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Philosophy of Mathematics: Theories and Defense

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Introduction

Mathematics may be characterized as the study of formal relations between properties of mathematical objects of many different kinds, including the natural numbers.[1] From a mathematician's perspective, asking questions about the existence or properties of mathematical objects is equivalent to asking whether these objects fall under a particular mathematical definition. For example, a number theorist takes for granted that prime numbers[2] really exist in some way, and instead concerns herself with deducing which numbers are prime in a particular number system. From a philosopher's perspective, however, questions about mathematical objects may include ontological questions addressing whether these objects exist as abstract entities, mental conceptions, or rooted in material objects. Philosophers also ask epistemological questions concerning our knowledge of mathematical objects. Philosophy of mathematics addresses a broad range of philosophical issues relating to logic, metaphysics, and epistemology.[3]

The philosophy of mathematics has been around at least since the time of Pythagoras.[4] Until relatively recently, mathematics was considered to be a philosophical subject. Philosophers such as Rene Descartes and Gottfried Leibniz debated the ultimate nature of mathematics.[5] In the early 20th century the philosophy of mathematics was arguably the most important topic treated by philosophers and it continues to be central for many contemporary philosophers.

In this paper I intend to discuss various theories in the philosophy of mathematics concerning the ontological status of mathematical objects. These theories include logicism,

[1] The natural numbers are the set of positive whole numbers: 1, 2, 3, …

[2] A prime number is an integer than can only be divided into evenly by itself and one.

[3] Michael Tavuzzi, "Central Problems and Directions in the Philosophy of Mathematics," 1988, in Nature, Science & Values, 268

[4] C.D.C. Reeve and Patrick Lee Miller, "Pythagoras and the Pythagoreans," 2006, in Introductory Readings in Ancient Greek and Roman Philosophy, Indianapolis, Hackett Publishing, 4

[5] Stewart Shapiro, *Thinking about Mathematics: The Philosophy of Mathematics*, 2000, Oxford, Oxford University Press, 3

intuitionism, formalism, platonism, structuralism, and moderate realism. I will also discuss problems that arise within these theories and attempts to solve them. Finally, I will attempt to harmonize the best feature of moderate realism and structuralism, presenting a theory that I take to best describe current mathematical practice.

Logicism

Logicism is the theory that the most fundamental part of mathematics, from which all other parts can be derived, is reducible to symbolic logic, and can be proven using the laws of logic.[6] Gottlob Frege, the founder of logicism, believed that all statements of arithmetic and real analysis are either laws of logic or can be proven with the laws of logic.[7] To justify his belief Frege developed a symbolic language of logic capable of expressing the basic arithmetical statements from which all others could be derived.[8] His definition of natural numbers begins with assigning 0 to a concept that does not apply to any object.[9] This assignment is appropriate since such a concept applies to zero objects. He continues, noting that "the number (*n*+1) applies to the concept *F* if there is an object *a* which falls under *F* and such that the number *n* applies to the concept 'falling under *F* but not [identical with] *a*.'"[10] For example, consider an object *a* which falls under the concept *F*, where *F* is the concept of being a vertex, or point, of a triangle. There are two other vertices that fall under this concept that are not identical to *a*. Thus the number 2+1, as an instance of *n*+1, applies to *F*. In defining the natural numbers, Frege also

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[6] Leon Horsten, "Philosophy of Mathematics," 2012, in *Stanford Encyclopedia of Philosophy*, section 2.1
[7] Shapiro, *Thinking about Mathematics*, 109
[8] Gottlob Frege, *The Foundations of Arithmetic: A Logico-mathematical Enquiry into the Concept of Number*, 1884, J.L. Austin (trans), Evanston, Northwestern University Press, 1980, cited in Horsten, section 2.1
[9] Frege, section 55, 67
[10] Ibid.

adopted what George Boolos has called Hume's principle[11]—the number of *F*s equals the number of *G*s if and only if the *F*s and *G*s have a one-to-one correspondence.[12] In this example the number of vertices of a triangle is equal to the number of wheels on a tricycle if and only if each tricycle wheel can be identified with one and only one triangle vertex such that all vertices have a corresponding wheel. Using his definition of the natural numbers and Hume's principle, Frege attempted to derive the foundations of arithmetic.[13]

In the process of defining natural numbers, Frege noticed that he has supplied no means by which to differentiate numbers from other objects. For example, there is no logical reason why Julius Caesar cannot be classified as a number in the same way that zero can.[14] He solved this "Caesar objection" by stipulating a way to distinguish numbers from non-numerical objects: "the number which applies to the concept *F* is the extension of the concept 'equinumerous with the concept *F*.'"[15] Here, an extension of a concept is a set that contains all of the objects to which the concept applies. Thus, the number three is defined as the set of all concepts, such as tricycle wheels and triangle vertices, under which exactly three objects fall—the set of all triples. Using this definition and the properties of extensions, Frege was able to rewrite Hume's principle so that the Caesar problem no longer arises. The result, which Frege calls Basic Law V, is paraphrased by Stewart Shapiro as follows: "for any concepts *F*, *G*, the extension of *F* is identical to the extension of *G* if and only if for every object *a*, *Fa* if and only if *Ga*."[16] Basic Law V, in conjunction with the previous definition of natural numbers, allowed Frege to derive all arithmetic using only logical laws and proofs. The apparent success of Frege's theory

[11] George Boolos, "The Standard of Equality of Numbers," 1990, in George Boolos (ed), *Meaning and Method: Essays in Honour of Hilary Putnam*, Cambridge, Cambridge University Press, 267

[12] Frege, section 63, 73

[13] Horsten, section 2.1

[14] Richard G. Heck, "The Julius Caesar Objection," 1997, in *Language, Thought, and Logic: Essays in Honour of Michael Dummett*, Oxford, Oxford University Press, 274

[15] Frege, section 68, 79

[16] Shapiro, *Thinking about Mathematics*, 114

suggested that all questions that philosophers can ask about mathematics are really questions about logic.

In 1902 Bertrand Russell wrote to Frege to inform him that his work was inconsistent.[17] The inconsistency that Russell discovered concerns Frege's reliance on extensionality and set theory to solve the "Caesar problem." What is now known as Russell's paradox shows that Basic Law V entails a contradiction:

> Let *R* be the concept that applies to an object *x* [if and only if] there is a concept *F* such that *x* is the extension of *F* and *Fx* is false. Let *r* be the extension of *R*. Suppose *Rr* is true. Then there is a concept *F* such that r is the extension of *F* and *Fr* is false. It follows from Basic Law V that *Rr* is also false (since *r* is also the extension of *R*). Thus if *Rr* is true, then *Rr* is false. So *Rr* is false. Then there is a concept *F* (namely *R*) such that *r* is the extension of *F* and *Fr* is false. So, by definition, *R* holds of *r*, and so *Rr* is true.[18]

In either case, we have that *Rr* is both true and false, a contradiction. Set theory allows for the formation of self-referring sets, but these sets always result in a contradiction when Basic Law V is applied. For example, let *S* be the set of all sets that do not contain themselves. Suppose *S* is itself a member of *S*. Then *S* contains itself hence *S* is not a member of *S*. If S is not a member of *S*, however, *S* does not contain itself. By the definition of *S*, *S* is a member of *S*. Therefore, *S* is both a member of itself and not a member of itself. Such self-referencing definitions are called impredicative. There was no obvious way for Frege to disallow impredicative cases without invoking intuition. His attempt at reducing mathematics to logic failed.

Russell, however, was not so quick to think that logicism was doomed. He noticed that the reason for the paradox was due to the impredicative definition of the concept *R*. *R* is defined with reference to any concept *F*, including itself. Russell considered circular, impredicative

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[17] Bertrand Russell, "Letter to Frege," 1902, in Jean van Heijenoort, *From Frege to Gödel: A Source Book in Mathematical Logic (1879-1931)*, 1967, Cambridge, Harvard University Press, 124-125, cited in Horsten, section 2.1

[18] Shapiro, *Thinking about Mathematics*, 114

statements absurd. [19] Instead of dismissing Basic Law V, Russell wanted to restrict the definition of *R* in such a way as to keep it from referring to itself. He created a supposedly non-intuitive way to exclude the problematic *R*—his ramified type structure of sets.[20] In this structure, different types of sets are assigned levels. Level 0 are mathematical entities and their properties, level 1 sets are those that contain level 0 entities, level 2 sets are those that contain level 1 sets, and so on.[21] This way of defining sets is predicative. That is, it does not allow for members of a set to share properties with the set itself.

Using this structure forced Russell to redefine the natural numbers, as Frege's definition caused them to be of different types.[22] According to Russell, "a number is anything which is the number of some class [or set]…The number of a class is the class of all those classes that are similar to it."[23] Thus 0 is the set of all sets with no members, 1 is the set of all sets with one member, and so on. Since each number is the set of a set of level 0 entities, all natural numbers are level 2 sets.

Russell's theory of types is not without problems. It allows that there exist only as many numbers as there are individual members that can belong to a set. David Hilbert noted that the universe is likely neither infinitely large nor infinitely divisible.[24] Since there may be only a finite number of entities, it is not necessarily the case that there are infinite natural numbers, as mathematicians have proven. While some radical philosophers of mathematics—the strict finitists—would allow for the finitude of natural numbers, the majority cannot accept this

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[19] Bertrand Russell, *Introduction to Mathematical Philosophy*, 1919, London, Allen & Unwin cited in Shapiro, *Thinking about Mathematics*, 116

[20] Horsten, section 2.1

[21] Russell, *Introduction to Mathematical Philosophy*, cited in Shapiro, *Thinking about Mathematics,* 117.

[22] Shapiro, *Thinking about Mathematics*, 118

[23] Russell, *Introduction to Mathematical Philosophy*, 19

[24] David Hilbert , "On the Infinite," 1925, in Paul Benacerraf and Hilary Putnam (ed), *Philosophy of Mathematics: Selected Readings*, 2nd ed., 1983, Cambridge, Cambridge University Press, 186

possibility.[25] Thus Russell's theory was widely thought to have failed to provide an adequate logicist theory of mathematics.

Intuitionism

Intuitionism is the theory that mathematical objects are abstract concepts, and mathematical operations and principles are mental constructions on these objects.[26] L.E.J. Brouwer, the founder of modern intuitionism, was an anti-realist about ontology and did not believe that the truth-value of mathematical statements derived from metaphysics. [27] He asserted that mathematics is a function of the mind—a way to interpret sense data. From Stewart Shapiro's perspective of Brouwer, "to think at all is to think in mathematical terms."[28] That is, Brouwer believed that intuition about mathematics begins with temporal perception. The world is understood as a discrete sequence of moments. The set of natural numbers can be abstracted by counting this sequence of discrete moments.[29] The sets of rational numbers (includes negative numbers and fractions) and real numbers (includes rational numbers and non-repeating decimals) are derived from the notion of "between" natural numbers. Arithmetic, real analysis, and geometry can all be derived in this manner.

One important result of rejecting platonic realism is that the law of excluded middle, '*A* or ¬*A'*, and its equivalents have no metaphysical basis. Arend Heyting, a student of Brouwer, formalized intuitionist logic without relying on such laws.[30] In Heyting's formalization what is allowed to be considered a proof of a proposition is determined by a set of rules. In Shapiro's

[25] Jean Paul Van Bendegem, "A Defense of Strict Finitism," 2012, in *Constructivist Foundations* 7, 141
[26] Horsten, section 2.2
[27] Shapiro, *Thinking about Mathematics,* 175
[28] Ibid.
[29] L.E.J. Brouwer, "Intuitionism and Formalism," 1912, in Paul Benacerraf and Hilary Putnam (ed), *Philosophy of Mathematics: Selected Readings,* 2nd ed., 1983, Cambridge, Cambridge University Press, 80
[30] Shapiro, *Thinking about Mathematics,* 185

words, for example, "a proof of a sentence of the form 'there is an x such that $\Phi(x)$' consists of the construction of an item n and a proof of the corresponding $\Phi(n)$," where n has the property Φ.[31] Intuitionists only allow for existence statements that include constructive proofs which provide either an example of an n such that $\Phi(n)$, or a method for deriving such an n. Existence, therefore, is synonymous with constructability.[32] Intuitionist mathematicians are restricted to working with only these constructive proofs.

Classically, mathematicians are not as restrictive and accept non-constructive proofs. One simple example is the proof of the following proposition:

> There exist non-rational [cannot be written as a fraction] numbers a and b such that a^b is rational.
>
> *Proof.* Take $b = \sqrt{2}$; so b is irrational. Either $\sqrt{2}^{\sqrt{2}}$ is rational, or it is not. If it is, then set $a = \sqrt{2}$. On the other hand, if $\sqrt{2}^{\sqrt{2}}$ is irrational, then take $a = \sqrt{2}^{\sqrt{2}}$, which makes $a^b = 2$ and thus rational. In either case, the conclusion holds.[33]

This proof does not show which of the two choices for a makes a^b rational. It neither constructs an example nor provides a method for constructing an example of irrational numbers with said property. Therefore, this does not count as a proof for the intuitionist. Note that the pivotal premise that '$\sqrt{2}^{\sqrt{2}}$ is either rational or not' relies on the law of excluded middle. In order to satisfy the intuitionists, much of classical mathematics must undergo expansion to include the neglected constructions or be revised to make it possible for construction to occur. It is not clear that all statements of mathematics can be revised in such a manner.

Brouwer does allow for classical mathematics to be used for science. However, he claims that it is less true than intuitionist mathematics due to its platonic assumptions. [34] Brouwer's demotion of mathematics used in science presents a problem for W.V.O Quine's claim that

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[31] Shapiro, *Thinking about Mathematics*, 186

[32] Arend Heyting, "Disputation," 1956, in Paul Benacerraf and Hilary Putnam (ed), *Philosophy of Mathematics: Selected Readings*, 2nd ed., Cambridge, Cambridge University Press, 67

[33] McKubre-Jordens, section 1a

[34] Brouwer "Consciousness, Philosophy, 1948, in Paul Benacerraf and Hilary Putnam (ed), *Philosophy of Mathematics: Selected Readings*, 2nd ed., 1983, Cambridge, Cambridge University Press,and Mathematics," 90

scientific theories are currently the epitome of rationality.[35] Moreover, James Brown notes that there are cases of useful scientific theories that cannot be revised to suit constructivist mathematics.[36] For example, suppose a physicist wants to check whether a remotely detonated bomb has discharged. The detonation signal will activate at temperature T. At time t_1, the physicist measures the temperature to be less than T. At time t_2, the temperature is measured to be greater than T. Assume that temperature changes continuously. The intermediate value theorem,[37] accepted in classical mathematics, allows the physicist to surmise that there was a time t between t_1 and t_2 when the temperature was T. This implies that the bomb has indeed detonated. In order to construct this time t, the physicist would need to take an infinite number of temperature measurements between t_1 and t_2. Since this is impossible, an intuitionist physicist cannot make any claim as to whether or not the bomb received the signal and was detonated. While it may not have concerned Brouwer, it appears as though contemporary science and intuitionism are at odds if not incompatible.

Formalism

Formalism is the theory that mathematics is a formal procedure of symbol manipulation.[38] In this procedure, mathematical objects are mathematical symbols whose manipulation is determined by rules and axioms.[39] David Hilbert crafted a version of formalism, known as deductivism, in which the symbols of mathematics make no reference to 'real' objects.[40] Any interpretation of

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[35] W.V.O. Quine, "Epistemology Naturalized," 1969, in W.V.O. Quine, *Ontological Relativity and Other Essays*, New York, Columbia University Press, cited in Horsten, section 3.2

[36] James Robert Brown, "Science and Constructive Mathematics," 2003, in *Analysis* 63, 49

[37] "If f is a continuous real-valued function on an interval I, then…whenever $a,b \in I$, $a<b$ and y lies between $f(a)$ and $f(b)$…there exists at least one x in (a,b) such that $f(x)=y$." Kenneth A. Ross, *Elementary Analysis: The Theory of Calculus*, 2nd Ed., 2013, New York, Springer, 134

[38] Shapiro, *Thinking about Mathematics*, 144

[39] Ibid., 142

[40] Ibid., 149

the symbols could be considered correct if the axioms correctly describe it. The number theorist's job is to develop theorems based on a set of axioms and definitions. Since theorems relating to the prime numbers, for example, fit well within the framework of programming, the computer scientist uses them to enhance computer security.[41] However, any application of mathematical theorems is inconsequential to the mathematician. Deductivism also asserts that knowledge of mathematics is knowledge of the rules of a procedure. For Hilbert, these rules amount to the laws of logic.[42] Hilbert does allow one small role for intuition—motivation for axiom choice. Since axioms cannot be proven, they cannot follow from a procedure. For Hilbert, axiom choice is outside the subject of mathematics.

Since mathematics does not depend on a particular material interpretation, the truth value of mathematical statements is tied to their consistency within a system.[43] Early on, formalists had managed to describe the systems with such rigor that they could be studied as mathematical objects themselves. Any operations performed on the systems taken as objects are considered meta-operations or meta-mathematics. According to Hilbert, using meta-mathematics to prove that a set of axioms is consistent is enough to guarantee that all statements within the system should be considered to be true.[44]

Hilbert's goal was to develop a program that would "establish once and for all the certitude of mathematical methods."[45] His program, as Johann von Neumann notes, consisted of

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[41] Kenneth H. Rosen, *Elementary Number Theory and Its Applications*, 6th ed, 2011, Boston, Addison-Wesley, 291-346

[42] Shapiro, *Thinking about Mathematics*, 159

[43] David Hilbert, *Foundations of Geometry*, 1899, E. Townsend (trans), La Salle, Open Court, cited in Shapiro, *Thinking about Mathematics*, 156

[44] Ibid.

[45] Hilbert, "On the Infinite," 184

four steps.[46] First, the symbols to be used within the system are specified.[47] For example, the symbol '+' is used in arithmetic. Second, formulas, or rules for symbol usage, are developed.[48] It is important to define how '+' will be used in certain circumstances. When adding natural numbers, a sentence like '12++' has no meaning, while a sentence like '16+2=20' does. At this stage, the truth value of the sentence is unimportant. Step three determines a formula construction method using deductive reasoning; that is, it develops proofs.[49] The final task is to show that formulas can be proven if and only if they are true.[50] Statements that are not consistent with the system are discarded as false. Until a system can be shown to be consistent within its own language it cannot be accepted.[51]

Hilbert's program was developed with finitary mathematics in mind. He did not believe that space and time were infinitely large nor infinitely divisible, and he wanted his mathematics to reflect the universe's finitude.[52] Since meta-mathematics has a real subject matter, and the real universe is finite, it was necessary for Hilbert to introduce the notion of 'finitary mathematics' to describe it. For Hilbert, a finite statement is any statement whose quantifiers are bounded in some way. Stewart Shapiro offers an example of the distinction between bounded and unbounded quantifiers:

(1) There is a number *p* greater than 100 and less than 101! + 2 such that *p* is prime. [53]
(2) There is a number *p* greater than 100 such that both *p* and *p* + 2 are prime.[54]

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[46] Johann von Neumann "The Formalist Foundations of Mathematics," 1931, in *Paul Benacerraf and Hilary Putnam (ed), Philosophy of Mathematics: Selected Readings, 2nd ed., Cambridge, Cambridge University Press,* 63

[47] Ibid.

[48] Ibid.

[49] Ibid.

[50] Ibid.

[51] Hilbert, *Foundations of Geometry*, cited in Shapiro, *Thinking about Mathematics,* 156

[52] Hilbert, "On the Infinite,"186

[53] "The number 101! is the result of multiplying 1, 2, 3, …, 101." Shapiro, *Thinking about Mathematics,* 159

[54] Ibid.

The first statement is bounded by the number 101! + 2, which is a number far too large to calculate by hand. Fortunately, Hilbert only requires that a bound be calculable in principle. Statement (2), on the other hand, has no bound.

Ultimately, Hilbert's program has been very useful in the standardization of contemporary mathematics.[55] It is unfortunate, then, that Hilbert's program was condemned to failure. Kurt Gödel proved two incompleteness theorems, showing that the fourth task of the program was impossible to actualize. The key insight has been summarized simply by Panu Raatikainen as follows:

> **First incompleteness theorem**
> Any consistent formal system *F* within which a certain amount of elementary arithmetic can be carried out is incomplete; i.e., there are statements of the language of *F* which can neither be proved nor disproved in *F*.
>
> **Second incompleteness theorem**
> For any consistent system *F* within which a certain amount of elementary arithmetic can be carried out, the consistency of *F* cannot be proved in *F* itself.[56]

This 'certain amount' of elementary arithmetic is what is needed to derive the other branches of mathematics. Gödel's first theorem shows that Hilbert would always be unsuccessful at finding the consistencies of at least some statements—in particular self-referring statements—in arithmetic. Thus, he would be unable to show that those statements were consistent in the other systems of mathematics as they are based on arithmetic. Hilbert takes a statement's consistency within a system to be its truth value, so it is impossible for him to show that truth is a sufficient condition for proof.[57] He is unable to complete the fourth step in his program. Hilbert also requires each system to be proven consistent within its own language. Therefore, Gödel's second theorem shows that Hilbert could not accept any non-trivial system as mathematics.

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[55] Shapiro, *Thinking about Mathematics*, 158

[56] Panu Raatikainen "Gödel's Incompleteness Theorems," 2015, *Stanford Encyclopedia of Philosophy*, section 1.1

[57] Hilbert, *Foundations of Geometry*, cited in Shapiro, *Thinking about Mathematics*, 156

Platonism

Platonism is the theory that takes mathematical objects to exist as abstract entities that are independent of human thought.[58] In considering their mind-independence, Kurt Gödel compares mathematical objects to physical objects. [59] Natural physical objects are discovered through observation; they are not human inventions. Although each subject may perceive them differently, physical objects present themselves to everyone in the same way. Humans have ideas of physical objects, but their existence is not exhausted by these ideas. Gödel concludes that mathematical objects are similar to natural objects in that they are discovered through observation, presented objectively, and existentially independent of perception. To illustrate this existential independence, for example, if five apples have fallen from a tree, the fact that the tree is now growing five fewer apples does not rely on there being someone around to observe this fact.

Mathematical intuition is analogous to the notion of 'perception' of physical objects.[60] Perceiving the physical correspondence of the five apples to five oranges, the gatherer becomes aware of the mathematical similarities between them. This person has thus been introduced to the notion of 'fiveness' that existed before her awareness of it. Once the gatherer has discovered 'fiveness,' she can then discover its properties by analogy with physical processes. If she takes two oranges from one tree and three oranges from another, she can learn that these two quantities grouped together make five, no matter what it is that she is grouping.

[58] Julian C. Cole, "Mathematical Platonism," 2010, in *The Internet Encyclopedia of Philosophy*, section 1

[59] Kurt Gödel, "Russell's Mathematical Logic," 1944, in Paul Benacerraf and Hilary Putnam (ed), *Philosophy of Mathematics: Selected Readings,* 2nd ed., Cambridge, Cambridge University Press, cited in Horsten, section 3.1

[60] Kurt Gödel, "What is Cantor's Continuum Problem?" 1947, in Paul Benacerraf and Hilary Putnam (ed), *Philosophy of Mathematics: Selected Readings,* 2nd ed., Cambridge, Cambridge University Press, cited in Horsten, section 3.1

Sensory perception and mathematical intuition are also both fallible.[61] The gatherer could pick five apples, drop one without noticing, quickly glance at her bounty, and deduce that she has enough to give one to each of her five children. She has made a perceptual error in not noticing the dropping of an apple, and an error in mathematical intuition in thinking that each of her children will receive an apple.

Gödel also points out a difference between mathematical and physical objects.[62] Mathematical objects do not exist spatially or temporally; they are abstract. The gatherer can notice that many groups of objects have five members, but she can never isolate the 'fiveness' sensorily from the group and inspect it alone.

An epistemological problem, as noted by Paul Benacerraf, arises when claiming that mathematical objects are abstract.[63] If human beings are purely physical and mathematical objects are purely abstract, then humans cannot causally interact with mathematical objects. Thus, platonists must come up with an explanation of how knowledge of mathematical objects is obtained. One potential solution, plenitudinous platonism, attempts to solve the problem by not requiring a causal relationship.[64] In this view, "all consistent candidates for mathematical theories are true" even if humans cannot access their objects.[65] Mark Balaguer argues that consistency, or obtainable justification, is enough to set platonists free from needing to bridge the impossibly wide gap between the world of mathematical objects and the physical world. Each theory necessarily refers to its own universe.[66] The mathematical realm is large enough to

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[61] Ibid.

[62] Gödel, "Russell's Mathematical Logic," cited in Horsten section 3.1

[63] Paul Benacerraf "Mathematical Truth," 1973, in Paul Benacerraf and Hilary Putnam (ed), *Philosophy of Mathematics: Selected Readings*, 2nd ed., 1983, Cambridge, Cambridge University Press, 415

[64] Mark Balaguer, *Platonism and Anti-Platonism in Mathematics*, 1998, Oxford, Oxford University Press, cited in Horsten, section 3.5

[65] Mary Leng, "Platonism and Anti-Platonism: Why Worry?" 2005, in *International Studies in the Philosophy of Science* 19, 68

[66] Balaguer, cited in Horsten, section 3.5

contain any and all objects with the properties seen in typical mathematical statements, and it does so as long as these objects are justifiably used within a given theory.[67] In other words, mathematicians are free to assume that any mathematical object of their liking exists, as long as it is logically possible for it to exist in a given theory. That pi is the ratio of a circle's circumference to its diameter is verifiable within Euclidean geometry, for example, so pi itself does exist even if humans cannot access its essence. Hence, mathematicians gain knowledge about mathematical objects by creating consistent theories about them.

Benacerraf noticed a second problem with platonism, called the "identification problem" that can be illustrated with the following example.[68] Where 0 = ∅, the empty set, the natural numbers can be defined within set theory in both of the following ways:

(i) [∅], [[∅]], [[[∅]]], …
and
(ii) [∅], [∅, [∅]], [∅, [∅], [∅, [∅]]], …[69]

Since platonism is a form of realism, and realists claim that propositions correspond with the unique, independent reality, there should be only one way of defining the natural numbers.[70] Therefore, if definitions (i) and (ii) differ in some way, they cannot both be correct. In this case, the sets are isomorphic in arithmetic, meaning that all arithmetic operations performed under definition (i) yield the same answers as when they are performed under definition (ii). However, in more complex mathematical systems they exhibit differences. For example, the number 3 in definition (i)—[[[∅]]]—contains one element but in definition (ii)—[∅, [∅], [∅, [∅]]]—contains three elements. Platonism requires there to be only one correct definition of natural numbers. However, it does not provide a criterion for which of these definitions, if either, to pick.

[67] Balaguer, cited in Cole, section 4

[68] Paul Benacerraf, "What Numbers Could Not Be," 1965, in Paul Benacerraf and Hilary Putnam (ed), *Philosophy of Mathematics: Selected Readings*, 2nd ed., 1983, Cambridge, Cambridge University Press, 279

[69] Benacerraf, "What Numbers Could Not Be," 278

[70] Lee Braver *A Thing of This World: A History of Continental Anti-Realism*, 2007, St. Evanston, IL, Northwestern University Press, 15-17

Despite possible difficulties with platonism, the Quine-Putnam Indispensability Argument (QPIA)[71], shows that platonism is essential to scientific realism. The QPIA is summarized in the following argument:

1. If a scientific theory is true, then the mathematical entities that it employs exist.

2. Some scientific theories are true.

3. Therefore, the mathematical entities employed by those true scientific theories exist. [72]

This argument assumes a background of scientific realism, which is the attitude that science attempts to describe the physical world as it really is.[73] Under realism, protons, neutrons, and electrons are real physical objects. It should be noted that the QPIA does not affect theoretical mathematics, only mathematics applied to scientific theories. In fact, Øystein Linnebo holds that theoretical mathematicians speak as though mathematical objects exist, but whether they, in fact, exist does not affect their work.[74] If platonism were false, nothing would change for them. However, science would be viewed very differently. By the QPIA, if scientific realism is true, then platonism is also true.[75] To illustrate this, consider Newton's second law of physics according to which: "the change of motion is proportional to the motive force impressed; and is made in the direction of the right line in which that force is impressed."[76] This law is known in physics by the formula **F**=m**a**, where **F** is the sum of forces acting on an object, m is the mass of the object, and **a** is the acceleration of the object. Newton's formulation of the law does not overtly refer to mathematical objects. However, forces, masses, and accelerations are real

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[71] Developed by Quine and Hilary Putnam in Hilary Putnam, *Philosophy of Logic*, 1972, London, Allen & Unwin, cited in Horsten, section 3.2

[72] Cole, section 2b

[73] Peter Godfrey-Smith, *Theory and Reality: An Introduction to the Philosophy of Science*, 2003, Chicago, University of Chicago Press, 176

[74] Øystein Linnebo, "Platonism in the Philosophy of Mathematics," 2013, in *Stanford Encyclopedia of Philosophy*, section 1.5

[75] Leng, 66

[76] Newton "Mathematical Principles of Natural Philosophy," 1687, in *The Portable Enlightenment Reader*, Issac Kramnick ed., New York, Penguin Books, 45

number magnitudes with forces and accelerations being tied to some direction. To use Newton's second law or to say it represents reality is in some sense to say that real numbers exist. To say that a real number exists is to say that platonism is true. If platonism is false, then, by simple *modus tollens*, scientific realism must also be false:

1. If scientific realism is true, then platonism is true.
2. Platonism is false.
3. Therefore, scientific realism is false.

In the words of Putnam, it follows that "the success of science [would be] a miracle."[77]

Structuralism

Structuralism, a theory backed by Stewart Shapiro and Paul Benacerraf, states that mathematical objects exist only as relationships within their systems.[78] For example, the natural number three is nothing more or less than its relation to the other natural numbers—one more than two, half of six, eight less than eleven, etc. In structuralist terminology, a system is any set of related objects. Mathematics is the study of the forms of relatedness that are abstracted from these systems, "highlighting the interrelationships among the objects, and ignoring any features of them that do not affect how they relate to other objects in the system."[79] These abstracted forms of mathematical study are called 'structures.' Structuralists do not believe that it is possible to discuss a mathematical object without at least implicitly referring to the structure in which it belongs. If, for example, one tried to divide three by two, the answer would make sense in the rational number structure since this structure contains all numbers that can be written as

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[77] Hilary Putnam, *Mathematics, Matter and Method: Philosophical Papers,* 1975, vol. 1, Cambridge, Cambridge University Press, 73

[78] Shapiro, *Thinking about Mathematics,* 258

[79] Ibid., 259

fractions, but it would not make sense in the natural number structure, which consists of non-negative whole numbers. Any system that permits this division cannot be an instance of the natural numbers structure. Rules of the structure determine which actions are allowed. All mathematics requires an implicit understanding of which structure is controlling the relation of objects.

Shapiro characterizes his position as "*ante rem* structuralism." As such, he believes that structures exist independently of and before the systems they instantiate.[80] This characterization puts him in a similar ontological position to the mathematical platonists. *Ante rem* structuralists, however, do not accept the platonist's idea of a universal's "explanatory primacy." A platonist would claim that a circle is round because it instantiates the universal 'roundness.' Shapiro states that *ante rem* structuralists "do not hold, for example, that a given system is a model of the natural numbers because it exemplifies the natural number structure.…What makes a system exemplify the natural number structure is that it is a model of arithmetic."[81]

Since *ante rem* structuralism is so similar to platonic realism, it must find a way to overcome the problems faced by platonists. Benacerraf's identification problem (described above in the platonism section) shows that two unique and incompatible set-theoretic systems of natural numbers can be instantiated by the natural number structure.[82] Asking a question about the set-theoretic properties of an arbitrary element in this structure can result in two different answers depending on which system is chosen. Structuralism dictates that the structure of natural numbers instantiates both set-theoretic systems, but neither system is identical to its structure.[83] Thus, Shapiro claims that asking questions external to a structure or even comparing systems that

[80] Ibid., 263
[81] Ibid., 264
[82] Benacerraf, "What Numbers Could Not Be," 279
[83] Horsten, section 4.2

share the same structure is not cogent. Asking these sorts of questions is "similar to asking whether the number 1 is funnier than the number 4, or greener."[84] By allowing only questions that are internal to a given structure, *ante rem* structuralists are able to avoid the identification problem altogether.

Benacerraf's epistemological problem (described above in the platonism section) is more difficult to solve. Since structures are abstract, non-physical entities, *ante rem* structuralists must provide some explanation as to how physical beings can know them.[85] Shapiro describes the abstraction process as follows:

> One first contemplates the finite cardinal structures as *objects* in their own right. Then we form a *system* consisting of the collection of these finite structures with an appropriate order. Finally, we discuss the *structure* of this system.[86]

Shapiro admits that this process does not resolve the epistemological problem. Ultimately, Shapiro relies on the success of *ante rem* structuralism at describing classical mathematics as evidence that structures exist.[87]

Moderate realism

Moderate realism is an often neglected theory that proposes that mathematical objects are mind-dependent concepts abstracted from physical objects that do not, however, incur subjectivity.[88] That the number three can be said of the wheels of a tricycle, for example, depends on the existence of a mind capable of abstracting three from the wheels. The threeness of the wheels presents itself to everyone in the same way as the threeness in the vertices of a triangle. Hence, three is the same regardless of what objects are presented as a triple. Michael Tavuzzi states that,

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[84] Shapiro, *Thinking about Mathematics,* 266

[85] Benacerraf, "Mathematical Truth," 415

[86] Shapiro, *Thinking about Mathematics,* 280

[87] Stewart Shapiro, *Philosophy of Mathematics: Structure and Ontology*, 1997, New York, Oxford University Press, cited in Shapiro, *Thinking about Mathematics,* 282

[88] Tavuzzi, 262

for the moderate realist "the objects of mathematics are no mere arbitrary fictions—they have an objective foundation in reality: the quantitative accidental forms of figure and number which inhere in existing material things and which are indirect objects of the external senses."[89] This objective foundation addresses Hilary Putnam's concern for the miracle status of the success of scientific prediction in non-platonic theories.[90] Under moderate realism, scientific theory would also be considered mind-dependent. It is fitting that the same abstraction process could be used for both mathematical and scientific objects and relations. The two subjects mesh so well together because they both have foundations in the same reality. The abstraction process also solves Benacerraf's epistemological problem which is concerned with how physical beings can have knowledge of abstract objects without being able to causally interact with them.[91] Since it is the human mind that performs the abstraction, humans do have a causal influence on mathematical objects.

Thomas Aquinas proposes a mechanism of abstraction called "simple apprehension."[92] Simple apprehension can be divided into eight steps: object, species, sense organ, impressed species, common sense, phantasm, active intellect, and potential intellect. All knowledge begins with objects. The species carries the intelligibility of the object to the subject. Sense organs collect these species. The species must leave an impression behind or it will be ignored. What is referred to as common sense notes the simultaneous agreement of the impressed species from different senses. These impressed species form a unity called the phantasm. Drawing on

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[89] Tavuzzi, 262

[90] Putnam, *Mathematics, Matter and Method*, 73

[91] Benacerraf, "Mathematical Truth," 415

[92] Thomas Aquinas, *Summa theologiae, Part I, question 85, art. 1,* Dominicans of the English Province (trans), Blackfriars and McGraw Hill, New York, 1964

Aristotelian doctrine,[93] Aquinas proposes that part of the intellect, called the "active intellect," changes the passive, material phantasm to an active, immaterial intelligibility. Finally, the potential intellect receives the abstract, active idea so that the concept is possessed by the knower.

An observation of Stewart Shapiro about deriving the natural numbers through abstraction could pose a problem for moderate realism.[94] In order for a number to exist, it must be able to be abstracted from some set of physical things. This implies that in order to use very large numbers or infinity, there must be enough physical things that can be grouped in order for someone to process the abstraction. It is not obvious that this is the case. Moreover, Tobias Dantzig notes that humans who lack number words and counting are incapable of perceiving the difference between groups of small numbers such as seven and eight.[95] There seems to be a numeric threshold for the mind's ability to identify the quantity of objects in a small group. On this account, it seems impossible for someone to be able to abstract large numbers in the first place.

Aquinas, however, would not agree that there is a problem with large numbers. He allows for the reflection on the active ideas acquired through simple apprehension. In fact, it is through this reflection that people are able to discover that there exist individual objects from which abstractions are made. [96] Dantzig notes that all humans can distinguish between one and two even South African Bushman who only have numerical words for the concepts 'one,' 'two,' and

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[93] Aristotle, *De Anima*, 430a15-20, Terence Irwin and Gail Fine (trans), in *Introductory Readings in Ancient Greek and Roman Philosophy*, C.D.C Reeve and Patrick Lee Miller (ed), Indianapolis, Hackett Publishing, 285

[94] Shapiro, *Thinking about Mathematics*, 66

[95] Tobias Dantzig, *Number: The Language of Science*, 2007, Joseph Mazur (ed), New York, Penguin Group, 5

[96] Thomas Aquinas, *Summa theologiae, Part I, question 86, art. 1,* Dominicans of the English Province (trans), Blackfriars and McGraw Hill, New York, 1964

'many.'[97] Thus, it is possible for everyone to abstract one and two from physical objects. From reflection on the difference between one and two, it seems as though developing a "plus one" principle would be the obvious next step. Then, it is only a matter of time before larger number words and counting are established. Similar to developing large numbers, of the infinite Aquinas states that "infinity is potentially in our mind through its considering successively one thing after another."[98] Thus, Shapiro's large number problem can be avoided with Aquinas's reflection allowance.

Criteria for Adequate Philosophy of Mathematics

In order to pick out which theories best describe current mathematical practice, a standard of judgment must be stipulated. Michael Tavuzzi's criteria for the acceptance of a philosophical theory of mathematics requires the theory to include the following:

> (1) The fact that doing mathematics consists largely of the rule-governed, formal, manipulation of symbol.
> (2) The fact that this seemingly mechanical manipulation of symbols is grounded in a rich dimension of subjective construction and daring intuition which is characteristic of the mathematical consciousness.
> (3) The fact that, nonetheless, mathematical consciousness is not concerned with some psychologistically interpreted subjective dimension but with objective truths and structures.
> (4) The fact that mathematics holds of the real world and therefore must have some intimate connection with reality.[99]

The necessity of criterion (1) is obvious, and all theories discussed above allow for rule-governed symbol manipulation. Criterion (2) is slightly more controversial; not everyone agrees that it is necessary to account for mathematical intuition. It does, however, seem uncontroversial that mathematicians often use intuition to pose conjectures that are not yet proved or disproved. For example, examining the early primes, 3 and 5, 5 and 7, 11 and 13, etc., one could conclude

[97] Dantzig, 5

[98] Thomas Aquinas, *Summa theologiae, Part I, question 86, art. 2,* Dominicans of the English Province (trans), Blackfriars and McGraw Hill, New York, 1964

[99] Tavuzzi, 269

that the twin prime conjecture, that "there are infinitely many pairs of primes p and $p + 2$" is intuitive.[100] Daniel Shanks notes that given that there have been over one hundred thousand confirmations of twin primes "the evidence for [the conjecture] is overwhelming."[101] However, it remains a matter of mathematical intuition because it has not been proven. The objectivity of mathematics from criterion (3) of Tavuzzi's criteria appears to be mostly uncontested. This is due in part to the existence of complex mathematics that follows from necessity from the universally accepted fundamental arithmetic. Criterion (4) is the most disputed of Tavuzzi's criteria, particularly with anti-realists who do not want to make mathematics dependent on an ontological position about reality.[102] Militating against the anti-realist position, however, are the innumerable outcomes which would be relegated to the status of coincidences by the anti-realists, such as the value that the study of large prime numbers adds to computer security.[103]

One immediate concern with using Tavuzzi's criteria as the standard of judgment for the above philosophical theories is that Tavuzzi is a moderate realist and likely crafted his criteria with his position of philosophy of mathematics in mind. Comparing Tavuzzi's criteria with another, more general set of criteria will showcase the level of objectivity that Tavuzzi's criteria employs. One such set of criteria is presented by Paul Ernest, an advocate of social constructivism which is not discussed in this paper. In his view, any theory of mathematics must consider the following:

1. Mathematical knowledge: its character, genesis and justification, with special attention to the role of proof
2. Mathematical theories, both constructive and structural: their character and development, and the issues in their appraisal and evaluation
3. The objects of mathematics: their character, origins, and relationship with the language of mathematics

[100] Rosen, 86
[101] Daniel Shanks, *Solved and Unsolved Problems in Number Theory*, 1993, 4th ed., Providence, AMS Chelsea Publishing, 219
[102] Braver, 13
[103] Rosen, 291-346

4. The applications of mathematics: its effectiveness in science, technology, and other realms and , more generally, the relationship of mathematics with other areas of knowledge and values
5. Mathematical practice: its character, and the mathematical activities of mathematicians, in the present and past
6. The learning of mathematics: its character, and its role in the onward transmission of mathematical knowledge and in the creativity of individual mathematicians.[104]

Ernest takes criteria 1 and 3 to be trivial notions that philosophical theories in general attempt to explain.[105] The fact that Tavuzzi does not mention these criteria should not invalidate his standard of judgment. Tavuzzi's criterion (4) is generally equivalent to Ernest's criterion 4, with the stipulation that Tavuzzi is committed to the claim that we can have knowledge of reality, a claim that Earnest does not explicitly make. The combination of Tavuzzi's criteria (1) and (2) comprise Ernest's criterion (5). Criterion (3) adopted by Tavuzzi has no explicit equivalent criterion within Ernest's standard. It is possible that Ernest would consider Tavuzzi's criterion (3) covered by some combination of his other criteria, or perhaps his position is one of the few that outright rejects mathematical objectivity. However, as mathematical theories are often considered the most certain among the natural sciences, I will continue to consider Tavuzzi's criterion (3) important to any description of mathematics.[106] Detailed discussion of Ernest's criteria 2 and 6 is outside the scope of this paper, but may prove important for future study. Despite possible disputes regarding some of Tavuzzi's criteria and in light of similar criteria given by Ernest, I shall compare Tavuzzi's criteria to the preceding philosophical theories in order to deduce which is the best description of present-day mathematics.

As noted above, Gottlob Frege's logicism ideally disallowed the use of intuition, going against criterion (2).[107] However, Frege was unsuccessful at eliminating paradox without relying

[104] Paul Ernest, *Social Constructivism as a Philosophy of Mathematics*, 1998, Albany, State University of New York Press, 56-57
[105] Ernest, 57
[106] Horsten, introduction
[107] Ibid., section 2.1

on intuition, as Bertrand Russell discovered. Self-referring sets are derivable in Frege's set theory. However, intuition dictates that in order to make Frege's system consistent, self-referring sets must be excluded. The removal of self-referring sets is not mandated by principles of logic. Russell similarly failed to describe a plausible logicist theory. His ramified type structure requires intuition to allow for an infinite set of natural numbers. Generally speaking, these results indicate that recourse to intuition is needed to derive infinite number from a finite universe. Disallowing human intuition, though methodologically desirable, ultimately was the cause of logicism's demise.

L.E.J. Brouwer's intuitionism opposes criteria (3) and (4). For Brouwer, mathematics is a completely subjective process occurring entirely in the mind.[108] As discussed above, the law of excluded middle is rejected by intuitionists as being dependent on a realist view. This, however makes the resulting mathematics incompatible with some useful scientific theories as James Brown notes.[109] Such an outcome would prompt W.V.O. Quine to say that intuitionism is the theory at fault for being at odds with reality.[110]

David Hilbert's deductive formalism was shown to be untenable as a philosophical theory. Under Hilbert's program, only the most trivial mathematical systems could avoid the result of Kurt Gödel's incompleteness theorems.[111] However, much non-trivial mathematics is quite useful, if incomplete. Presumably a correct description of reality would have to be complete, so most of Hilbert's theory cannot be a description of reality and is in conflict with criterion (4).

[108] Shapiro, *Thinking about Mathematics*, 175
[109] Brown 2003, 49
[110] Quine, cited in Horsten, section 3.2
[111] Raatikainen, section 1.1

Both Gödel's platonism and Stewart Shapiro's *ante rem* structuralism fall prey to Paul Benacerraf's epistemological problem. Neither theory provides an adequate explanation as to how humans can know abstract objects with which they have no causal relation. Mark Balaguer's plenitudinous platonism's attempt at solving the problem makes platonism sound much like Hilbert's failed formalism. Under plenitudinous platonism, Balaguer claims that object consistency within a theory implies object existence in some platonic realm.[112] However, this does not address Gödel's first incompleteness theorem in which not all statements could be proved consistent within a theory.[113] Thus, Balaguer's attempt to avoid the epistemological problem fails in this respect. Shapiro's attempt to describe an abstraction process, by his own admission, also fails to solve the epistemological problem.[114] Since neither platonism nor *ante rem* structuralism can explain how humans know mathematical objects, the theories do not describe the reality of the human relationship with mathematics, conflicting with criterion (4).

Platonism has another problem, Benacerraf's identification problem, which it shares with moderate realism. Benacerraf notes two different set-theoretic definitions of natural numbers which both yield the same answers to all arithmetic questions, but yield different answers about some set theory questions.[115] Realists, both platonic and moderate, claim that there is one unique reality and therefore only one definition of natural number.[116] Neither platonism nor moderate realism is capable of explaining which of the set-theoretic definitions is correct. To this extent these theories are in conflict with criterion (4) as it is not clear which natural number definition refers to reality, or in what sense they both do.

[112] Balaguer, cited in Horsten, section 3.5
[113] Raatikainen, section 1.1
[114] Shapiro, *Philosophy of Mathematics*, cited in Shapiro, *Thinking about Mathematics*, 282
[115] Benacerraf "What Numbers Could Not Be," 279
[116] Braver, 15-17

A structuralist approach to solving the identification problem is to apply platonism or moderate realism to the natural number structure, for example, instead of the natural numbers themselves. Structuralists are able to avoid the identification problem by observing that the structure of natural numbers instantiates both of Benacerraf's set-theoretic definitions, but neither definition is identical to its structure.[117] *Ante rem* structuralism has already been disqualified due to its conflict with criterion (4).

Moderate Realist Structuralism

What might be called "moderate realist structuralism" has not yet been synthesized as a position in the philosophy of mathematics. Drawing on the strengths of *ante rem* structuralism and moderate realism could overcome the challenges that they face individually. As noted above, structuralism takes mathematical structures, or relations between mathematical entities to be objects.[118] Instead of taking mathematical structures to be platonic objects as *ante rem* realists do, moderate realist structuralism takes mathematical structures to be moderate realist objects; that is, they are mind-dependent abstractions formed when one recognizes some pattern in a system of related physical objects and ignores any features unrelated to the pattern.[119] For example, in considering an analog clock, one could abstract from it the underlying group[120] structure of the relationships between the numbers. Under moderate realist structuralism, structures are concepts, but they are not subjective. They exist as the intelligibility of the

[117] Horsten, section 4.2

[118] Shapiro, *Thinking about Mathematics*, 259

[119] Tavuzzi, 262

[120] Let G be a set together with a binary operation (usually called multiplication) that assigns to each ordered pair (a, b) of elements of G and element in G denoted by ab. We say G is a *group* under this operation if the following three properties are satisfied.

1. *Associativity*. The operation is associative; that is, $(ab)c = a(bc)$ for all a, b, c, in G.
2. *Identity*. There is an element e (called the *identity*) in G such that $ae = ea = a$ for all a in G.
3. *Inverses*. For each element a in G, there is an element b in G (called an *inverse* of a) such that $ab = ba = e$.

Joseph A. Gallian, *Contemporary Abstract Algebra*, 7th ed., 2010, Belmont, Brooks/Cole, Cengage Learning, 41 relations of physical objects in a system. This intelligibility is available to be known by everyone in the same way. The abstraction process serves as a causal gateway between physical humans and nonphysical structures, thereby solving Benacerraf's epistemological problem, as discussed above in the moderate realism section.[121] Also, as detailed above in the structuralism section, taking mathematical objects to be structures frees the moderate realist from Benacerraf's identification problem. [122]

In light of these solutions, one might consider whether this "moderate realist" or "*in re*" structuralism could be a philosophical theory that does not conflict with Tavuzzi's criteria:

(1) Moderate realist structuralism would allow for rule-governed symbol manipulation, as moderate realism and structuralism do.

(2) The moderate realist abstraction process allows for reflection on abstractions.[123] This reflection conforms to the need for mathematical intuition.

(3) Even though the reflection is a subjective, mental practice, the abstraction process begins in the objective reality, and mathematics has an objective subject.[124]

(4) Since the abstraction process begins in the objective reality, moderate realist structuralism is fundamentally rooted in reality. By following the structuralist's defense against the identification problem, Benacerraf's seemingly problematic case is avoided, and there is no conflict regarding which of two or more incompatible instantiations of a system is correct in representing reality.

It seems as though out of all the theories discussed, this theory is the most adherent to Tavuzzi's criteria.

[121] Benacerraf, "Mathematical Truth," 415
[122] Horsten, section 4.2
[123] Aquinas, *Summa theologiae*, Part I, question 86, art. 1
[124] Aquinas, *Summa theologiae*, Part I, question 85, art. 1

Despite its apparent adequacy to the above standard of judgment, moderate realist structuralism may still face other criticisms. One possible rejection of this theory could result from the fact that there is currently no accepted psychological account of the brain's pattern recognition mechanism.[125] For example, Aquinas's simple apprehension would not be a satisfactory mechanism to those who identify the mind with the brain, as they would reject the proposed immaterial steps.[126] However, nothing about moderate realist structuralism requires that simple apprehension be the adopted abstraction method. Any cogent mechanism of abstraction that begins in the sensory world and ends with human knowledge of mathematics could be proposed in its place.

On a similar note, one could reject that humans abstract structures in place of mathematical objects. As Shapiro notes, “a child can learn much about the number 2 while knowing next to nothing about other numbers like 6 or 6,000,000.”[127] This appears to suggest that the number 2 is not dependent on its place in the natural numbers structure. However, by analogy with material objects, Shapiro notes that epistemic independence does not imply ontological independence.[128] Many drivers are unaware of their cars’ complex engine mechanics, but the cars still depend on their engines to be drivable. It also seems as though abstracting quantity is impossible without first noticing some difference between a single object and multiple objects. This difference is the beginning of the structure of natural numbers.

Perhaps other criticisms could be made of moderate realist structuralism. It is beyond the scope of this paper to attempt to address all of them. Nonetheless, it has been my aim to propose

[125] Shapiro, *Thinking about Mathematics*, 276
[126] Aquinas, *Summa theologiae*, Part I, question 85, art. 1
[127] Shapiro, *Thinking about Mathematics*, 258
[128] Ibid.

that of all the theories discussed, it is the most adherent to Tavuzzi's criteria and provides the best description of current mathematical practice.

Conclusion

In this paper, I discussed six philosophical theories of mathematics and some of their corresponding problems. Logicism takes the position that foundational mathematics is purely about symbolic logic and logic relations. Intuitionism is an anti-realist theory that takes mathematics to be about mental constructions. Formalism proposes that mathematical objects are nothing more than the symbols with which they are written, with formal procedures governing the symbol manipulation. Platonism theorizes that mathematical objects are real, mind-independent, abstract entities. Structuralism takes mathematical objects to be relationships within structures. Moderate realism appropriates mathematical objects as objective, mind-dependent, abstractions from material objects.

None of these theories is without criticism. Logicism incurs Russell's paradox. Intuitionism is at odds with scientific theory. Formalism breaks down in view of Gödel's incompleteness theorems. Platonism and *ante rem* structuralism incur Benacerraf's epistemological problem. Moreover, platonism and moderate realism incur Benacerraf's identification problem, and moderate realism is left to overcome the problem of providing an accepting psychological account of the brain's mechanism of pattern recognition. Combining moderate realism with structuralism, however, seems to yield a theory which is able to overcome these problems.

The preceding has been an attempt to evaluate various philosophical theories of mathematics and select the option that best describes current mathematical practice. It does not presume to have presented every theory nor to have addressed every problem that could arise for the theories presented. However, it does claim to have treated problems that any philosophical theory of mathematics will need to address, and proposes an avenue towards a solution to these problems.

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