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Using a Coupled Bio-Economic Model to Find the Optimal Phosphorus Load in Lake Tainter, WI

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1 Abstract

In Dunn County, Wisconsin the lakes suffer from algae blooms due to excess phosphorus runoff. A coupled bio-economic model is studied with the intent of finding the optimal level of phosphorus that should be allowed into the lake depending on certain biologic and economic parameters. We model the algae and phosphorus concentration in the lake over time based off the phosphorus input. Community welfare is modeled by comparing the costs and benefits of phosphorus fertilizer. This model is proposed to find the phosphorus level that maximizes community welfare and then determine how certain environmental and social change initiatives will affect the phosphorus, algae, and welfare level.

1.1 Motivation

In late summer, Tainter Lake, WI is plagued by algae blooms caused by the cyanobacteria *Microcystis aeruginosa* [17]. These algae blooms create a multitude of problems including decreased housing prices and increased toxin levels [12, 21]. Algae growth is fueled from the high phosphorus concentration in the lake, which has phosphorus levels that are 3-10 times over the levels deemed eutrophic. Lakes that are eutrophic are unhealthy and have an over-abundance of nutrients which leads to excess growth of algae and a depletion of oxygen which kills both plant and aquatic life. This high phosphorus concentration is caused from phosphorus in the sediment, from agricultural runoff, and from other eutrophic lakes in the watershed that funnel into Tainter Lake [17].

To better understand the problems of Tainter Lake, we expand upon the analysis of the shallow lake problem. Shallow lakes are lakes with a mean depth of 3 meters or less. They can either be in a low-polluted or a high-polluted state based on the amount of phosphorus that goes into the lake. These lakes can only be in a high-polluted or a low-polluted state because of how they process phosphorus, as described in Section 1.2. The shallow lake problem is to determine whether a certain shallow lake should be managed to be in the low-polluted or high-polluted state based on the social welfare of using phosphorus. Shallow lakes have been studied both ecologically [21] and economically [25, 14, 6]. However, more analysis and modeling needs to be done, in particular to understand how the phosphorus level in shallow lakes responds to phosphorus input and how utility responds to increases in pollution.

1.2 Ecological Background

Phosphorus enters the water mainly through the runoff from agricultural land, where fertilizer and manure are used to load the soil with nitrogen and phosphorus for crop growth. Wastewater treatment plants also exude some phosphorus in the outflow, though it is only a small percentage of the total phosphorus input. In recent years runoff from farms has been increasing due to the combination of increased spring rain storms from climate change and an increasing need to load unhealthy soil as a result of overproduction [23, 8, 2]. Phosphorus pollution has continued to be a problem because phosphorus from farm runoff is nonpoint pollution, making it impossible to tell from which farms the pollution is coming from.

Shallow lakes are more difficult to manage than deep lakes because of their tendency to display catastrophic flipping behavior. This means that small increases in the input of phosphorus have the ability to switch the lake from a low-polluted oligotrophic state to a high-polluted eutrophic state. Lakes that are oligotrophic are the opposite of eutrophic; they are healthy and have low nutrient levels, low algae growth, and high oxygen levels. Although it has been observed that shallow lakes display this nonlinear switching behavior, it is still unclear as to what causes the flipping behavior due to the many factors influencing these lakes [21].

When the lake switches to the high-pollution state, many qualitative changes happen to the lake including a decrease in the vegetation and water clarity, and an increase in the amount of phytoplankton and suspended solids. These changes are reinforcing feedback mechanisms. When there is low vegetation cover on the bottom of the lake, sediment is more easily disturbed from fish swimming in the lake. The disturbance of sediment decreases the water clarity and introduces more phosphorus into the water column that was previously in the sediment. When water clarity is low, less light can reach the vegetation, and this causes it to grow less. In addition, when there are fewer plants they use less phosphorus for photosynthesis. This allows the phytoplankton to consume more of the phosphorus allowing their population to grow, further decreasing the water clarity. The combination of these conditions help enforce the two opposing states of the lake of either water clarity or turbidity.

These feedback mechanisms also help explain why to go back from a high-polluted state to a low-polluted state, the phosphorus input level has to decrease below the input level that originally caused the lake to flip to the high-polluted state. Another way to induce a shift back to a low-polluted state is to introduce new species of plants or mussels through biomanipulation to decrease the amount of phosphorus in the lake [21]. This has been tried in a few lakes. While in the short term the lake becomes more clear, in the long term overpopulation of the new species may occur and disrupt the natural ecosystem, which can cause more costs than benefits as shown by Lake Erie [23, 7].

One of the many problems with phosphorus pollution in lakes is the growth of algae. Some amount of algae is considered good for a healthy, biodiverse ecosystem. However, the algae in polluted lakes tends to be dominated by toxic cyanobacteria. Cyanobacteria has the advantage of being able to take in more phosphorus than necessary for biological production and store it for later use [11]. In order to grow and dominate, cyanobacteria need a high nitrogen-to-phosphorus ratio, as well as sunlight, optimum temperature, low turbulence, and long water residence time. In eutrophic lakes, there is typically enough nitrogen and phosphorus for cyanobacteria to grow and produce organic matter, which further increases eutrophication and bloom potential. An algae bloom occurs when there is a rapid increase in the algae population. Shallow lakes are particularly plagued with cyanobacteria since shallow lakes warm up faster than deep lakes, creating a perfect environment for growth.

These blooms can produce liver toxins that are potentially deadly to pets and people who go into the lake [19]. These toxins can force beaches to close, having a negative impact on tourism and on the housing prices near the dirty and toxic lake [12, 4]. If the lake is also used for drinking water, then these toxins could prevent townspeople from having access to clean drinking water for several days, similar to what happened in Toledo, Ohio [22]. Even if the water does not have enough toxins to close the beaches, blooms can cover the lake in thick biomass, causing the lake to produce an awful smell and appear unsightly. The blooms around the edges prevent townspeople from swimming and taking their canoes or boats out onto the lake, thereby having a negative impact on tourism and recreation for the local townspeople.

1.3 Literature Review

Shallow lakes have many, conflicting uses. A shallow lake can serve primarily as a waste sink that is only used for agricultural runoff. Or a shallow lake can be treated as a resource for recreation and fishing. The balance between these uses depends on the value of farming and on the value the townspeople hold for natural beauty, recreation, and tourism. This paper finds the optimal phosphorus level to maximize utility. Policies can then be designed around the best way to reach the optimal level of phosphorus. With nonpoint pollution, there is no accurate, cost-effective way to measure the exact phosphorus load from each plot of land utilizing chemicals which makes it difficult to apply common price and quantity based policies to hold farms accountable for the runoff they produce. Several policies have been adopted to combat this problem including taxing fertilizer, using a permit system, and subsidizing best management practices.

Previously, Wagener and Mäler have looked at balancing these costs and benefits by considering a single social planner who determines the optimal pollution level both over time and in a single time period [25, 14]. They find that the optimal phosphorus loading level occurs where the marginal benefits from polluting equal the marginal costs. Depending on the parameters, there are either 1 or 2 solutions. If there are 2 solutions, then one corresponds to a low-pollution equilibrium and the other corresponds to a high-pollution equilibrium. Both solutions need to be checked to find the largest total welfare.

Suzuki has expanded the literature by introducing multiple communities using the lake that have different social values. Each community plays a pollution game against one another for the ability to use the resource [24]. They find that when more players are cooperative and choose a lower pollution level this increases social pressure and induces other players to be more cooperative.

Carpenter and Zeng have expanded upon the biological model [26, 5]. Carpenter runs numerical simulations on a lake by changing the variability over time of the level of phosphorus input. They find that if the phosphorus input variability decreases in the short-term then long-term variability increases and makes the lake more susceptible to flip from an oligotrophic to a eutrophic state. Zeng expands this and examines how the variability over time in the phosphorus input and phosphorus recycling affects the state of the lake. They find that if these two variabilities or noises are positively correlated then this can cause the lake to shift more quickly from an oligotrophic to a eutrophic state. In addition, if they are negatively correlated then this can cause the lake to shift more quickly from a eutrophic to an oligotrophic state.

Due to the many problems surrounding phosphorus pollution and algae blooms, this model was created to find the optimal phosphorus input level for Tainter Lake given its specific biological characteristics and how well the townspeople and farmers can tolerate pollution. This lake was chosen to be calibrated with the model due to the large amount of interdisciplinary research done on the lake by the University of Wisconsin-Stout through Research Experiences for Undergraduates (REU) so both the economic and ecologic parameters can be estimated.

This paper expands upon the shallow lake literature by considering the townspeople's disutility of pollution depending on the amount of algae in the lake instead of the phosphorus level in the lake. This was done because the townspeople care more about the algae level which is clearly visible and directly contributes to the aesthetic value and toxin level of the lake. In addition, we will also consider whether the community would be better off by implementing a tax on excess phosphorus use.

This paper is organized as follows. Section 2 describes the lake dynamics by using ordinary differential equations to model phosphorus and algae concentration over time. These equations are analyzed by finding fixed points and phase planes based off different parameter sets. Section 3 examines the social welfare by accounting for the utility of the farmers based on their phosphorus use and the disutility of townspeople around the lake due to the resulting algae. This constrained optimization problem of maximizing welfare subject to the lakes biological dynamics is solved using optimal control theory. Section 4 uses data from the Department of Natural Resources (WDNR) and previous REUs to run simulations for Lake Tainter with and without a tax. Since the initial condition is uncertain we find that the community is most likely better off with a tax. In addition, if the community can enact social or ecological change, altering the loss rate of phosphorus has the biggest positive impact on social welfare.

2 Coupled Biological System

Modeling phosphorus and algal dynamics was done by using a coupled ordinary differential equation model modified from Scheffer's algae growth model [21] and Carpenter's phosphorus model [6]. Both are influenced by the background pollution in the lake, how quickly the sediment and algae can process excess phosphorus, and phosphorus input. We chose to expand upon the standard shallow lake model by adding a term for algae growth to make this system more realistic. It should be noted that these equations only accurately describe shallow lake dynamics and other models are used for deeper lakes.

The lake dynamics over time are described by:

$$\begin{cases} \frac{dP}{dt} = I - l_p P + \frac{r_p P^2}{h_p^2 + P^2} \\ \frac{dA}{dt} = r_a A \frac{s}{s + D(e_a A + E_b)} \frac{P - p_a A}{h_a + P - p_a A} - l_a A \\ P(0) = P_0 \\ A(0) = A_0, \end{cases} \tag{1}$$

where the functions P and A represent the concentration of phosphorus and algae in the water, respectively.

The first equation models how the concentration of phosphorus in the water column is changing over time. The term I represents the phosphorus input from agricultural runoff. The term $l_p P$ represents the phosphorus loss from the water column due to the natural flushing of the lake, phosphorus entering the sediment, and algae absorbing phosphorus for growth. The term $\frac{r_p P^2}{h_p^2 + P^2}$ represents the internal recycling pattern in shallow lakes where phosphorus leaves the sediment and enters the water column.

The second equation models how the algae concentration in the water column is changing over time. The term $r_a A$ represents the maximum growth algae can attain when it is not restricted by light or nutrients. The term $\frac{s}{s + D(e_a A + E_b)}$ represents growth limitation due to light and the term $\frac{P - p_a A}{h_a + P - p_a A}$ represents growth limitation due to phosphorus. Both of the limitation terms are smaller than one, meaning that the algae will never achieve maximum growth. Therefore, the term $r_a A \frac{s}{s + D(e_a A + E_b)} \frac{P - p_a A}{h_a + P - p_a A}$ expresses the bounded growth of algae taking into consideration the light and nutrient availability. The term $l_a A$ represents the loss of the algae due to the flushing of the lake and natural mortality.

Two growth limitation terms are needed because light and phosphorus are the main limiters in growth. Including only one limitation term would result in exponential growth but including both results in more realistic long term growth trends. To simplify system (1), we propose replacing the term $\frac{s}{s + D(e_a A + E_b)}$ that describes light attenuation with a constant, α. When we nondimensionalize and linearize (1) this allows us to solve for fixed points analytically and prove their stability. In addition, the system still displays non-exponential growth when we replace the light attenuation term with a fraction.

The parameter I represents the concentration of phosphorus in the inflowing water. Loss rates are represented by l with l_a representing the loss rate of algae due to flushing and mortality and l_p representing the loss rate of phosphorus due to flushing, sedimentation, and algal use. Maximum growth rates are represented by r with r_a denoting the maximum growth rate algae can biologically attain and r_p denoting the maximum rate of recycling, or the rate which phosphorus moves from the sediment to the water column. Half-saturation is represented by h with h_a representing half of the maximum phosphorus concentration for algae with p_a representing the actual phosphorus content of algae and h_p representing the phosphorus concentration at which phosphorus recycling reaches half of it's maximum. The background turbidity or how polluted the water is without algae is represented by E_b. The light attenuation coefficient is e_a which tells how deep the light can go through the water column and s represents the shade tolerance of the algae. The average depth of the lake is represented by D. The range of values for each parameter is listed in Table 1 along with their data sources.

Parameter	Description	Value	Units	Source
Phosphorus Equation				
P	Phosphorus concentration	0.05 - 0.4	mgL^{-1}	WDNR [18]
I	Inflow phosphorus concentration	0.12	mgL^{-1}day^{-1}	WDNR, Hein [16, 18, 9]
l_p	Phosphorus loss rate	0.1 - 0.9	day^{-1}	WDNR [18]
r_p	Max phosphorus recycling rate	0.4 - 0.9	mgL^{-1}day^{-1}	WDNR, Genkai [18, 10]
h_p	Half recycling phosphorus concentration	0.05 - 0.4	mg L^{-1}	WDNR [18]
Algae Equation				
A	Algae concentration	0.1 - 0.7	mgL^{-1}	REU [3]
l_a	Algae loss rate	0.1 - 0.9	day^{-1}	Scheffer, WDNR [21, 18]
r_a	Max algae growth rate	0.4 - 0.6	day^{-1}	Scheffer [21]
h_a	Half saturation	2 - 4	mgL^{-1}	Scheffer, REU [21, 3]
p_a	Phosphorus content percentage	1	—	Scheffer [21]
E_b	Background turbity	0 - 1	m^{-1}	Scheffer [21]
e_a	Light attenuation coefficient	1 - 2.2	m^{2}g^{-1}	WDNR, Hein [18, 9]
s	Shade tolerance	1	—	Scheffer [21]
D	Average lake depth	3.4	m	WDNR [18]
α	Light attenuation	0.1 - 0.75	—	WDNR, REU [18, 3]

Table 1: Ranges of variable and parameter values used for simulations using data from 1989 - 2017 for Lake Tainter.

Using the following rescaling

$$T = \frac{r_p}{h_p}t,\ X = \frac{1}{h_p}P,\ Y = \frac{p_a}{h_p}A,$$

system (1) becomes

$$\begin{cases} \frac{dX}{dT} = a - bX + \frac{X^2}{1+X^2} = f(X,Y) \\ \frac{dY}{dT} = \alpha c Y(\frac{X-Y}{h+X-Y}) - dY = g(X,Y) \\ X(0) = X_0 \\ Y(0) = Y_0. \end{cases} \tag{2}$$

The nondimensional parameter groupings for (2) are

$$a = \frac{I}{r_p},\ b = \frac{h_p l_p}{r_p},\ c = \frac{r_a h_p}{r_p},\ d = \frac{l_a h_p}{r_p},\ h = \frac{h_p}{h_a}.$$

The most important nondimensional parameters for the analysis and interpretation are a, b, c, d. Here they represent the rescaled phosphorus loading, loss rate of phosphorus, growth rate of algae, and mortality rate of algae, respectively.

2.1 Model Analysis

To analyze the behavior of (2) we find the fixed points of the system. A fixed point occurs at values that make the right hand side of the differential equation equal to 0. This also means that fixed points represent where the system is not changing. Fixed points can be classified as either stable or unstable. If a fixed point is stable then solutions that start near the fixed point stay near the fixed point. If a fixed point is unstable then at least one solution moves away from the fixed point over time. For a system of differential equations it is important to find fixed points because we would like to know where the system ends up over time.

First, we look at the dynamics of the phosphorus and algae equations individually before analyzing the coupled behavior. The first equation in (2) does not depend on Y so we begin by understanding its dynamics. In previous literature [6, 25, 14], only the first equation in (2) was used to describe lake biology. This equation has been studied extensively. We refer the reader to those papers for more details. Here we briefly explain the phosphorus dynamics. Based on different values of b the equation can either have 1 or 3 fixed points. If there are 3 fixed points the lake displays a switching behavior. Figures 1, 2, and 3 display the number of fixed points and their stability as a function of a. In these figures the dashed lines represent unstable fixed points and the solid lines represent stable fixed points. These graphs are read by following the solid lines as loading level changes to find the phosphorus level the system goes to.

When $0.5 < \mathrm{b} \leq 0.65$ the system displays a reversible switching behavior, which is shown in Figure 1. This is called a switching behavior because if the system starts on the lower branch (solid line) and the phosphorus loading is increased, once the loading level reaches a threshold value, a_h, the phosphorus level will jump up from the lower branch to the upper branch. The phosphorus level will have to decrease past the original threshold value to a_l to get back on the lower pollution branch. This is a graphical representation of the reinforcing effects and the difficulty of switching from a eutrophic to an oligotrophic state.

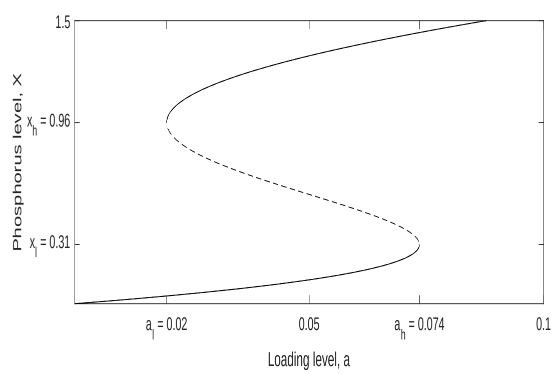

Figure 1: Bifurcation diagram for b = 0.52. The system displays discontinuous though reversible switching behavior.

In some cases if the system is on the high pollution branch then the switch is irreversible and there is no way to go back down to the low pollution branch. This happens when $0 < \mathrm{b} \leq 0.5$, as shown in Figure 2. If there is no switching behavior then small increases in the phosphorus loading level always lead to small increases in the phosphorus level in the lake. This happens when $\mathrm{b} \geq 0.65$, as shown in Figure 3.

Larger sedimentation rates mean that phosphorus is leaving the water column and entering the sediment relatively quickly. This indicates that the system can handle increases in phosphorus naturally and it is why there is no abrupt switching behavior once b gets large enough. We are mostly concerned in the cases where there are 3 fixed points, because these

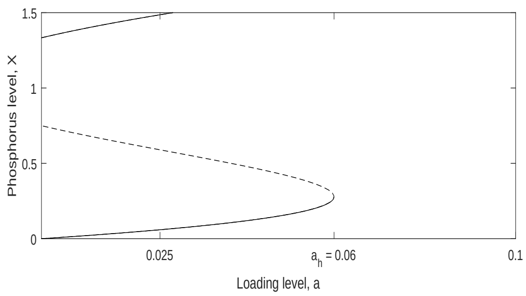

Figure 2: Bifurcation diagram for b = 0.48. The switching behavior is irreversible.

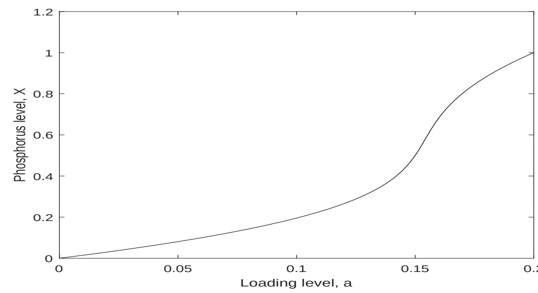

Figure 3: Bifurcation diagram for b = 0.7. The system displays a continuous response to loading level.

cases provide the most interesting behavior and should be the ones we concern ourselves with for policy analysis since allowing phosphorus loading to increase too much could lead to catastrophic shifts in those ecosystems.

Let $\bar{x}$ represent a fixed point to the first equation in (2). To find the Y-component of the fixed points of (2) we consider the equation $\dot{Y} = Y(\frac{\bar{x}-Y}{h+\bar{x}-Y} - \frac{d}{\alpha c}) = 0$. We see that $(\bar{x}, 0)$ is a fixed point. To find the nonzero Y values satisfying this equation we check where $m(Y) = \frac{\bar{x}-Y}{h+\bar{x}-Y}$ intersects the horizontal line $m = \frac{d}{\alpha c}$. The graph of $m(Y)$ is shown in Figure 4 with 2 different values of $m = \frac{d}{\alpha c}$. It can be seen that the graph has a vertical asymptote at $Y = \bar{x} + h$ and a horizontal asymptote at $m = 1$. Also, the graph crosses the Y-axis at $\bar{x}$ and crosses the m-axis at $\frac{\bar{x}}{\bar{x}+h}$. This means that the values for the Y-coordinate of the fixed point are constrained between $(0, \bar{x})$ and $(\bar{x}+h, \infty)$. The value of $\frac{d}{\alpha c}$ determines which interval the Y coordinate of the fixed point is located.

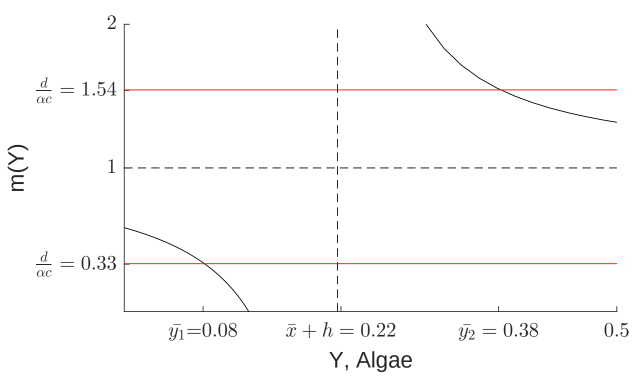

Figure 4: Varying values of $\frac{d}{\alpha c}$ to determine the Y-coordinate of the fixed point for the second equation in (2).

If $0 < \frac{d}{\alpha c} < \frac{\bar{x}}{\bar{x}+h}$ then $0 < \bar{y} < \bar{x}$. If $\frac{d}{\alpha c} > 1$ then $\bar{y} > \bar{x} + h$. If $\frac{\bar{x}}{\bar{x}+h} < \frac{d}{\alpha c} < 1$ then $\bar{y} < 0$. While this is fine from a mathematical point of view, because we cannot have negative population sizes this fixed point cannot happen realistically. Therefore, the system only has the fixed point $(\bar{x}, 0)$.

It should be noted that as $\bar{x} \to 0$, $\frac{\bar{x}}{\bar{x}+h} \to 0$, which implies for smaller values of $\bar{x}$, the graph has a smaller m intercept. This also means that if the smallest $\bar{x}$ value corresponds to a positive $\bar{y}$ then all the other $\bar{x}$'s for the system will correspond to positive $\bar{y}$ values. When $\frac{d}{\alpha c} < 1$ this leads to lower $\bar{y}$ values than when $\frac{d}{\alpha c} > 1$. In Figures 5 and 6, we show two possible phase planes for (2). Figure 5 corresponds to $\frac{d}{\alpha c} < 1$ and Figure 6 corresponds to $\frac{d}{\alpha c} > 1$.

In Figure 4, $\bar{x}$ was found by using the parameters $a = 0.05$ and $b = 0.52$ and the smallest fixed point $\bar{x}_l = 0.1264$ was used. The other parameters used were $d = 0.25$, $h = 0.09$, $\alpha = 0.6$, and $c = 0.27$ for the upper horizontal line, $\frac{d}{\alpha c} = 1.54$, and $c = 1.25$ for the lower horizontal line, $\frac{d}{\alpha c} = 0.33$.

2.2 Linearization

Next, we linearize around the fixed points to determine the local behavior of the solutions. We will then verify this numerically.

We let $(\bar{x}, \bar{y})$ denote a fixed point to system (2). We know a fixed point $(\bar{x}, \bar{y})$ satisfies $a - bX + \frac{X^2}{X^2+1} = 0$ and $\alpha c Y(\frac{X-Y}{h+X-Y}) - dY = 0$. We cannot solve analytically for $\bar{x}$ but we can solve analytically for $\bar{y}$ in terms of $\bar{x}$. After some algebraic manipulation we find $\bar{y} = \frac{dh}{(d-\alpha c)} + \bar{x}$, assuming that $\bar{y} \neq 0$. This also implies that if there are 3 values for $\bar{x}$ satisfying $\dot{X} = 0$ then there are 3 nonzero values for $\bar{y}$. Similarly, if there is 1 value for $\bar{x}$ satisfying $\dot{X} = 0$ then there will be 1 nonzero value for $\bar{y}$. It should also be noted that $(\bar{x}, 0)$ is always a fixed point and the number of fixed points of this form also depends on the number of values for $\bar{x}$.

Next, to find the stability of the fixed points to (2), we linearize the system. The linearization around the fixed point $(\bar{x}, \bar{y})$ is

$$\begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} \frac{\partial f(\bar{x},\bar{y})}{\partial X} & \frac{\partial f(\bar{x},\bar{y})}{\partial Y} \\ \frac{\partial g(\bar{x},\bar{y})}{\partial X} & \frac{\partial g(\bar{x},\bar{y})}{\partial Y} \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix},$$

where

$$\frac{\partial f(\bar{x},\bar{y})}{\partial X} = -b + \frac{2\bar{x}}{(\bar{x}^2+1)^2} = \lambda_1$$

$$\frac{\partial f(\bar{x},\bar{y})}{\partial Y} = 0$$

$$\frac{\partial g(\bar{x},\bar{y})}{\partial X} = \frac{\alpha h c \bar{y}}{(h+\bar{x}-\bar{y})^2} = \beta$$

$$\frac{\partial g(\bar{x},\bar{y})}{\partial Y} = \alpha c \frac{\bar{x}-\bar{y}}{h+\bar{x}-\bar{y}} - \frac{\alpha c h \bar{y}}{(h+\bar{x}-\bar{y})^2} - d = \lambda_2.$$

We can rewrite the system as

$$\begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} \lambda_1 & 0 \\ \beta & \lambda_2 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}. \tag{3}$$

Since we have a lower triangular matrix we know the eigenvalues of this system are the entries on the diagonal.

To check if the fixed points for the system are stable we must check the sign of λ_1 and λ_2. For any (a, b) parameter pair chosen, if $\bar{x}$ is a stable fixed point to the first equation in

(2) then $\lambda_1 < 0$. Similarly, if $\bar{x}$ is an unstable fixed point then $\lambda_1 > 0$. For any parameter grouping with $\bar{y} > 0$ then $\lambda_2 < 0$. For $\bar{y} = 0$ then the value of λ_2 depends on the parameters chosen. When $d < \alpha c$ then $\lambda_2 > 0$ and when $d > \alpha c$ then $\lambda_2 < 0$.

This tells us that if $\bar{x}$ is an unstable fixed point, then $(\bar{x}, \bar{y})$ and $(\bar{x}, 0)$ are unstable nodes for all parameters. If $\bar{x}$ is a stable fixed point, then the stability of $(\bar{x}, \bar{y})$ depends on the value of the parameters chosen. If $d > \alpha c$, then $(\bar{x}, \bar{y})$ and $(\bar{x}, 0)$ are both stable nodes. If $d < \alpha c$ then $(\bar{x}, \bar{y})$ is a stable node and $(\bar{x}, 0)$ is an unstable node. We should note that the behavior around $(\bar{x}, \bar{y})$ is the same for equations (2) and (3) only for (X, Y) pairs that are close to $(\bar{x}, \bar{y})$ and may not give reliable results for (X, Y) pairs far away from the fixed point.

To see how the trajectories behave away from the fixed points requires some further analysis using phase planes to break up the XY coordinate plane. The expression $\dot{Y}$ changes sign when $\alpha c Y(\frac{X-Y}{h+X-Y}) - dY$ changes sign, which can be rewritten as

$$(Y)(\alpha c(X - Y) - d(h + X - Y))\left(\frac{1}{h + X - Y}\right).$$

This implies that to know when $\dot{Y}$ changes sign we must find where each expression in the parentheses is equal to 0 or is undefined.

$$\begin{cases} Y = 0 \\ \alpha c(X - Y) - d(h + X - Y)) = 0 \implies Y = \frac{dh}{(d-\alpha c)} + X \\ h + X - Y = 0 \implies Y = h + X \end{cases}$$

These 3 equations along with the horizontal lines representing X values where $\dot{X} = 0$ break up the XY coordinate plane into sections. The behavior of the trajectories in each section is checked to determine where the trajectories end up. The last 2 equations that make $\dot{Y} = 0$ are parallel to each other. We have 2 different types of phase planes depending on which equation has the larger intercept. It should also be noted that $\frac{d}{d-\alpha c}$ is either smaller than 0 or greater than 1. All parameter groupings are positive and this quantity will never be equal to 1 since neither α nor c is ever equal to 0. This implies the two intercepts will never be the same and one intercept will always be larger than the other. Based on the values of a and b, there will be either 3 fixed points or 1 fixed point. Here we show illustrations for the case where there are 3 fixed points to get a better understanding of the dynamics of the system. If there is just 1 fixed point then the fixed point is stable and the system displays the same behavior as around one of the stable fixed points in the 3 fixed points case.

The two phase planes have the same X-coordinate values for the fixed points. This is because the parameters determining the phase plane behavior are α, c, and d which influence $\bar{y}$ but not $\bar{x}$. Also, nonzero $\bar{y}$ in Figure 5 are smaller than the $\bar{y}$ in Figure 6. This poses a curious problem, if the initial values of X and Y are low enough then the system will reach lower nonzero fixed points in Figure 5. This however poses a risk since if the X or Y values get too large then the system will be unable to reach the fixed points.

If

$$\frac{d}{d - \alpha c} < 0 \implies d < \alpha c \implies l_a < \alpha r_a,$$

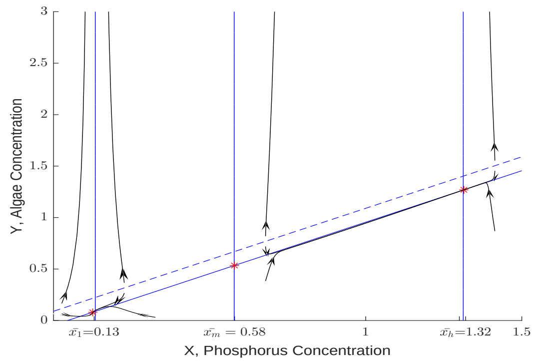

Figure 5: The parameter values used are: a = 0.05, b = 0.52, c = 1.25, d = 0.25, h = 0.09, α = 0.6. The nonzero fixed points here are (0.1264, 0.0814), (0.5791, 0.5341), and (1.314, 1.2688). Here $\bar{x_l}$, $\bar{x_m}$, and $\bar{x_h}$ denote the low, medium, and high pollution fixed point X-coordinate respectively.

then we see a phase plane like the one depicted in Figure 5.

This happens when the fixed loss rate is less than the growth rate bounded by the available light resources. Since the bounded growth rate is larger than the loss rate it makes sense that for algae values that start far away from the fixed points in the positive direction, the algae level blows up to infinity. Also if the system starts off with a small enough level of phosphorus and algae, then the system can naturally restrict the growth and the algae population goes to the nonzero fixed points.

This unbounded behavior occurs because we use a simplified model. When we chose the constant α to simplify the light attenuation term, nothing is bounding the growth when the population gets large. If the full light attenuation term was left in the model then the growth would be bounded when there is a larger population since the algae would have to compete for the light resource. Therefore, there would be another larger nonzero fixed point the system would go to that is not present in this simplified model.

If

$$\frac{d}{d-\alpha c} > 0 \implies d > \alpha c \implies l_a > \alpha r_a,$$

then we see a phase plane like the one depicted in Figure 6.

This happens when the fixed loss rate is greater than the growth rate bounded by the available light resources. Since the bounded growth rate is smaller than the loss rate it makes sense that for algae values that start far away from the fixed points the algae level goes to the nonzero fixed points. Also if the system starts off with a small enough level of phosphorus and algae, the system can naturally handle this and the algae population goes to 0.

In general, solutions move away from the dashed line and move toward the solid line representing the nonzero fixed point. The middle vertical line representing $X = x_m$ is the separatrix. This means that (X, Y) pairs that start to the left of the line stay on the left of the line, in the low phosphorus area. Similarly, (X, Y) pairs that start to the right of the line stay to the right of the line, in the high phosphorus area. In the case of only 1 fixed point, the vertical line representing $\bar{x}$ is the separatrix.

The seperatrix also implies the system is dependent on the initial condition. If the system

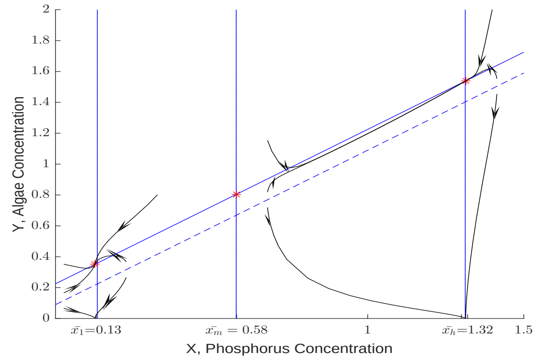

Figure 6: The parameter values used are: a = 0.05, b = 0.52, c = 0.27, d = 0.25, h = 0.09, α = 0.6. The nonzero fixed points here are (0.1264, 0.3821), (0.5791, 0.8347), and (1.314, 1.5695).

starts at a point to the right of the seperatrix, meaning a higher phosphorus concentration, then the system will go to the high-pollution fixed point. The only way to get to a low-pollution fixed point is to change one of the parameters. This can be done by decreasing the phosphorus input, increasing the loss rate of phosphorus, decreasing the growth rate of algae, or increasing the mortality rate of algae.

3 Economic Model

Similar to most environmental problems, a trade off occurs by allowing phosphorus to enter the lake. Fertilizer is used to make farmland more productive and increase the crop yield. However, phosphorus use can also cause pollution. To solve this trade off problem, we follow the approach used in Wagener and Mäler [14, 25]. We assume that there is a single social planner that is trying to maximize the town's welfare and is allowed to choose a level of phosphorus loading that maximizes this welfare. Since decision makers consider the benefits over time, we use a dynamic approach to determine how the phosphorus loading should change over time to achieve the optimal welfare. We will explore this problem without a tax in Section 3.2 and with a tax in Section 3.3.

We assume that phosphorus loading, $a(T)$, is allowed to change over time and phosphorus and algae concentration in the lake change over time according to system (2). We model total welfare at a given time by $w(a, Y) = \ln(a) - CY^2$. Here $\ln(a)$ represents the benefits from using fertilizer, which is a standard functional form that displays decreasing marginal benefits. The costs from using fertilizer is CY^2, where C is a parameter that describes how much people value the lake. This is also a standard functional form used to model increasing marginal costs. In addition, similar to the majority of environmental problems, the benefits of pollution go to the polluters, here the farmers, while someone else pays the cost, here the townspeople.

We calculate the welfare over time with welfare in the future being discounted by a constant discount rate δ. The goal of the planner is

$$\max_{a,Y} W = \max_{a,Y} \int_0^\infty e^{-\delta T} w(a, Y) dT = \max_{a,Y} \int_0^\infty e^{-\delta T} (\ln(a) - CY^2) dT, \tag{4}$$

where system (2) is satisfied.

3.1 Optimal Control Theory Background

In order to find the maximum of this integral, we use optimal control theory [15, 13]. Optimal control theory problems can be considered as a type of calculus of variations problems. In calculus of variations instead of finding values that maximize functions, we find functions that maximize functionals. Here, we try to find the extreme values of the integral by finding functions that maximize the integral. This problem can be written as

$$\max_{\mathbf{r}} \int_{T_1}^{T_2} F(\mathbf{r}(T), \dot{\mathbf{r}}(T), T)dT, \tag{5}$$

where T is the independent variable and $\mathbf{r}(T)$ is a vector-valued function of T. We use r_i to denote the i^{th} component of $\mathbf{r}$. In problem (4) we have two functions of T, $\ln(a(T))$ and $CY^2(T)$, but none of their derivatives.

To find the function $\mathbf{r}$ that maximizes (5), the Euler-Lagrange equation must be solved for every function r_i

$$\frac{\partial F}{\partial r_i} - \frac{d}{dT}\frac{\partial F}{\partial \dot{r}_i} = 0. \tag{6}$$

If there is a constraint on (5), such as an equality constraint of the form $P(T, \mathbf{r}(T)) = s$, then a Lagrangian is constructed

$$L(\mathbf{r}(T), \dot{\mathbf{r}}(T), T) = F(\mathbf{r}(T), \dot{\mathbf{r}}(T), T) - \lambda(T)(P(T, \mathbf{r}(T)) - s).$$

Here, the Lagrange multiplier $\lambda(T)$ is referred to as the co-state variable. It is a function of T because the constraint must hold for all values of T considered. To find the extremum of the constrained integral, we apply the Euler-Lagrange equation to the Lagrangian. This results in replacing F by L in (6). Also, since λ is a function of T, we must also solve the Euler-Lagrange equation for λ. The Euler-Lagrange equations produce 2^{nd} order differential equations that must be solved to find the function $\mathbf{r}$ that maximizes the integral.

Optimal control theory problems deal with maximizing an integral where one or more of the variables can be controlled over time or space to reach the optimum. This formulation is preferred in economic applications since optimal control theory can handle more general constraints, specifically linear constraints and functions. To put (5) into the framework of optimal control, we can rewrite (5) as

$$\max_{\mathbf{r}} \int_{T_1}^{T_2} F(\mathbf{r}(T), \mathbf{u}(T), T)dT,$$

subject to the constraint $\dot{\mathbf{r}}(T) = \mathbf{u}(T)$. Here $\mathbf{r}$ is referred to as the state variable and $\mathbf{u}$ is referred to as the control variable, since it describes how the state variable is changing over T. In problem (4) we have 2 state variables, X and Y that denote the phosphorus and algae concentration in the water column. And we have one control variable a since how much phosphorus that goes into the lake controls the phosphorus and algae concentration in the water column.

To find the maximum, we follow the process above and the Lagrangian

$$L(\mathbf{r}, \mathbf{u}, T, \dot{\mathbf{r}}) = F(\mathbf{r}, \mathbf{u}, T) - \boldsymbol{\lambda}(T) \cdot (\dot{\mathbf{r}}(T) - \mathbf{P}(\mathbf{r}, \mathbf{u}, T)) = F - \boldsymbol{\lambda} \cdot \dot{\mathbf{r}} + \boldsymbol{\lambda} \cdot \mathbf{P},$$

is constructed by allowing $\mathbf{P}(\mathbf{r}, \mathbf{u}, T) = \mathbf{u}$. For this Lagrangian, $\boldsymbol{\lambda}$ is a vector that has the same number of elements as $\dot{\mathbf{r}}$.

Problem (4) has two constraints which are the differential equations for X and Y. Following [15], we construct the current value Lagrangian for problem (5) by ignoring the exponential discount term, which yields

$$L = \ln(a) - CY^2 - \lambda_1(\dot{X} - f(a, X)) - \lambda_2(\dot{Y} - g(X, Y)).$$

We will illustrate the process of finding the Euler-Lagrange equation for Y, the equations for X, a, λ_1, and λ_2 are found by a similar process. The new Euler-Lagrange equation is

$$\frac{\partial L}{\partial Y} - \frac{d}{dT}\frac{\partial L}{\partial \dot{Y}} = 0 \implies \frac{\partial L}{\partial Y} = -\dot{\lambda}_2. \tag{7}$$

Most problems in optimal control only depend on the state and control variables and not their derivatives. We would like to consider a function that does not depend on these derivatives like L does.

In optimal control problems, one introduces the Hamiltonian defined as $H(\mathbf{r}, \mathbf{u}, T) = F + \boldsymbol{\lambda} \cdot \mathbf{P}$. The Hamiltonian for our problem is

$$H = \ln(a) - CY^2 + \lambda_1 f(a, X) + \lambda_2 g(X, Y). \tag{8}$$

We can rewrite the Lagrangian as $L = H - \boldsymbol{\lambda} \cdot \dot{\mathbf{r}}$. We must relate the partial derivatives of L and H to find necessary conditions for the optimal control problem. Substituting $\frac{\partial L}{\partial Y} = \frac{\partial H}{\partial Y}$ into (7) gives

$$\frac{\partial H}{\partial Y} = -\dot{\lambda}_2.$$

To solve this system fully initial conditions need to be specified and in infinite time problems a transversality must also be specified.

3.2 The Discounted Welfare Case

Problem (4) is called in control theory an autonomous infinite time horizon problem since time is only explicitly stated in the exponential discount term. Due to the discount factor, the welfare is in discounted terms instead of current value terms. A discounted Hamiltonian is used and then transformed into a current value Hamiltonian. This is because in economic problems it often makes more sense to evaluate welfare in present value.

In terms of problem (4), the discounted Hamiltonian can be written as

$$H_d = e^{-\delta T}(\ln(a) - CY^2) + q_1 f(a, X) + q_2 g(X, Y),$$

where $q_i = e^{-\delta T}\lambda_i$ is a discounted co-state variable. The discounted Hamiltonian can be related to the current value Hamiltonian, $H_d = e^{-\delta T} H$. The current value Hamiltonian is given in(8).

To solve (4), we must find the Euler-Lagrange equations in the discounted case and then convert to the equations in the current value case. To do this, we relate the partial derivatives of H_d and H and the time derivatives of q_i and λ_i. Continuing with the equation for Y, we know $\frac{\partial H_d}{\partial Y} = e^{-\delta T}\frac{\partial H}{\partial Y}$ and $\dot{q}_i = e^{-\delta T}\dot{\lambda}_i - \delta e^{-\delta T}\lambda_i$. This leads to

$$e^{-\delta T}\frac{\partial H}{\partial Y} = \frac{\partial H_d}{\partial Y} = -\dot{q}_2 = -e^{-\delta T}\dot{\lambda}_2 + \delta e^{-\delta T}\lambda_2$$

$$\implies \delta\lambda_2 - \frac{\partial H}{\partial Y} = \dot{\lambda}_2.$$

The Euler-Lagrange equation for a gives

$$\frac{\partial H}{\partial a} = \frac{1}{a} + \lambda_1 = 0.$$

The equation for $\dot{\lambda}_1$ is replaced with one for $\dot{a}$ since we care more about how the loading level is changing over time compared to the shadow price of phosphorus over time. The condition for λ_1

$$\delta\lambda_1 - \frac{\partial H}{\partial X} = \dot{\lambda}_1,$$

is rewritten in terms of a by using the relation $\lambda_1 = -\frac{1}{a}$ and $\dot{\lambda}_1 = \frac{\dot{a}}{a^2}$. From the Euler-Lagrange equations we have the following system of 4 ODEs in X, Y, a, and λ_2

$$\begin{cases} \dot{a} = (-\delta + \frac{\partial f}{\partial X} - \lambda_2\frac{\partial g}{\partial X}a)a \\ \dot{\lambda}_2 = \delta\lambda_2 - \frac{\partial H}{\partial Y} = \delta\lambda_2 - (-2CY + \lambda_2\frac{\partial g}{\partial Y}) \\ \dot{X} = \frac{\partial H}{\partial \lambda_1} = a - bX + \frac{X^2}{X^2+1} \\ \dot{Y} = \frac{\partial H}{\partial \lambda_2} = c\alpha Y(\frac{X-Y}{h+X-Y}) - dY. \end{cases} \tag{9}$$

To solve these equations, we first find the steady states of (9) by setting the right-hand side of each equation to 0. The values of λ_2, Y, and a can be rewritten as functions of X. Assuming the values of Y are neither 0 nor infinite, we have that

$$\begin{cases} Y = \frac{dh}{(d-\alpha c)} + X \\ \lambda_2 = \frac{-2CY}{\delta - \frac{\partial g}{\partial Y}} = h_1(X) \\ a = bX - \frac{X^2}{X^2+1} = h_2(X). \end{cases} \tag{10}$$

The values of X can be found by solving for the zeros of

$$Z(X) = h_2\left(-\delta + \frac{\partial f}{\partial X} - h_1\frac{\partial g}{\partial X}h_2\right). \tag{11}$$

It can be shown that $Z(0) = 0$ and $\lim_{x\to\infty} Z(X) = \infty$, which means that the zeros that we find in a finite interval are the only zeros of (11). Based on different parameter sets there are either 1, 2, or 3 zeros to (11) in addition to $X = 0$. This implies that there are 1, 2, or 3 non-zero fixed points to the system. Due to the complex non-linearities of this system it is difficult to solve analytically for sets of parameters that lead to different scenarios.

To find solutions in the unstable manifolds we go forward in time from the fixed points and we go backward in time to find solutions in the stable manifolds. We use the manifolds to see the behavior of the solutions as they approach the steady state solutions. 2-dimensional phase plane analysis of the control and the state variable is typically done to get graphical look at the solution behavior, however since we have two state variables we have 3-dimensional phase portraits.

To run these simulations the initial value of λ_2 was held constant. For a given initial condition, simulations were run to find the functions $a(t), \lambda_2(t), X(t)$, and $Y(t)$. These solutions are then plotted in X, Y, a space.

Due to the complexities of the model, we run numerical experiments for one set of parameters and provide a discussion. The biological parameters chosen are $b = 0.52$, $c = 0.27$, $d = 0.25$, $h = 0.09$, $\alpha = 0.6$, which are the same used in Figure 6. The economic parameters chosen are $\delta = 0.03$, $C = 0.1$ [20]. Figure 7 shows the fixed points along with solutions in the stable and unstable manifolds for the no tax model. Here the nonzero fixed points are: $(0, 0.2557, 0)$, $(0.3355, 0.5912, 0.0733)$, $(0.9081, 1.1638, 0.0203)$, $(3.1701, 3.4527, 0.7389)$. These can be considered the zero, low, middle, and high-pollution fixed points. The black lines describe solutions in the unstable manifold and the gray lines describe solutions in the stable manifold. The X and Y values of the fixed points are different here than in Figure 6 even though the same parameter values are used. This happens because now the phosphorus loading level is allowed to vary over time where before it was held constant.

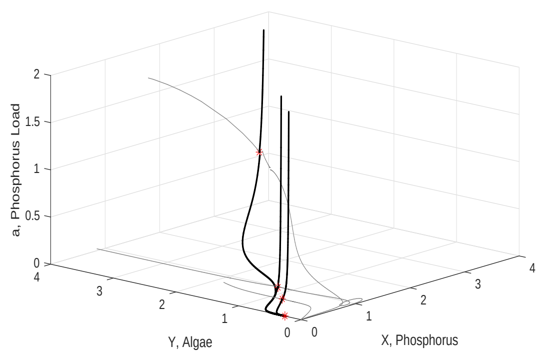

Figure 7: Solutions in the stable and unstable manifolds for the no tax model with the fixed points represented by stars.

3.3 The Constant Tax Case

Typically, farmers have no incentive to use the amount of phosphorus that is socially optimal, which is where the social planner comes in. If a water body does not meet water quality standards designated by the state then the Clean Water Act requires that the EPA and the state government design and approve a Total Maximum Daily Load (TMDL) plan. TMDL plans identify the type of pollutant that is causing damage and the maximum amount of pollutant that should be allowed in the water to meet water standards. For point source pollution, like wastewater treatment plants, the EPA can see how much pollution each plant is creating and give a set of permits that allows a specified amount of pollutant. For non-point source pollution, like agricultural runoff, enforcement is more difficult and obeying the targets set by TMDL plans are voluntary [1].

We propose a constant tax on loading that goes above the target and a constant subsidy for loading that is below the target to incentivize obeying the regulation. Assuming the social planner can know exactly how much phosphorus each farm is using, this tax is levied on all farmers based off their phosphors use. This is an ad valorem tax, since the farmer is charged a percentage on the amount of phosphorus used.

The modified model including the tax is now

$$\max_{a,Y} W = \max_{a,Y} \int_0^\infty e^{-\delta T}(\ln(a) + \tau(\bar{a} - a) - CY^2)dT, \tag{12}$$

where system (2) is satisfied, $\tau > 0$ is the constant tax rate and $\bar{a}$ is the loading level set by the TMDL plan.

We follow the same process as before by finding the necessary conditions. The new Euler-Lagrange equation for a gives

$$\frac{\partial H}{\partial a} = \frac{1}{a} - \tau + \lambda_1 = 0.$$

This is rewritten so we can eliminate λ_1 in terms of a by using the relations $\lambda_1 = \tau - \frac{1}{a}$ and $\dot{\lambda_1} = \frac{\dot{a}}{a^2}$. From the Euler-Lagrange equations we have the following system of 4 ODEs in X, Y, a, and λ_2

$$\begin{cases} \dot{a} = (-\delta + \frac{\partial f}{\partial X} - \lambda_2 \frac{\partial g}{\partial X} a)a + (\delta - \frac{\partial f}{\partial X})\tau a^2 \\ \dot{\lambda_2} = \delta\lambda_2 - \frac{\partial H}{\partial Y} = \delta\lambda_2 - (-2CY + \lambda_2 \frac{\partial g}{\partial Y}) \\ \dot{X} = \frac{\partial H}{\partial \lambda_1} = a - bX + \frac{X^2}{X^2+1} \\ \dot{Y} = \frac{\partial H}{\partial \lambda_2} = c\alpha Y(\frac{X-Y}{h+X-Y}) - dY. \end{cases} \tag{13}$$

The only condition that has been changed from (9) is the condition on a. Now, the left hand side of the condition is shifted either up or down depending on the sign of $\delta - \frac{\partial f}{\partial X}$. If the sign of that expression is negative, δ is small compared to $\frac{\partial f}{\partial X}$. A small δ implies that the community considers more heavily the future value of the lake, compared to a larger value of δ. This incentivizes the community to pollute less to keep the lake pristine for future use. A large value of $\frac{\partial f}{\partial X}$ implies the lake is sensitive to changes in the phosphorus level. This incentivizes the community to pollute less since the lake will respond rapidly to changes in the phosphorus level. Both of these factors lead the community to pollute less so it makes sense that a negative sign of $\delta - \frac{\partial f}{\partial X}$ causes $\dot{a}$ to be less than before and leads the community to pollute less. If the sign of that expression is positive then the factors above are reversed and $\dot{a}$ is greater than before and leads the community to pollute more.

To solve these equations, we first find the steady states of (13). The values of λ_2, Y, and a can be rewritten as functions of X and are the same as in (10). To find the new values of X that represent the fixed points of the system, we must find the zeros of a new equation,

$$Z_2(X) = Z(X) + \left(\delta - \frac{\partial f}{\partial X}\right)\tau h_2^2. \tag{14}$$

We relate the graph of Z_2 to the graph of Z, which is defined in (11). The graph of Z_2 is shifted either up or down depending on the sign of $\delta - \frac{\partial f}{\partial X}$. The term h_2^2 compresses Z_2 to be closer to the origin when $h_2 < 1$, which is the range we concern ourselves with in Section 4. This causes the zeros of Z_2 to be smaller than the zeros of Z, which makes sense since a positive tax rate should induce a community to load less phosphorus into the water which then leads to a lower phosphorus concentration in the water. The value of $\bar{a}$ does not influence the values of the zeros of Z_2 or how the loading changes over time, it only influences the value of total welfare.

Figure 8 shows the fixed points along with solutions in the stable and unstable manifolds for the constant tax model. Here the fixed points are: $(0, 0.2557, 0)$, $(0.3352, 0.5909, 0.0733)$, $(0.9081, 1.1638, 0.0203)$, $(2.8016, 3.0573, 0.5699)$. The low, medium, and high-pollution fixed points in Figure 8 are smaller than those in Figure 7 in the sense that all the components are smaller and thus closer to the origin. Fixed points closer to the origin represent a success from an environmental point of view since they mean lower phosphorus and algae concentration. The parameters used are the same as in Figure 7, now with the addition of $\tau = 0.5$. Figures 7 and 8 display similar behavior around their respective fixed points, but the tax model produces fixed points closer to the origin.

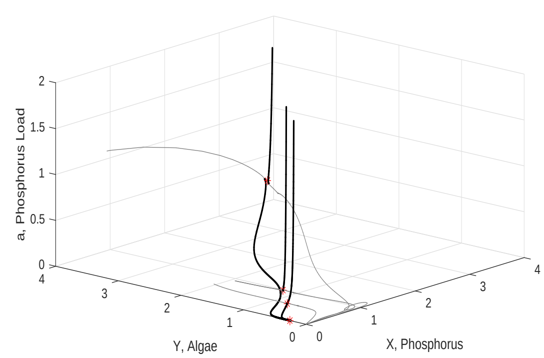

Figure 8: Solutions in the stable and unstable manifolds for the constant tax model with $\tau = 0.5$.

4 Policy Analysis

Now that we have looked at both the tax and no tax model, a policy maker may interpret the results and make a decision. First, they should decide whether or not to impose the tax based on their initial condition and parameter values. As shown before, imposing the tax results in fixed points closer to the origin but the planner must also assess which fixed point the system will go to and how easy it is to reach the fixed points. To look at the sensitivity of the system we look at trajectories starting from different initial conditions to see their behavior as $t \to \infty$.

The unstable manifolds show us how the system is evolving forward in time. As could be seen in Figures 7 and 8, the unstable manifolds show that $\lim_{t\to\infty} a(t) = \infty$. This leads to the algae and phosphorus levels to also approach an infinite value if a trajectory begins near the unstable manifold. From an environmental point of view this represents a catastrophic failure when the lake is allowed to become completely polluted and no one is allowed to use it except for polluting purposes. We check initial conditions in the 3-dimensional space to find which initial conditions lead to this catastrophic "blow-up" behavior. To quantify the sensitivity of the system, we try to quantify the ratio of initial conditions that lead to catastrophe compared to the total number of initial conditions.

We form a grid in $\mathbb{R}^3$ of initial conditions. For this grid, we use a mesh size of 0.1 for X and Y and a mesh size of 0.05 for a. We chose the ranges of $X \in (0,3)$, $Y \in (0,3)$, $a \in (0,0.6)$ based on data from Lake Tainter. The X and Y bounds were found by looking at the maximum and minimum values from 1989 - 2017. The a bound was estimated from the loading level calculated from the TMDL plan in 2012, which the Wisconsin Department of Natural Resources estimated using Army Corps Engineering models. Since they only reported one value of $a = 0.25$ we included a large margin of error in our range due to the possible variation in this value. This range restricts our attention to only the low and medium non-zero fixed points.

In Figures 9 and 10 the black dots represent the initial conditions that lead to infinite final values. Figure 9 displays the density of the 3D grid that leads to catastrophic behavior for the no tax model. The percentage of initial conditions that lead to this behavior is 17.71%. Here we define density as the number of initial conditions in the 3D grid that lead to catastrophic behavior compared to the total number of initial conditions in the grid. Figure 10 displays the density of the 3D grid that leads to this behavior for the constant tax model. The percentage of initial conditions that lead to this behavior is 18.19%.

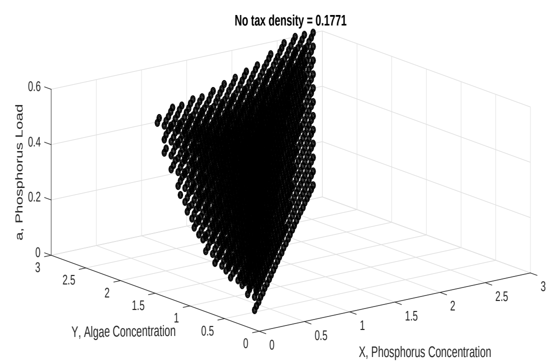

Figure 9: Initial conditions that lead to infinite final values for the no tax model.

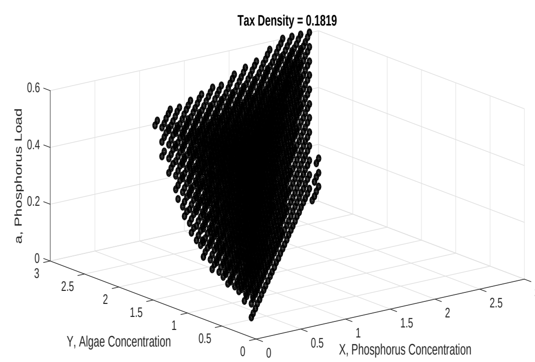

Figure 10: Initial conditions that lead to infinite final values for the constant tax model.

The black dots in Figures 9 and 10 appear to form a solid geometric figure that we can approximate the boundaries of. To find the boundaries of these geometric figures, we hold a constant and look at XY planes. We find that as we increase a the number of black dots in the XY plane increases. Lower loading levels lead to less risk of catastrophic behavior because the system has not absorbed much phosphorus so there is still the possibility of preserving the lake for both recreation and farming.

We look at the XY plane where $a = 0.6$ to find curves that bound the region since the largest number of black dots occur at the largest value of a. Using straight lines provides a reasonable approximation to the boundaries with the chosen mesh size. The gray dots represent initial conditions that lead to finite final values. Figures 11 and 12 show the XY

planes for a values of 0.05 and 0.6 for the no tax model. These figures show the difference in catastrophic behavior between the two a values based on the density of the black dot region. Here we define density as the ratio of the number of black dots in a given plane to the total number of dots in that plane. Figures 13 and 14 show the XY planes for the tax model. Both the tax and no tax model have the same bounding lines for $a = 0.6$ of $Y = X + 0.1$ and $Y = X + 1.5$ enclosing black dot areas of approximately 27%.

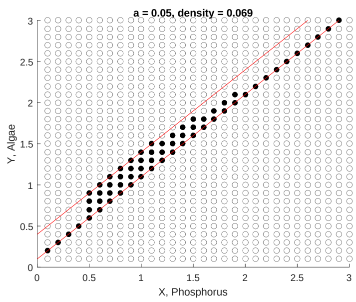

Figure 11: XY plane with $a = 0.05$ for the no tax model with bounding lines of $Y = X + 0.1$ and $Y = X + 0.4$ enclosing 6.9% of initial conditions that lead to catastrophe.

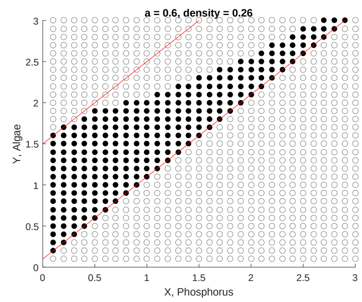

Figure 12: XY plane with $a = 0.6$ for the no tax model with bounding lines of $Y = X + 0.1$ and $Y = X + 1.5$ enclosing 26% of initial conditions that lead to catastrophe.

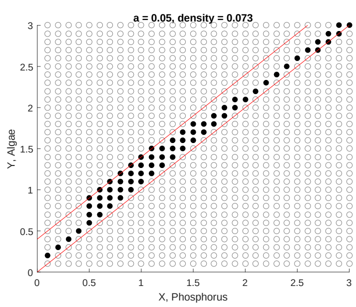

Figure 13: XY plane with $a = 0.05$ for the tax model with bounding lines of $Y = X$ and $Y = X + 0.4$ enclosing 7.3% of initial conditions that lead to catastrophe.

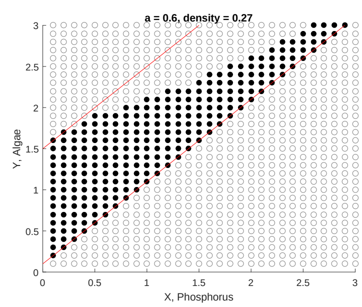

Figure 14: XY plane with $a = 0.6$ for the tax model with bounding lines of $Y = X + 0.1$ and $Y = X + 1.5$ enclosing 27% of initial conditions that lead to catastrophe.

The bounding planes can act as a warning zone for the social planner. If the system starts inside this region it could lead to catastrophe if the planner does not manage the lake properly. Since we have the planes enclosing the boundary of the initial conditions that lead to catastrophe, we plot the bounding planes with the trajectories. Figure 15 displays bounding planes and trajectories for the no tax model and Figure 16 displays this for the tax model. The area bounded by the planes includes the fixed points and some area around the fixed points. The lower bounding plane is almost tangent to the unstable manifold but never touches it. Initial conditions near the unstable manifold should display the same behavior as the unstable manifold.

Even though the tax model leads to fixed points closer to the origin, the system is more sensitive and more initial conditions lead to catastrophe. This poses an interesting problem

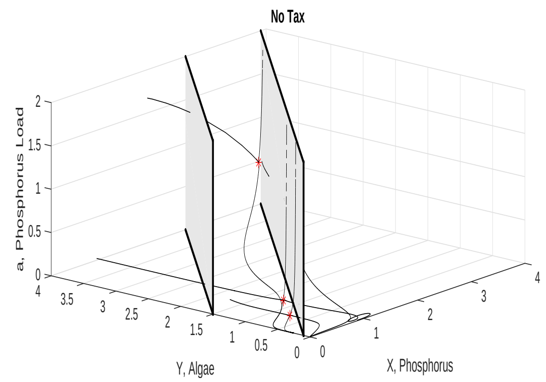

Figure 15: Solutions in the stable and unstable manifolds for the no tax model plotted with the bounding planes of $Y = X + 0.1$ and $Y = X + 1.5$.

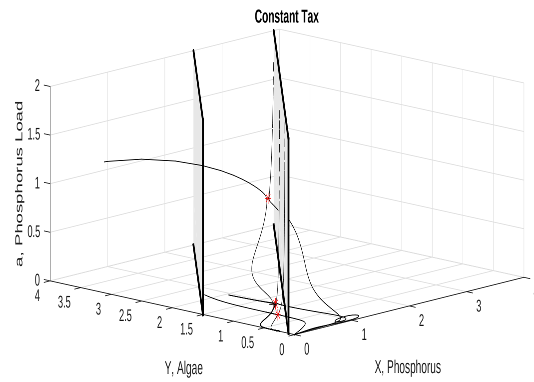

Figure 16: Solutions in the stable and unstable manifolds for the tax model plotted with the bounding planes of $Y = X + 0.1$ and $Y = X + 1.5$.

for the planner since there is a higher probability of environmental disaster but if the planner can successfully reach the lower fixed points, then this will lead to less pollution. However, there may be some cases where the system becomes too sensitive and implementing a tax will backfire from an environmental point of view and lead to large levels of phosphorus loading and algae. With our parameter set, the difference in the amount of initial conditions that lead to catastrophic behavior with the tax is only 0.48%, so there is not that much risk in imposing the tax.

We have found what percentage of initial conditions lead to catastrophic behavior but we also want to know which initial conditions go to which fixed points and how imposing a tax changes this behavior. Initial conditions to the left of the upper boundary, $Y = X + 1.5$, go to the low or middle fixed point or the fixed point $(0, 0.25, 0)$. Initial conditions that start to the right of the lower boundary, $Y = X + 0.1$, go to the origin. Figure 17 displays the initial conditions that go to the medium fixed point and Figure 18 displays the initial conditions that go to the origin for the no tax model. This behavior is consistent with both the no tax and constant tax model.

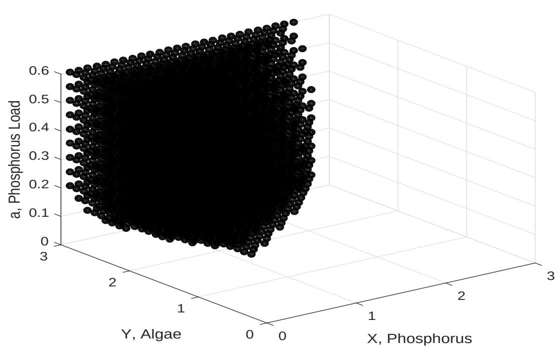

Figure 17: Initial conditions from the no tax model that go to the medium fixed point $(0.9081, 1.1638, 0.0203)$.

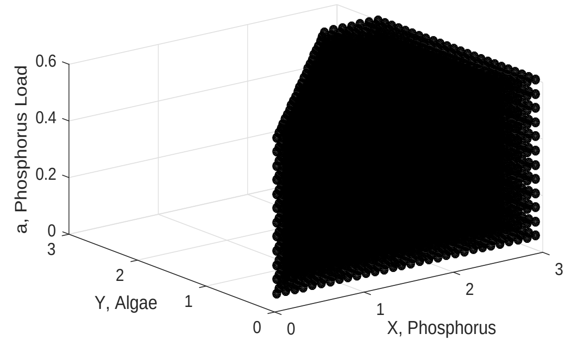

Figure 18: Initial conditions from the no tax model that go to the origin.

Another way to measure the sensitivity of the system is to examine the percentage of vertical lines where all initial conditions on the line lead to finite loading. We can assume that the planner can get reliable measurements of the phosphorus and algae level in the lake since there are many tools to examine water quality and it is done frequently. We also assume that they are uncertain about the current phosphorus loading level, since complicated simulations need to be done to get an estimate on this value and it can vary depending on the season. Once the planner has accurate phosphorus and algae concentration values, then they just have to pick an initial loading level. In $\mathbb{R}^3$, they are focused on a vertical line with fixed X and Y coordinates and varying levels of a to choose from.

If the planner starts at an a value that leads to finite loading without a tax then he will continue without a tax if adding a tax leads to an infinite loading level. In addition, depending on how risk averse the planner is, he will continue without a tax if there are more initial conditions on the corresponding vertical line that lead to catastrophic behavior with a tax. For the no tax model the percentage of lines without catastrophic initial conditions is 73.78% and for the tax model the percentage is 72.56%. There are fewer lines that do not lead to catastrophic behavior with a tax since the tax model had more initial conditions that lead to catastrophe. Also, if the planner starts on an initial line that is outside the area bounded by the planes, then there is no difference in the sensitivity of the system between the tax and no tax model.

In Figures 19 and 20 we look at the vertical line with $X = 0.4$ and $Y = 1.5$ for the tax and no tax model. These values were selected because they were the average phosphorus and algae concentration in 2012. Next to the initial conditions their final loading values are listed if the final value is finite. If $a > 0.35$, then both models display catastrophic behavior. For a values below this threshold, both models display finite behavior though the final loading values are different. The tax model has the same or lower loading level than the no tax model.

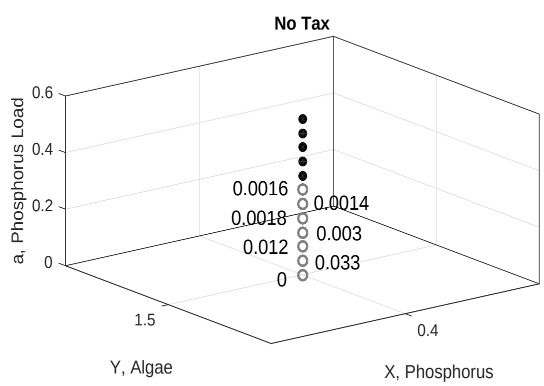

Figure 19: Vertical line representing $X = 0.4$ and $Y = 1.5$ with final loading values for the no tax model.

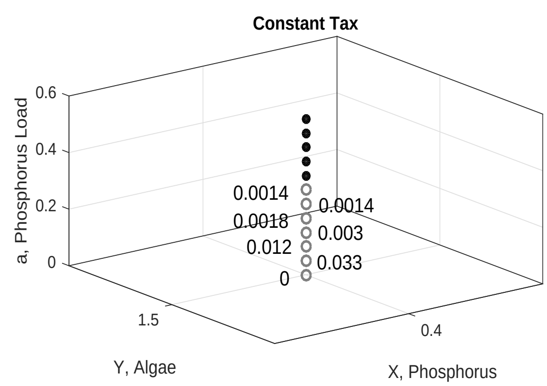

Figure 20: Vertical line representing $X = 0.4$ and $Y = 1.5$ with final loading values for the tax model.

In Figure 21 the initial and final loading values from Figures 19 and 20 are plotted. The initial loading values are plotted against the final loading values for both the tax and no tax model. Larger initial loading levels up to $a = 0.35$ result in smaller final loading levels. The

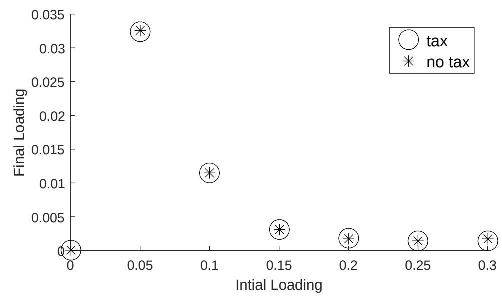

Figure 21: Initial loading level plotted against the final loading level for both the no tax and tax model with the values taken from Figures 19 and 20.

larger loading levels are more quickly reduced possibly because there is a larger penalty from the tax and the loading is not yet so large that the system is irrecoverable.

Perhaps the defining factor in decision making is the total welfare of imposing these policies and making changes to the lake. To calculate welfare we must have a reasonable initial condition. Then using the maximum conditions in (9) and (13), we find $a(t)$ and $Y(t)$, which we can plug back into the welfare function. We use the initial condition $X = 0.4$, $Y = 1.5$, $a = 0.25$ since the X and Y values are averages from the 2012 season and the a value is the loading level calculated from the TMDL plan for the 2012 season. We use the value of $\bar{a} = 0.1$ based off the suggested loading from the TMDL plan. The *trapz* function in MATLAB is used to calculate the welfare since the functions we get from the maximum conditions are not analytically tractable and must be integrated numerically. The welfare for the no tax model is -159.21 and the welfare for the tax model is -157.32. In addition, for every initial condition on the line where $X = 0.4$ and $Y = 1.5$ below $a = 0.35$, there is a larger welfare associated with implementing the tax.

If initial conditions are uncertain, we can also look at regions where imposing a tax results in larger welfare levels than without a tax. Some of the initial conditions lead to infinite loading in both models and just comparing the total welfare between the tax and no tax model does not work since we would be comparing two infinite values. However, we can compare welfare values if they are finite. The Figures 23 and 24 displays the region where the community is better off with a tax. Two figures were used to better see the pattern of initial conditions. Figure 23 displays initial conditions to the left of the upper boundary line $Y = X + 1.5$ and Figure 24 displays initial conditions to the right of the lower boundary line $Y = X + 0.1$. Figure 22 displays the region where the community is worse off with a tax. Ignoring the initial conditions that lead to infinite loading, we find that the percentage of initial conditions that are better with the tax is 87.55%.

For every initial condition to the right of the line $Y = X + 0.1$, the community is better off with the tax. This occurs because over time the loading level goes to 0, so the community would get a large benefit from the subsidy since they would be loading phosphorus below the target value. For some initial conditions to the left of the line $Y = X + 1.5$, the community is better off with a tax and for others the community is worse off with a tax. This could happen because these initial conditions are approaching the medium fixed point. Once the system reaches the medium fixed point, the community will receive a benefit from the subsidy because the medium fixed point loading level is below the target loading level.

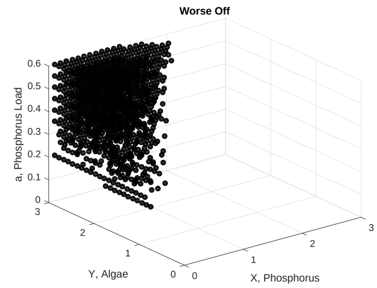

Figure 22: Initial conditions for which the community is worse off by implementing a tax.

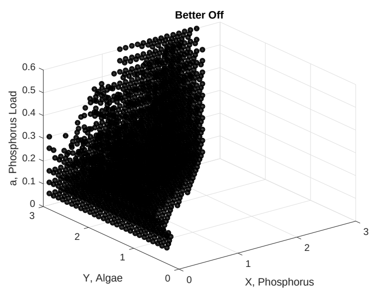

Figure 23: Initial conditions to the left of the boundary line $Y = X + 1.5$ for which the community is better off by implementing a tax.

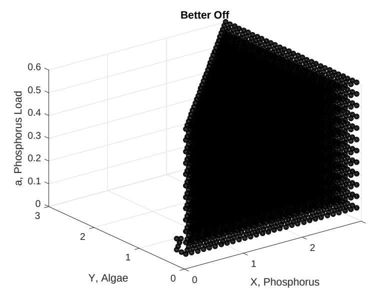

Figure 24: Initial conditions to the right of the boundary line $Y = X + 0.1$ for which the community is better off by implementing a tax.

However, depending on how quickly the system reaches this fixed point, this benefit could be offset by the cost from the tax since the initial loading is larger than the target loading value.

The planner also has the ability to change certain parameters. The planner can promote ecological change by increasing the value of b, the loss rate of phosphorus. They can do this by altering the lake through adding vegetation and aquatic life that use phosphorus for growth or by adding aluminum to the lake which removes phosphorus. The planner can also promote social change by increasing the value of C, how much the people value the lake. This can occur by holding more community events at the lake and spreading awareness about environmental impacts. Altering the value of C most likely also has an effect on δ, how much people discount the future. This could happen because if the townspeople care more about the lake they might want to keep the lake clean for future generations so they can learn to also appreciate the lake. Each of these policies has its own set of benefits and costs associated with them. Before the planner makes a decision they must understand both the lake and the townspeople to know if it is easier to change the initial value of the loading level or if it easier to change a parameter value to reach the desired outcome.

Changes in the parameters lead to changes in the zeros of (10) and (14). Increasing b shifts the curves down which leads to larger loading values in the fixed points. People have an incentive to increase the phosphorus load because the lake can recycle phosphorus faster. Increasing C shifts the curves up which leads to smaller loading values for the fixed points. When people value the lake more they want smaller levels of phosphorus and algae in the

Parameter Change	Med Fixed Point	Infinity	Origin	Med Fixed Point
No Tax				
Original Set	(0.9081, 1.1638, 0.0203)	17.71%	50.79%	27.73%
b = 0.57	(0.8193, 1.0750, 0.0654)	15.25%	51.31%	1.11%
C = 0.11	(0.9091, 1.1647, 0.0202)	16.65%	50.81%	28.58%
δ = 0.027	(0.9143, 1.17, 0.0201)	17.71%	50.77%	27.78%
Constant Tax				
Original Set	(0.9081, 1.1638, 0.0203)	18.19%	50.47%	27.38%
b = 0.57	(0.8204, 1.0761, 0.065)	16.08%	50.91%	1.01%
C = 0.11	(0.9092, 1.1648, 0.02)	17.1%	50.50%	28.21%
δ = 0.027	(0.9144, 1.701, 0.0201)	18.18%	50.45%	27.38%

Table 2: Fixed point simulation results for different sets of parameters for the no tax and tax model.

lake. Decreasing δ shifts the curves up which leads to smaller loading values for the fixed points. As people value the future more they care about keeping the lake clean for future generations.

Changes in the parameters also lead to changes in the sensitivity of the system. These parameter changes decrease the sensitivity of the system by decreasing the proportion of initial conditions that lead to catastrophic behavior. The reasons why this happens are similar to the reasons why fixed points change. Increasing b decreases the sensitivity since the lake can recycle phosphorus faster. More phosphorus can be added to the lake because the lake can naturally handle it without leading to catastrophic behavior. Changing C and δ change people's preferences to keep the lake cleaner.

Since certain parameters decrease the number of initial conditions that approach infinity, these initial conditions must go somewhere else. Increasing b leads to nearly all of the initial conditions that went to the medium fixed point now go to the low fixed point. Increasing C leads to a larger proportion of initial conditions going to the middle fixed point.

Table 2 records the value of the medium fixed point, the percentage of initial conditions that lead to catastrophic behavior, to the origin, and to the medium fixed point for different parameter sets. We mainly concern ourselves with the medium fixed point because for $(X, Y) \in (0, 3)$ no initial conditions lead to the high fixed point and 2% of initial conditions lead to the low fixed point. For the simulations, we change the parameters by 10% of their original value. Increasing b still results in the reversible flipping behavior that resulted before when $b = 0.52$.

Since the changes in parameters lead to changes in the behavior of the system this also leads to different percentages of initial conditions that are better off with a tax. Table 3 records the value of total welfare starting at the 2012 average initial condition (0.4, 1.5, 0.25) and following the loading level function over time given by the maximum conditions. It also records the percent of initial conditions that result in a higher level of welfare for the community with a tax implemented. Starting from the estimated initial condition, every parameter set except for the change of $\delta = 0.027$ is better off implementing a tax. Also, every parameter set has at least 86% of the initial conditions where the community is better off with a tax.

We can conclude that for this system with our original parameter set if we can be certain

Parameter Change	Welfare from (0.4, 1.5, 0.25)	% Better Off With a Tax
No Tax		
Original Set	-159.21	—
b = 0.57	-105.43	—
C = 0.11	-158.14	—
$\delta = 0.027$	-165.16	—
Constant Tax		
Original Set	-157.32	87.55%
b = 0.57	-105.27	86.73%
C = 0.11	-158.46	87.42%
$\delta = 0.027$	-164.62	87.65%

Table 3: Welfare simulation results for different parameter sets for the no tax and tax model.

that the initial loading is below 0.35 with $X = 0.4$ and $Y = 1.5$ or if $Y < X + 0.1$, then the community would be better off with a tax. If we are unsure of our initial condition, 87.55% of initial conditions make the community better off with a tax. Although with a tax a larger percentage of initial conditions lead to catastrophe the increased percentage is only 0.48%. In addition, the medium fixed point for both models are the same and the low and high fixed points are closer to the origin when the tax is implemented.

If it was easier to change a parameter value than to implement a tax, then the best parameter to change would most likely be b, the loss rate of phosphorus, assuming that changing any of the parameters by 10% costs the same. Compared to the original parameter set, 26% more initial conditions go to the low fixed point, which leads to a lower phosphorus and algae concentration in the water. Though the medium fixed point loading value is larger than with the original parameter set, the resulting phosphorus and algae concentration in the lake is smaller and only 1.11% of initial conditions go to this fixed point. Also, a smaller proportion of initial conditions leads to catastrophic behavior. With our chosen initial condition $(0.4, 1.5, 0.25)$ the increased b value gives the community a higher welfare. However, before trying to change parameters the planner must first learn the costs of trying to change this parameter. Then the planner can run a benefit cost analysis with the estimated welfare and cost value and if the benefit outweighs the cost then the parameter should be altered.

4.1 Conclusion

In this paper, we proposed a coupled bio-economic model to find the optimal level of phosphorus that should be allowed in the lake. To model the lakes biological response to phosphorus input, the phosphorus and algae concentration over time are modeled. We assume a single social planner that is trying to maximize the communitys utility and can choose a level of phosphorus loading that maximizes this utility. Total welfare is modeled by accounting for the utility of the farmers based on their phosphorus use and the disutility of townspeople around the lake due to the resulting algae. We also study what happens when a tax is implemented if the farmer uses over the phosphorus input amount allowed from water quality standards. These models are run with parameters from Tainter Lake and run with and without a tax. We find that with most initial conditions, the community is better off by implementing the chosen tax rate. In addition, if the community had the ability to enact environmental change then increasing the loss rate of phosphorus would have the biggest positive impact on the community's welfare.

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