, if $k \in$ S, then $G_i = T_i$.

Let $\{n|T_n\}$ be large. Then, S $\cap\{n|T_n\}$ is large. But, S $\cap\{n|T_n\} \subset \{n|G_n\}$, as S $\cap\{n|T_n\}$ is the set of all statements such that T_n is true and $T_n = G_n$. So, $\{n|G_n\}$ is large.

Let $\{n|T_n\}$ not be large. Then, $\{n|T_n$ is false$\}$ is large. Then, $\{n|T_n$ is false$\}$ $\cap S$ is large. Finally, $\{n|T_n$ is false$\}$ $\cap S \subset \{n|G_n$ is false$\}$. So, $\{n|G_n\}$ is not large.

□

Theorem: The Transfer Principle Let G^* be a formula of R^*. Let G_n be any corresponding nth formula. Then G^* is true if and only if $\{n|G_n\}$ is large.

Proof. This proof is a proof by contradiction. A formula can only be of type 1, 2, or 3. Furthermore, the theorem can only be false for a formula of type 2 or 3, as that is the definition of what it means for a statement of type 1 to be true. But, note, every formula must be decomposed into a finite number of quantifiers, logical operators, and relation symbols - we cannot have a formula infinitely receeding into a formula, they have to be "constructed" out of previous formulas. So, assume that the theorem does fail, then that must mean there is a "smallest formula" (i.e., the smallest amount of quantifiers and $\sim\wedge$ operators possible), because else, if there was not, there could be a formula with a negative amount of quantifiers or operators.

Assume the statement is of type 2. Assume that K^* is true, while $\{n|K_n\}$ is not large. Then, as $K^* = F^* \sim\wedge G^*$, that must mean that either F^* is false or G^* is false. Assume then that G^* is false. G^* is "smaller" than K^*, so the theorem works, thus, $\{n|G_n$ is false $\}$ is a large set. But $\{n|G_n$ is false $\} \subset \{$n $|$ $G_n \sim\wedge F_n\}$, which is $\{n|K_n\}$. So, $\{n|K_n\}$ is a large set. Contradiction. The reasoning is the same in the case that K^* is false, while $\{n|K_n\}$ is large.

Assume the statement is of type 3, so $K^* = \exists x G^*$. Assume K^* is true, and yet, $\{n|K_n\}$ is not large. However, at least one hyperreal must satisfy G^*, so let F^* be the statement G^* where x is replaced by a hyperreal that makes the formula true. Then, as F^* is smaller than K^*, the theorem must work for F^*.

Then the set $\{n|F_n\ \}$ is large. But, that means $\{n|\exists x_n\ G_n(x_n)\ \}$ is large, which means $\{n|K_n\ \}$ is large. Contradiction. The reasoning is the same in the case that K^* is false, while $\{n|K_n\ \}$ is large.

$\square$

Chapter 2: Sequences

In this section, sequences will be discussed. The notion of convergence and divergence will be defined, as well as bounded and unbounded. Then, basic limit theorems will be established. The notion of monoticity will be explained. This chapter will then be ended by the non-standard Cauchy criterion for convergence. Throughout this whole work, only sequences from $\mathbb{N} \to \mathbb{R}$, and their hyperextensions, will be considered.

2.1 Definition A sequence, $\{a_n\}_{n=1}^{\infty}$, is a function from the naturals to the real (standard) numbers.

2.2 Definition If there exists a standard number, a, such that $st(a_N^*) = a$ for all hyperinfinite N. It is denoted as

$$\lim_{n\to\infty} a_n = a$$

. If the limit of a sequence exists, the sequence is said to converge [2].

2.3 Theorem The limit of a sequence is unique.

Proof. Suppose both a and b are the limit of a sequence $\{a_n\}_{n=1}^{\infty}$. Then, for hyperinfinite N, $st(a_N^*) = a$ and $st(a_N^*) = b$. Of course, this means that $a = b$. □

2.4 Standard Definition The standard definition of the limit of a sequence, $\{a_n\}_{n=1}^{\infty}$, is that a is the limit of a sequence if for all ϵ greater than zero, there exists an r in the real numbers such that $n > r$ implies that $|a_n - a| < \epsilon$.

2.5 Theorem The standard and non-standard definitions of a sequence are equivalent.

Proof. Assume $\forall\epsilon \in \mathbb{R}^+ \ \exists N \in \mathbb{R}$ such that $n > N$ implies $|a - a_n| < \epsilon$. Fix ϵ. Let $n > r$ so that $a - \epsilon < a_n < a - \epsilon$. Thus by the Transfer Principle, $a - \epsilon < a_N^* < a - \epsilon$ for all hyperinfinite N. Note that ϵ was an arbitrary positive number, so $st(a_N^*) = a$.

Then, assume that the standard limit is not a. That is, $\exists \epsilon \in \mathbb{R}^+$ such that $\forall r \in \mathbb{R}\ \exists n > r$ such that $|a_n - a| \geq \epsilon$. By the transfer principle, this means, for all R in $\mathbb{R}^{\nleftarrow}$ there exists an N such that $|a_N^* - a| \geq \epsilon$, in particular for an infinite R and thus an infinite N. But then, $st(a_N^*) \neq a$. Thus the non-standard limit is not a.

Thus, the definitions are equivalent.

□

2.6 Definition A sequence, $\{a_n\}_{n=1}^{\infty}$, is said to diverge to infinity if $\forall N \in \mathbb{N}^*/\mathbb{N}$, $st(a_N^*) = \infty$. A sequence, $\{a_n\}_{n=1}^{\infty}$, is said to diverge to negative infinity if $\forall N \in \mathbb{N}^*/\mathbb{N}$, $st(a_N^*) = -\infty$.

2.7 Standard Definition A sequence, $\{a_n\}_{n=1}^{\infty}$, diverges to infinity if $\forall m \in \mathbb{R}$, $\exists R \in \mathbb{R}$ so that $n > r$ implies $a_n > m$. It is similar for diverging to negative infinity.

2.8 Theorem The standard and non-standard definitions of divergence are equivalent.

Proof. Let $\{a_n\}_{n=1}^{\infty}$ be a sequence.

Let $\forall m \in \mathbb{R}\ \exists r \in \mathbb{R}$. Fix m. Thus, when $n > r$, $a_n > m$. Then, by the Transfer Principle, it is the case that for all hyperinfinite N, $a_N^* > m$. But m was an arbitrary real number, so a_N^* is greater than every real number. Thus, for all N, $st(a_N^*) = \infty$.

Let the standard definition be false. Thus, there exists an $m \in \mathbb{R}$ so that $a_n < m$ for all natural n. But then by the Transfer Principle, $a_N^* < m$ for all hyperinfinite N. So the non-standard definition is also false. □

2.9 Definition A sequence, $\{a_n\}_{n=1}^{\infty}$, is bounded from above if for all hyperinfinite naturals, N, $st(a_N^*) < \infty$. A sequence, $\{a_n\}_{n=1}^{\infty}$, is bounded from below if for all hyperinfinite naturals, N, $st(a_N^*) > -\infty$. A sequence, $\{a_n\}_{n=1}^{\infty}$, is bounded if it is bounded from above and below. A sequence which is not bounded is unbounded.

2.10 Standard Definition A sequence, $\{a_n\}_{n=1}^{\infty}$, is said to be bounded from above if $\exists k \in \mathbb{R}$ such that $\forall n \in \mathbb{N}\ a_n < k$.

2.11 Theorem The standard and non-standard definitions of boundedness are equivalent.

Proof. Let $\{a_n\}_{n=1}^{\infty}$ be a sequence.

Let $\exists k \in \mathbb{R}$ such that $a_n < k$ for all natural n. So, by the Transfer Principle, $a_N^* < k$, for all hypernatural infinite N. Thus, $st(a_N^*) < \infty$

Let the standard definition for bounded fail. That is, $\forall k \in \mathbb{R}$, there exists an $n \in \mathbb{N}$such that $a_n > k$. Construct the sequence of real numbers, $\{n_l\}_{l=1}^{\infty}$

such that $a_{n_l} > l$ and $a_{n_{l+1}} > a_{n_l}$. Clearly, the limit of the sequence $\{a_{n_l}\}_{l=1}^{\infty}$ is infinity, so there is a hyperinfinite N such that $st(a_N^*) = \infty$.

It is similar for bounded from below.

□

Consider the sequences $\{n\}_{n=1}^{\infty}$, $\{(-1)^n n\}_{n=1}^{\infty}$, and $\{n^2\}_{n=1}^{\infty}$. Each of these sequences is obviously unbounded from above, and for each we see that, when evaluated at the infinite hypernaturals, they are definitely not infinitely close to one another (in direct contrast to sequences that approach a limit). The question arises then, does this generalize to all unbounded sequences, or is it possible to have a sequence that, when evaluated at any infinite hypernatural, are always infinitely close to some infinite N? The following theorem tells us that the former is the case.

2.12 Theorem If a sequence, $\{a_n\}_{n=1}^{\infty}$ is unbounded from above or from below, there are hyperininfinite natural L, N such that a_L^* is not infinitely close to a_N^*.

Proof. Let $\{a_n\}_{n=1}^{\infty}$ be a sequence, unbounded from above.

Then, $\forall m \in \mathbb{R}$ there exists an $n \in \mathbb{N}$ such that $a_n > m$. So, by the Transfer Principle, $\forall M \in \mathbb{R}^*$ there exists an $N \in \mathbb{N}^*$ such that $a_N^* > M$. In particular, if M is an infinite, there is an L such that a_L^* is greater than M. It should be clear that L must itself be infinite. There must also be an N so that a_N^* is greater than $M + a_L^*$, and it is also clear that this N is an infinite. Thus, there are hyperininfinite natural L, N such that a_L^* are not infinitely close to a_N^*.

□

2.13 Theorem Convergent Sequences are bounded.

Proof. Suppose $\{a_n\}_{n=1}^{\infty}$ is a convergent sequence. Then there is some standard number, a, such that for all infinite hypernatural N, $st(a_N^*) = a$. Thus, all a_N^* are finite, and it is always that $st(a_N^*) < \infty$. □

Now that one has the infrastructure about boundedness, convergence, and divergence, it is important to establish basic facts about limits.

2.14 Theorem Let k be a real number, and $\{a_n\}_{n=1}^{\infty}$ converge to a. Then, $\{k * a_n\}_{n=1}^{\infty}$ converges to $k * a$.

Proof. Consider $\{k * a_n\}_{n=1}^{\infty}$. Its hyperreal extension is $\{k * a_N^*\}_{n=1}^{\infty}$. So, letting N be infinite, $st(k * a_N) = k * st(a_N) = k * a$.

□

2.15 Theorem If $\{a_n\}_{n=1}^{\infty}$ converges to a, and $\{b_n\}_{n=1}^{\infty}$ converges to b, then $\{a_n * b_n\}_{n=1}^{\infty}$ converges to $a * b$.

Proof. The hyperextension of $\{a_n * b_n\}_{n=1}^{\infty}$ is $\{a_n^* * b_n^*\}_{n=1}^{\infty}$. Let N be infinite, $st(a_N^* * b_N^*) = st(a_N^*) * st(b_N^*) = a * b$. □

2.16 Theorem If $\{a_n\}_{n=1}^{\infty}$ converges to a, and $a \neq 0 \neq a_n$, then $\{\frac{1}{a_n}\}_{n=1}^{\infty}$ converges to $\frac{1}{a}$. If $\{a_n\}_{n=1}^{\infty}$ diverges, $\{\frac{1}{a_n}\}_{n=1}^{\infty}$ is 0.

Proof. The extension of $\{\frac{1}{a_n}\}_{n=1}^{\infty}$ is $\{\frac{1}{a_N^*}\}_{n=1}^{\infty}$. So for infinite N $st(\frac{1}{a_N^*}) = \frac{1}{st(a_N^*)} = \frac{1}{a}$.

However, if a_N^* is always infinite, then $\frac{1}{a_N^*}$ is infinitesimal, so $st(\frac{1}{a_N^*})$ is 0. □

Theorem 2.17 If $\{a_n\}_{n=1}^{\infty}$ converges to a, and $a \neq 0 \neq a_n$, and $\{b_n\}_{n=1}^{\infty}$ converges to b, then $\{\frac{b_n}{a_n}\}_{n=1}^{\infty}$ converges to $\frac{b}{a}$.

Proof. This is a combination of Theorem 2.15 and 2.16. □

2.18 Theorem

1. if $p > 0$
$$\lim_{n\to\infty} \frac{1}{n^p} = 0$$
2. if $|a| < 1$
$$\lim_{n\to\infty} a^n = 0$$
3.
$$\lim_{n\to\infty} n^{\frac{1}{n}} = 1$$
4. if $a > 0$
$$\lim_{n\to\infty} a^{\frac{1}{n}} = 0$$

Proof. 1. Let N be infinite and p be positive. Then N^p is infinite. Thus $st(\frac{1}{N^p}) = 0$.

2. If $a = 0$ this is obvious, so assume $0 < |a| < 1$. Let $b = \frac{1}{a}$. Assume that for all infinite N, b^N is finite, thus, there is a standard r such that $|b^N| < r$. As r and b are standard, there is a standard x such that $r = |b|^x$. That is, $|b|^N < |b|^x$, or $|b|^{N-x} < 1$. But, $N - x$ ¿ 1, $b > 1$, so $|b|^{N-x} > 1$. Contradiction. Thus, b^N is infinite, so $st(a^N)$ is zero.

3. The same presentation as Ross [4].

4. Same presentation as in Ross [4].

□

2.19 Theorem Suppose $a_n < b_n$ for integers greater than n, and $\{a_n\}_{n=1}^{\infty}$ converges to a, and $\{b_n\}_{n=1}^{\infty}$ converges to b.

Proof. By Transfer Principle, for all infinite N, $a_n^* < b_n^*$. So $st(a_n^*) < st(b_n^*)$, or $a < b$.

□

2.20 Theorem Let $\{a_n\}_{n=1}^{\infty}$ diverge to infinity and $\{b_n\}_{n=1}^{\infty}$ be such that $\lim_{n\to\infty} b_n > 0$. Then $\lim_{n\to\infty} a_n * b_n = \infty$

Proof. The extension of $a_n * b_n$ is $a_n^* * b_n^*$, and for infinite N, b_N^* is either a positive infinite or a positive finite, so $a_N^* * b_N^*$ is infinite. □

2.21 The Squeeze Theorem Let $\{a_n\}_{n=1}^{\infty}$, $\{b_n\}_{n=1}^{\infty}$, and $\{c_n\}_{n=1}^{\infty}$ be sequences, and t a real number, such that $\lim_{n\to\infty} a_n = \lim_{n\to\infty} c_n = t$, and for all natural n, $a_n \leq b_n \leq c_n$ (these may also be made into strict inequalities), then $\lim_{n\to\infty} b_n = t$.

Proof. We have $a_n \leq b_n \leq c_n$, so by the Transfer Principle, we have $a_N^* \leq b_N^* \leq c_N^*$. As a_N^* and c_N^* are finite, b_N^* is finite, and thus we have $st(a_N^*) \leq st(b_N^*) \leq st(c_N^*)$. But $\lim_{n\to\infty} a_n = \lim_{n\to\infty} c_n = t$, so we have $t \leq st(b_N^*) \leq t$, which means, for all hyperinfinite N, we have $st(b_N^*) = t$

□

Now, we can move on to the notion of a monotonic sequence.

2.22 Definition A sequence, $\{a_n\}_{n=1}^{\infty}$, is called an increasing sequence if $a_n \leq a_{n+1}$. It is called decreasing if $a_n \leq a_{n-1}$. A sequence with strict inequality is called strictly increasing or decreasing. A monotonic sequence is a sequence that is either increasing or decreasing.

2.23 Theorem All bounded monotonic sequences converge.

Proof. Assume $\{a_n\}_{n=1}^{\infty}$ is an increasing sequence bounded from above. Define $A = \{a_n | n \in \mathbb{N}\}$. A is bounded from above, so $a = supA$ exists. Let $\epsilon \in \mathbb{R}^+$. As a is the supremum, there is a natural r such that $a - \epsilon < a_r$. $\{a_n\}_{n=1}^{\infty}$ is increasing, so $u - \epsilon$ is a lower bound on all $n > r$. So, for all such n, $a - \epsilon < a_n \leq a$. So by the Transfer Principle, letting N be any infinite, $a - \epsilon < a_N^* \leq a$. But, ϵ was arbitrary, so it is true for all positive standard reals, thus $st(a_N^*) = a$. The proof for decreasing, bounded from below is similar. □

2.24 Theorem If $\{a_n\}_{n=1}^{\infty}$ is unbounded from above and increasing, $\lim_{n\to\infty} a_n = \infty$. If it is unbounded from below and decreasing, $\lim_{n\to\infty} a_n = -\infty$

Proof. Clearly, for the first part of the theorem, the sequence cannot diverge to negative infinity. Let there be an infinite N such that $st(a_N^* = a$, for a standard a. Then, as the sequence is increasing, it is the case for all $n < N$, infinite or standard, that $a_n < a$. But then, the sequence is bounded by a, which is a contradiction.

The proof for unbounded from below and decreasing is similar.

□

2.25 Corollary All monotonic sequences either converge to a real number, or diverge to infinity or negative infinity.

Finally, this section can discuss the Cauchy criterion for convergence.

2.26 Definition A sequence, $\{a_n\}_{n=1}^{\infty}$, is Cauchy if for all hyperinfinte M, N, $a_M^* \approx a_N^*$ [3].

2.27 Standard Definition A sequence, $\{a_n\}_{n=1}^{\infty}$, is Cauchy if $\forall \epsilon \in \mathbb{R}^+$ $r \in \mathbb{R}$ such that $m, n > r$ implies $|a_n - a_m| < \epsilon$.

2.28 Theorem The standard and non-standard definitions of Cauchy sequences are equivalent.

Proof. Assume that $\forall \epsilon \in \mathbb{R}^+$ $r \in \mathbb{R}$ such that $m, n > r$ implies $|a_n - a_m| < \epsilon$. Then, for infinite N, M, $|a_N^* - a_M^*| < \epsilon$ by the Transfer Principle. But ϵ was an arbitrary positive real, so $a_N^* \approx a_M^*$.

Assume that the standard definition fails. That is, $\exists \epsilon \in \mathbb{R}^+$ such that for all real r, there exists m,n greater than r so that $|a_n - a_m| \geq \epsilon$. Then, by the Transfer Principle, there is an infinite R, there exists $M, N > R$ so that $|a_N^* - a_M^*| \geq \epsilon$. Then, a_N^* is not infinitely close to a_M^*.

□

The following proof, concerning the boundedness of Cauchy sequences, follows immediately from Theorem 2.12. Usually, to prove this result, one looks at the standard definition of a Cauchy sequence. With this, for any positive real r, there is a natural n such that l, p greater than n implies that the distance between the sequence evaluated at l and p is less than r. Then, you fix p to be equal to $n + 1$. You group up all terms less than $n + 1$ into a set, add one to them, and the maximum of this set turns out to be a bound on the sequence [4].

2.29 Theorem Cauchy sequences are bounded.

Proof. Let $\{a_n\}_{n=1}^{\infty}$ be an unbounded Cauchy sequence. Then, by Theorem 2.12, there exists infinite L, N such that $a_L^* \not\approx a_N^*$. But, because it is Cauchy, $a_L^* \approx a_N^*$. Contradiction.

□

2.30 Theorem A sequence is Cauchy if and only if it is convergent.

Proof. Let $\{a_n\}_{n=1}^{\infty}$ be a sequence.

If $\{a_n\}_{n=1}^{\infty}$ converges to a, for $\forall N, M$ that are infinite, $a_N^* \approx a \approx a_M^*$. Thus, the sequence is Cauchy.

If $\{a_n\}_{n=1}^{\infty}$ is Cauchy, then for any infinite N, a_N^* is finite. Thus, there is a standard number a such that $a_N^* \approx a$. As the sequence is Cauchy, $a_N^* \approx a_M^*$, so $a_N^* \approx a \approx a_M^*$. Thus, for all infinite numbers L, $st(a_L^*) = a$, that is, it converges to a.

□

Chapter 3: Subsequences

3.1 Definition Let $\{n_k\}_{n=1}^{\infty}$ be a strictly increasing, natural valued sequence. A subsequence of $\{a_n\}_{n=1}^{\infty}$ is then a sequence of the form $\{a_{n_k}\}_{n=1}^{\infty}$.

In standard analysis, there is a condition that is both necessary and sufficient for determining if there is a subseqence converging to some limit. It is that there is a subseqence convering to s if and only if, for every positive real r, the set $\{n : |s_n - s| < r\}$ is infinite. The following theorem gives an analogous equivalent condition.

3.2 Theorem Let t be either a standard number, the symbol ∞, or the symbol -∞. Let $\{a_n\}_{n=1}^{\infty}$ be a real-valued sequence. Then there exists an infinite hypernatural N such that $st(a_N^*) = t$ if and only if there is a subsequence of $st(a_n^*)$ that converges (diverges) to t. Furthermore, if t is real, then there is a subsequence converging to t if and only if the set $\{n \in \mathbb{N} : |a_n - t| < r\}$ is infinite for all positive r. Finally, $\{a_n\}_{n=1}^{\infty}$ has a divergent subsequence if and only if $\{a_n\}_{n=1}^{\infty}$ is unbounded.

Proof. For the first part of the proof, assume that t is real.

Now, consider the set $\{n \in \mathbb{N} : |a_n - t| < r\}$. Either these sets are infinite for all positive real r, or it is finite for at least one positive r.

If one such set is finite, there exists a real m, r such that for all $n > m$, $|a_n - t| > r$. Obviously then, no subsequence can converge to t, and for all infinite natural N, $|a_N^* - t| > r$, that is, there is no $|a_n - t| > r$ infinitely close to t.

Now, suppose that these sets are infinite for all positive r. We can formulate that as $\forall r \in \mathbb{R}^+ \ \forall m \in \mathbb{N} \ \exists n \in \mathbb{N}$ such that $n > m$ and $|a_n - t| < r$. By the Transfer Principle, this means that "for all positive infinitesimal ε, for every $M \in \mathbb{N}^*$ there exists an $N \in \mathbb{N}^*$ so that $|a_N^* - t| < \varepsilon$ is consequentially a true statement. But this means that there is a hyperinfinite N such that $st(a_N^*) = t$. All that remains to be shown is that there is a subsequence converging to t, and the proof is near-identical to the one as presented by Ross.

For the second part of the proof, I will only be considering subsequences diverging to ∞, as the proof for -∞ is similar.

For all sequences $\{a_n\}_{n=1}^{\infty}$, either $\forall r \in \mathbb{R} \ \exists n \in \mathbb{N}$ so that $a_n > r$ is true or false. If it is false, then by Theorem 2.11, then there is no hyperinfinite N so that $st(a_N^*) = \infty$, and clearly, no subsequence can diverge to ∞ (to formally prove this, assume that it would, meaning that it is not bounded, but then we assumed that it is bounded, a contradiction).

So, assume that for $\{a_n\}_{n=1}^{\infty}$, $\forall r \in \mathbb{R}\ \exists n \in \mathbb{N}$ so that $a_n > r$ is true. We simply construct the same sequence as Ross.

□

3.3 Theorem If $\{a_n\}_{n=1}^{\infty}$ converges (diverges), then every subsequence converges (diverges) to the same limit.

Proof. Let a be a real number, the symbol $-\infty$, or the symbol ∞. As $\{a_n\}_{n=1}^{\infty}$ converges (diverges) to a, that means, for any hypernatural N, $st(a_N^*) = a$. Then, no hyperreal may be infinitely close to t, where t may be a real number, the symbol $-\infty$, or the symbol ∞, so long as it distinct from a. Thus, there is no subsequence that could converge to t, and all subsequences must converge to a. □

3.4 Theorem Every sequence contains a monotonic subsequence. (for the proof, see Ross) [4].

Theorem 3.5, the Bolzano-Weierstrass Theorem Every bounded sequence contains a convergent monotonic subsequence.

Proof. By Theorem 3.4, every sequence contains a monotonic subsequence. Because the original sequence was bounded, this one is. By Theorem 2.23, we know this sequence converges, as it is bounded and monotonic. □

3.6 Definition Let C denote the set of all subsequential limits of $\{a_n\}_{n=1}^{\infty}$. Let limsup(a_n) denote the supremum of C, and liminf(a_n) denote the infimum of C. From this definition alone, it is clear that the limit of a sequence exists if and only if its liminf and limsup are equal [2].

3.7 Theorem Let $\{s_n\}_{n=1}^{\infty}$ be a sequence. There exists a subsequence that converges to limsup(s_n) and liminf(s_n).

Proof. We are only considering the case of bounded from above and limsups, as the reasoning for bounded from below and liminf is exactly identical.

First, let $\{s_n\}_{n=1}^{\infty}$ be unbounded. Then, there is an infinite hypernatural N such that $st(s_N^*) = \infty$, so clearly, limsup(s_n) $= \infty$. Now, assume that $\{s_n\}_{n=1}^{\infty}$ is bounded. Then, limsup(s_n) exists (and is real), denote it by t. Let $\{s_{n_k}\}_{n=1}^{\infty}$ be a sequence defined by s_{n_k} being an element of $\{s_m : m > n_k\}$ that satisfies $t - \frac{1}{k} \leq s_m$. Such an element is guaranteed to exist, for if there was no such element such that $t - \frac{1}{k} \leq s_m$, then $t - \frac{1}{k}$ would be an upperbound on the sequence, and thus when this sequence is evaluated at a hyperreal, it would always be less than $t - \frac{1}{k}$. But then the set of subsequential limits (remembering theorem 3.2) would be less than $t - \frac{1}{k}$, and t could not be its supremum. Hyperextending the subsequence $\{s_{n_k}\}_{n=1}^{\infty}$, and evaluating it at some hypernatural K, we get that $t - \frac{1}{K} \leq s^*_{n_K}$, in particular, $t \leq st(s^*_{n_K})$. and we automatically have that $st(s^*_{n_K}) \leq t$, so $t = st(s^*_{n_K})$. Thus there is a subsequence converging to the limit supremum. □

3.8 Theorem Consider the sequence defined by $\{s_n\}_{n=1}^{\infty}$. The limit of the sequence $a_n = \sup \{s_m : m \geq n\}$ is limsup(s_n), and the limit of the sequence defined by $b_n = \inf \{s_m : m \geq n\}$ is liminf(s_n).

Proof. First, it is obvious that $\lim a_n$ and $\lim b_n$ exists, as these are monotonic sequences (so if they are bounded they converge, if they are unbounded they diverge to ∞ or $-\infty$.

We will only consider the first part of the theorem, as the second part of the proof is exactly identical. Let S be the set of all subsequential limits, then by theorem 3.7, this set has a maximum, and it suffices to show that the limit of $\{a_n\}_{n=1}^{\infty}$ is the maximum of S. First, it will be shown that $\lim a_n$ is greater than every element of S, then it will be shown that it is also an element of S. We let $\lim a_n = a$.

Note that, by the very definition of a_n, we have that $s_n \leq a_n$ for all natural n. Then, by properties of hyperextension, $s^*_N \leq a^*_N \approx a$ (or a is the symbol ∞ or $-\infty$), so every subsequential limit is less than or equal to a.

To prove that $\lim a_n$ converges to an element in S, it suffices to show there is a subsequence of $\{s_n\}_{n=1}^{\infty}$ that converges to $\lim a_n$.

If $\lim a_n$ is ∞ or $-\infty$, then it is clear that $\{s_n\}_{n=1}^{\infty}$ is unbounded (either from above or from below), and thus, has a divergent subsequence (either to ∞ or $-\infty$). So, assume that a is a real number.

Fix $r_o \in \mathbb{R}^+$, and fix $n \in \mathbb{N}$. By properties of supremum, $\exists s_l \in sup(\{s_m : m \geq n\})$ such that $s_l + r_o > a_n$. So, the statement: "$\forall n \in \mathbb{N}\ \exists l \in \mathbb{N}$ such that $l > n$ implies $s_l + r_o > a_n$" is true. Then, by the Transfer Principle, it must be the case that "$\forall N \in \mathbb{N}^*\ \exists L \in \mathbb{N}^*$ such that $L > N$ implies $s^*_L + r_o > a^*_N$" is true. So, going further, we have that "$\forall N \in \mathbb{N}^*/\mathbb{N}\ \exists L \in \mathbb{N}^*/\mathbb{N}$ such that $L > N$ implies $s^*_L + r_o > a^*_N \approx a$" is true. The last part of that statement is the same as "$s^*_L + r_o \geq a$," and if we let r be any real greater than r_o, we then have $s^*_L + r > a$. Recalling that $s^*_N \leq a^*_N$, we then automatically have the statement $s^*_L - r < a$. So, then the statement "$\forall N \in \mathbb{N}^*/\mathbb{N}\ \exists L \in \mathbb{N}^*/\mathbb{N}$ such that $L > N$ implies $|s^*_L - a| < r$" is true.

Then, the set $\{N \in \mathbb{N}^* : |s^*_N - a| < r\}$ is infinite for all real positive r (as r simply had to be greater than the already arbitrary positive r_o). So, by Transfer, the set $\{n \in \mathbb{N} : |s_n - a| < r\}$ is infinite for all positive real r. Thus, by theorem 3.2, there is a subsequence of s_n converging to a.

□

Chapter 4: Integration

4.1 Definition Let P be a partition of $[a,b]$ (that is, a finite subset of $[a,b]$). The Riemann Sum of f associated with P is a sum of the form $\sum_{k=1}^{n} f(x_k)(t_{k+1} - t_k)$ where $x_k \in [t_k, t_{k+1}]$, and $t_i \in$ P.

4.2 Definition A function is said to be Riemann integrable if $\exists r \in \mathbb{R}$ such that $\forall \epsilon \in \mathbb{R}^+$ $\exists \delta \in \mathbb{R}^+$ such that $mesh(\mathrm{P}) < \delta$ implies $|S - r| < \epsilon$ for every Riemann Sum S associated with P. We write $\int_a^b f(x)dx = r$ [4].

The negation of the definition of the Riemann Integrable is: if for all $r \in \mathbb{R}$ there exists an ϵ such that $\forall \delta \in \mathbb{R}^+$ there is a Riemann Sum, S, associated with a partition P such that mesh(P) ¡ δ and $|S - r| \geq \epsilon$. If one merely wants to say the Riemann Integral does not equal r, replace "if for all $r \in \mathbb{R}$" with "if for r". Note as well, in this negation, ϵ may depend on r.

4.3 Theorem Let $\{\mathrm{P_n}\}_{n=1}^{\infty}$ be a sequence of partitions of $[a, b]$ such that $\lim_{n\to\infty} mesh(\mathrm{P_n}) = 0$, let S_n be a Riemann Sum of a bounded function f associated with $\mathrm{P_n}$. Then, all such S_n converge to the same value, r, if and only if $\int_a^b f(x)dx = r$.

Proof.
Case 1: Let $\exists r \in \mathbb{R}$ such that $\forall \epsilon \in \mathbb{R}^+$ $\exists \delta \in \mathbb{R}^+$ such that $mesh(\mathrm{P}) < \delta$ implies $|S - r| < \epsilon$ for every Riemann Sum S associated with P. We have that $\forall \delta$ $\exists n_o \in \mathbb{N}$ such that $n > n_o$ implies $mesh(\mathrm{P_n}) < \delta$. Which means $\forall \epsilon \in \mathbb{R}^+$ $\exists n_o \in \mathbb{N}$ such that $n > n_o$ implies $|\int_a^b f(x)dx - S_n| < \epsilon$.

Case 2: Assume the negation of the existence of the Riemann Integral, that is for all $r \in \mathbb{R}$ there exists an ϵ such that $\forall \delta \in \mathbb{R}^+$ there is an Riemann Sum, S, associated with a partition P such that $mesh(\mathrm{P}) < \delta$ and $|S - r| \geq \epsilon$. Fix r, which guarantees a particular ϵ. Construct a sequence of partitions, $\mathrm{P_n}$ such that $mesh(\mathrm{P_n}) < \frac{1}{n}$. Clearly, its limit is zero. Construct a sequence of S_n's associated with $\mathrm{P_n}$ such that $|S_n - r| \geq \epsilon$ (by the hypothesis, for each $\frac{1}{n}$ one such S_n must exist). Thus $\forall n \in \mathbb{N}$, $|\int_a^b f(x)dx - S_n| \geq \epsilon$. Therefore $\lim_{n\to\infty} S_n \neq r$. Note that r was arbitrary, so it cannot be the case that every sequence of such

Riemann sums converges to an r. Also, note, each Riemann sum is bounded, so it may not diverge either. $\square$

The previous theorem allows one to say that a bounded function on [a,b] is Riemann integrable iff all sequences of Riemann sums, S_n, such that their associated partitions, $\mathrm{P_n}$, satisfy $\lim n \to \infty mesh(\mathrm{P_n}) = 0$, converges to the same value r, and furthermore, if such r exists, $\int_a^b f(x)dx = r$. So, that is the definition of the Riemann integral presented here.

Robinson defines his integral in the following way. Let S be a Riemann sum of f whose associated partition is an "internal fine partition," if all such S's have the same standard part, their standard part equals the integral. Basically, the way one would find an internal fine partition would be to take a sequence of Riemann sums such that the mesh goes to zero, extend this sequence in the usual way, and then evaluate the extended sequence at a hyperinfinite natural. Then, one would check to see if all such "infiniteth Riemann sums" have the same standard part. So, the sequential definition presented here captures the essence of Robinson's integral, while still being valid using just the machinery of standard analysis. [1]

This definition is different than the definition presented by Henle in his book. His integral is equivalent to the Riemann integral if the Riemann integral exists, but it may be able to integrate functions which are not Riemann integrable. It is relatively easy to see their integral is equivalent to a sequence of Riemann Sums whose mesh goes to zero, whose partitions are uniform, and that the evaluation points are the endpoints. That is, only uniform partitions are considered, and of those, only the endpoints are "selected" as evaluation points for the function. [3]

We now consider conditions to determine if a function is Riemann integrable.

4.4 Lemma 1 Let f be a bounded function on $[a,b]$ such that for every partition $\mathrm{P_n}$ such that $\lim_{n\to\infty} mesh(\mathrm{P_n}) = 0$, if S_n and R_n are both associated with $\mathrm{P_n}$, then $\lim_{n\to\infty} S_n - R_n = 0$. Then, for all Riemann sums T_n and B_n such that their associated partitions $\mathrm{Q_n}$ and $\mathrm{M_n}$ satisfy $\lim_{n\to\infty} mesh(\mathrm{Q_n}) = \lim_{n\to\infty} mesh(\mathrm{M_n}) = 0$, we have $\lim_{n\to\infty} T_n - B_n = 0$.

Proof. We begin with an induction argument.

Let $\mathrm{P_n}$ be a sequence of partitions of $[a,b]$, such that $\lim_{n\to\infty} mesh(\mathrm{P_n}) = 0$. Let $v \in [a,b]$ but never in $\mathrm{P_n}$. Define $\mathrm{Q_n} = \mathrm{P_n} \cup \{v\}$, that is if $\mathrm{P_n} = \{a = t_1 < t_2 ... < t_{k-1} < t_k < .... < t_n = b\}$ then $\mathrm{Q_n} = \{a = t_1 < t_2 ... < t_{k-1} < v < t_k < .... < t_n = b\}$. Let S_n be a Riemann sum associated with $\mathrm{P_n}$, and let R_n be a Riemann sum associated with $\mathrm{Q_n}$, such that, when their partitions coincide, they have the same evaluation points. Then R_n - S_n = $f(x_{v,k-1})(t_{k-1} - v) + f(x_{v,k})(v - t_k) - f(x_k)(t_k - t_{k-1})$ - where $x_{v,k-1}$ is between t_{k-1} and v, and $x_{v,k}$

is between v and t_k. Note that f is bounded by b, so the absolute value of this quantity is less than $3bmesh(\mathrm{P_n})$. Thus, $\lim_{n\to\infty} R_n - S_n = 0$. Furthermore, define $R_n^\star$ to be any Riemann sum associated with $\mathrm{Q_n}$ (that is, they may have different evaluation points). Consider $|R_n^\star - S_n| = |R_n^\star - R_n + R_n - S_n| \leq |R_n^\star - R_n| + |R_n - S_n|$. By the previous sentence and the hypothesis, both converge to zero, so $R_n^\star - S_n$ converges to zero. That settles the base case.

Now, Suppose $\mathrm{Q_n}$ has m more points than $\mathrm{P_n}$, suppose that both have meshes which converge to zero, and suppose that any Riemann sums R_n, associated with $\mathrm{Q_n}$, and S_n, associated with $\mathrm{P_n}$, satisfy $lim_{n\to\infty} R_n - S_n = 0$. Let $\mathrm{V_n}$ have 1 more point than $\mathrm{Q_n}$ (and share all other points), and T_n be a Riemann sum associated with $\mathrm{V_n}$. The first paragraph shows that $R_n - T_n$ converges to zero. Consider $|S_n - T_n| = |S_n - R_n + R_n - T_n| \leq |S_n - R_n| + |R_n - T_n|$. Both go to zero, so $S_n - T_n$ converges to zero. More generally put, this induction argument shows that if $\mathrm{P_n} \subset \mathrm{V_n}$, and the mesh of $\mathrm{V_n}$ goes to zero, any Riemann sum T_n associated with $\mathrm{V_n}$ satisfies $\lim_{n\to\infty} T_n - S_n = 0$, where S_n is a Riemann sum associated with $\mathrm{P_n}$.

Finally, let $\mathrm{M_n}$ and $\mathrm{Q_n}$ be partitions of $[a,b]$ such that their meshes converge to zero. Let T_n be associated with $\mathrm{Q_n}$, B_n be associated with $\mathrm{M_n}$, and R_n be associated with $\mathrm{P_n} \cup \mathrm{Q_n}$. Consider $|B_n - T_n| = |B_n - R_n + R_n - T_n| \leq |B_n - R_n| + |R_n - T_n|$. By the previous paragraph, these two quantities go to zero, so $\lim_{n\to\infty} B_n - T_n = 0$. □

4.5 Lemma 2 Let f be a bounded function on $[a,b]$ such that for every partition $\mathrm{P_n}$ such that $\lim_{n\to\infty} mesh(\mathrm{P_n}) = 0$, if S_n and R_n are both associated with $\mathrm{P_n}$, then $\lim_{n\to\infty} S_n - R_n = 0$. Then f is Riemann integrable.

Proof. Since f is bounded, by the previous lemma, it suffices to find one sequence of Riemann sums (whose associated mesh converges to zero) that converges. This is because any other Riemann sum minus this convergent Riemann sum converges to zero, so that other Riemann sum must also have the same limit as the convergent Riemann sum.

Let $\mathrm{P_n}$ be a partition of $[a,b]$ such that its mesh converges to zero. Let S_n be a Riemann sum associated with $\mathrm{P_n}$. As f is bounded, this sequence is bounded, so by the Bolzano-Weierstrass Theorem, there exists a convergent subsequence of S_n. This subsequence is a convergent sequence of Riemann sums such that the associated mesh goes to zero. Thus, the proof is complete. □

In Ross, and in many standard texts on elementary analysis, Darboux Sums are used as a way to establish theorems about Riemann integrals, as well as its existence. This relies on the least upper-bound property of the real numbers - and was avoided in this text as hyperreals lack that property. However, the existence of the Riemann integral relies on the Bolzano-Weierstrass theorem, which in this formulation, is equivalent to the least-upper-bound property of the real numbers.

4.6 Theorem Every monotonic function on [a,b] is Riemann integrable.

Proof. Let f be increasing, and let $f(a) < f(b)$. As it is monotonic, it is clearly bounded by $f(a)$ from below, and $f(b)$ from above. Let $\{mathrmP_n\}_{n=1}^{\infty}$ be a sequence of partitions of [a,b] such that its mesh converges to zero. Consider the Riemann sums $L_n = \sum_{k=1}^{n} f(t_k)(t_{k+1} - t_k)$ and $U_n = \sum_{k=1}^{n} f(t_{k+1})(t_{k+1} - t_k)$. Clearly, L_n is less than every Riemann sum associated with $\mathrm{P_n}$, and U_n is greater than all such Riemann sums.

Consider $L_n - U_n = \sum_{k=1}^{n}(f(t_{k+1}) - f(t_k))(t_{k+1} - t_k) \leq \sum_{k=1}^{n}(f(t_{k+1}) - f(t_k))mesh(\mathrm{P_n})$. This equals $mesh(\mathrm{P_n}) * \sum_{k=1}^{n} f(t_{k+1}) - f(t_k)$. The sum is a telescoping sum, so the whole thing equals $mesh(\mathrm{P_n})(f(b) - f(a))$, and this clearly converges to zero. Thus by Lemma 2, f is Riemann integrable. (The proof for a decreasing function is similar).

□

4.7 Theorem Every continuous function f on [a,b] is Riemann integrable.

Proof. Let $\{mathrmP_n\}_{n=1}^{\infty}$ be a sequence of partitions of [a,b] such that its mesh converges to zero. Let S_n and R_n be Riemann sums associated with $\mathrm{P_n}$. $S_n - R_n$ then equals $\sum_{k=1}^{n}(f(y_k) - f(x_k))(t_{k+1} - t_k)$. Recall that f must be uniformly continuous, which means $\forall\epsilon \in \mathbb{R}^+ \ \exists\delta \in \mathbb{R}^+$ such that $|y_k - x_k| < \delta$ implies $|f(y_k) - f(x_k)| < \frac{\epsilon}{b-a}$ and at the same time $\forall\delta \in \mathbb{R}^+ \ \exists n_o \in \mathbb{N}$ such that $n > n_o$ implies $mesh(\mathrm{P_n}) < \delta$.

So, choose a suitable n_o so that $n > n_o$ implies $mesh(\mathrm{P_n}) < \delta$ implies $|f(y_k) - f(x_k)| < \frac{\epsilon}{b-a}$. That is, $n > n_o$ implies $|S_n - R_n| < \sum_{k=1}^{n} \frac{\epsilon}{b-a}(t_{k+1} - t_k)$, which equals ϵ. Thus, $\lim_{n\to\infty} S_n - R_n = 0$. By Lemma 2, f is integrable.

□

We now prove the Fundamental Theorem of Calculus.

Fundamental Theorem of Calculus Part 1 Let f be differentiable on (a, b). Let $f'(t)$ be integrable on [a,b]. Then $\int_a^b f'(x)dx = f(b) - f(a)$.

Proof. Let S_n be a sequence of Riemann sums whose associated partition's mesh converges to zero. Let S_n be of the form $\sum_{k=1}^{n} f(x_k)(t_{k+1} - t_k)$.

Consider $\sum_{k=1}^{n} f(x_k)(t_{k+1} - t_k)$ -$\sum_{k=2}^{n-1} f(x_k)(t_{k+1} - t_k) = f(x_n)(t_{n+1} - t_n) + f(x_1)(t_2 - t_1)$ As the mesh converges to zero, this converges to zero, so $\sum_{k=1}^{n} f(x_k)(t_{k+1} - t_k)$ and $\sum_{k=2}^{n-1} f(x_k)(t_{k+1} - t_k)$ have the same limit.

Let $\mathrm{P_n}$ be a uniform partition such that $\lim_{n\to\infty} mesh(\mathrm{P_n}) = 0$, so that is $\Delta x = \frac{b-a}{n}$.

Consider $\sum_{k=2}^{n-1} f'(x_k)\Delta x$ - $\sum_{k=2}^{n-1} \frac{f(x_k+\Delta x)-f(x_k)}{\Delta x}\Delta x$ when the two Riemann sums have the same partition, $\mathrm{P_n}$, and x_k's (the evaluation points) are the same. This then equals $\sum_{k=2}^{n-1}(\frac{f(x_k+\Delta x)-f(x_k)}{\Delta x} - f'(x_k))\Delta x$. Note that for every $\epsilon \in \mathbb{R}^+$ $\exists\delta \in \mathbb{R}$ such that $\Delta x < \delta$ implies $|\frac{f(x_k+\Delta x)-f(x_k)}{\Delta x} - f'(x_k)| < \frac{\epsilon}{b-a}$, as well as

$\forall \delta \in \mathbb{R}\ \exists n_o \in \mathbb{N}$ such that $n > n_o$ implies $\Delta x < \delta$. So, by selecting an appropriate n_o, $|\sum_{k=2}^{n-1}(\frac{f(x_k+\Delta x)-f(x_k)}{\Delta x} - f'(x_k))\Delta x| \leq \sum_{k=2}^{n-1}|(\frac{f(x_k+\Delta x)-f(x_k)}{\Delta x} - f'(x_k))|\Delta x < \sum_{k=2}^{n-1}\frac{\epsilon}{b-a}\Delta x = \frac{n-2}{n}\epsilon = (1-\frac{2}{n})\epsilon$. This converges to zero as n goes to infinity, so the difference between these two sums converges to zero. That means$\sum_{k=2}^{n-1}\frac{f(x_k+\Delta x)-f(x_k)}{\Delta x}\Delta x$ and $\sum_{k=2}^{n-1} f'(x_k)\Delta x$ converge to the same limit, which by the first paragraph, means $\sum_{k=1}^{n-1}\frac{f(x_k+\Delta x)-f(x_k)}{\Delta x}\Delta x$ converges to $\int_a^b f'(x)dx$.

Take note that the bounds needed to be from $k = 2$ to $k = n-1$, else, the step allowing the summand to be less than $\frac{\epsilon}{b-a}$ would not have been valid, as f is not necessarily differentiable at a or b.

Consider now $\sum_{k=1}^{n-1}\frac{f(x_k+\Delta x)-f(x_k)}{\Delta x}\Delta x$, where x_k is fixed to be kth term in the partition, and the partition has constant distance between all neighboring points, that is $\Delta x = \frac{b-a}{n}$. This sum, by basic algebra, is

$$\sum_{k=1}^{n-1} f(x_k+\Delta x) - f(x_k) = \sum_{k=1}^{n-1} f(x_{k+1}) - f(x_k)$$

This is a telescoping sum, so it is equal to $f(x_n) - f(x_1) = f(b) - f(a)$. As this sequence converges to the integral, $\int_a^b f'(x)dx = f(b) - f(a)$. □

Fundamental Theorem of Caclulus Part 2 Let f be a bounded Riemann Integrable function of [a,b], then $F(t) = \int_a^b f(x)dx$ is continuous on [a,b]; furthermore, if f is continuous [a,b], then $F'(t) = f(t)$.

Proof. Let (c_m) be a sequence converging to c in (a,b). Let R_n be a sequence of Riemann sums associated with $\mathrm{P_n}$ be a partition of $[a,b]$ such that $\lim_{n\to\infty} mesh(\mathrm{P_n}) = 0$. Let $\{\mathrm{B}_n\}_{n=1}^{\infty}$ be a sequence of partitions of [c,c_m] such that $\lim_{n\to\infty} mesh(B_n) = 0$. Define $\mathrm{Q_n} = \mathrm{P_n} \cup \mathrm{B_n}$, it is easy to see that $mesh(\mathrm{Q_n})$ converges to zero. Let S_n be a sequence of Riemann sums associated with $\mathrm{Q_n}$ such that all chosen x_k's $\in [t_{k+1}, t_k]$, when $t_{k+1}, t_k \in \mathrm{P_n}$, are the same x_k's chosen for R_n. Therefore, $S_n - R_n$ is a Riemann sum on $[c, c_m]$. By assumption f was bounded from below by m, and above by M, so this Riemann sum is bounded from below by $m*(c_m - c)$ and from above by $M*(c_m - c)$. As m goes to infinity, these go to zero, so that must mean $\lim_{n\to\infty} S_n - R_n = 0$. In other words, $\lim_{m\to\infty}\int_a^{c_m} f(x)dx - \int_a^c f(x)dx = 0$, which is to say the integral is continuous.

Let f also be continuous, then the proof done by ross in *Elementary Analysis* page 298-295 suffices [4].

□

Conclusion

Non-standard analysis requires a fair bit of set theory in order to establish the existence of the hyperreals. After this, the theorems follow rather smoothly from the definitions, and it is arguable that they appeal to a certain intuition. Many of the theorems presented in this text were shorter than the corresponding theorems of standard analysis.

An understanding of non-standard analysis yields a deeper understanding of standard analysis, necessary to demonstrate the equivalence of the respective definitions and concepts. Indeed, the definitions of non-standard and standard analysis yield similar theorems and results. The proofs of their equivalence are simple enough that they are the same as the “intuitive reason” that these definitions say the same thing.

Throughout this text we attempted to handle sequences within the non-standard approach. The proof of theorem 3.8 was the only exception that relied on a standard result. Non-standard reasoning was used in order to prove that for every positive real number, r, the set $\{N \in \mathbb{N} : |s_n - a| < r\}$ was shown to be infinite. Only then was the standard result used. In this way, the theorem can be thought of as being like a theorem that establishes an equivalence between the standard and non-standard definitions of limit suprema and infima (even though that both definitions were ”standard”).

In Chapter 4, we only used standard reasoning. However, integration was presented differently than it would be otherwise. Typically, Darboux sums are used to prove properties of and theorems concerning integrals. It is still not immediately clear how one would hyperextend the Darboux sums, as they are functions of suprema and infima. In other attempts to develop the Riemann integral - in Goldblatt’s and Henle’s works - only uniform partitions are considered. So, in their treatment of the Riemann integral, they develop an integral which is only equivalent to the Riemann integral if the Riemann integral exists (and is in that sense a sort of generalization of the Riemann integral). In the way that I have handled the Riemann integral it is really equivalent to the Riemann integral. Two lemmas in chapter 4 appear to be original results and are used to establish theorems concerning Riemann integration. Presumably one would need to rely on these lemmas if, instead of working with limits of a sequence of sums, they directly hyperextended these sums.

Throughout this honor thesis, I learned a lot. Because I needed to construct the hyperreals, I had to learn what Zorn's lemma is, what a chain is, and what an ultrafilter is. Not merely repeating the definitions and statements, but understanding what they are saying so that I could actually work with them. Then, I developed the skills necessary to see how one shows the equivalence between non-standard analysis and standard analysis. This is important if I would want to continue a study of non-standard analysis for topology, distribution theory, measure theory, etc. I have also gained a proficiency in mathematical writing, LateX, and mathematical citations - something I would not have developed as an undergraduate if I did not work on this thesis.

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